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EXACT CONSTANTS IN JACKSON-TYPE INEQUALITIES FOR L_2-APPROXIMATION ON AN AXIS

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We investigate exact constants in Jackson-type inequalities in the space L_2 for the approximation of functions on an axis by the subspace of entire functions of exponential type.

Let L_2 be the space of real-valued functions f defined and measurable on $(-\infty, \infty)$ that satisfy the condition

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 < \infty$$

and let L_2^r, $r \geq 0$, be the set of all functions f such that their $(r-1)$th derivatives on the axis are locally absolutely continuous and $f^{(r)} \in L_2$ (if r is not an integer, then $f^{(r)}$ is the derivative in the sense of Weyl).

Let E_σ denote the class of entire functions of exponential type $\geq \sigma$, let

$$B_\sigma = L_2 \cap E_\sigma,$$

and let

$$A_\sigma(f) = \inf\{\|f - g_\sigma\| \mid g_\sigma \in B_\sigma\} \tag{1}$$

be the approximation of a function $f \in L_2$ by the set B_σ.

Denote the pth integral modulus of smoothness of a function f by

$$\omega_p(f;t) = \sup\left\{\left\|\Delta_\eta^p f(\cdot)\right\| \,\middle|\, |\eta| \leq t\right\}, \tag{2}$$

where $\Delta_\eta^p f(x)$ is the difference of order p of a function f at a point x with step η.

As usual,

$$F(f;\omega) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \exp(-i\omega t)\, f(t)\, dt \tag{3}$$

is the Fourier transform of a function f.

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Inequalities of the form

$$A_\sigma(f) \;\le\; \frac{\aleph}{\sigma^r}\,\omega_p\big(f^{(r)};\delta/\sigma\big) \tag{4}$$

are called Jackson-type inequalities. In these inequalities, the least constant

$$\aleph \;=\; \aleph_{\sigma,r,p}(\delta) \;=\; \sup_{\substack{f\in L_2^r\\ f\neq\text{const}}} \frac{\sigma^r A_\sigma(f)}{\omega_p\left(f^{(r)};\delta/\sigma\right)} \tag{5}$$

is called exact.

The problem of the determination of exact constants in Jackson-type inequalities in the space L_2 was studied in many works (see, e.g., [1–5] and the bibliography therein).

The aim of the present paper is to generalize exact Jackson-type inequalities for the best approximations of periodic functions by trigonometric polynomials in the space L_2 (which were investigated in [3, 4]) to the case of approximation of functions by entire functions of exponential type on the entire axis in the space L_2.

The following theorem is true:

Theorem 1. *For any* $a > 1$, $\sigma > 0$, $r \ge 0$, *and* $p = 1, 2, \ldots$ *and any nonzero nonnegative summable function* $\theta(t)$, $0 < t < b < \pi$, *the following inequalities are true:*

$$\sup_{\substack{f\in L_2^r\\ f\neq\text{const}}} \frac{\sigma^{2r}A_\sigma^2(f)}{\int_0^b \omega_p^2\left(f^{(r)};t/\sigma\right)\theta(t)dt} \;\le\; \frac{a^{2r}}{a^{2r}-1}\left\{2^p \inf_{1\le y\le a}\Phi_{b,r,p}(\theta;y)\right\}^{-1}, \tag{6}$$

where

$$\Phi_{b,r,p}(\theta;y) \;=\; y^{2r}\int_0^b (1-\cos yt)^p\,\theta(t)\,dt. \tag{7}$$

Proof. It is known [1] that, for any function $f \in L_2$, one has

$$A_\sigma^2(f) \;=\; \int_{|\omega|\ge\sigma} |F(f;\omega)|^2\,d\omega. \tag{8}$$

Hence, taking into account that the function $|F(f\,;w)|$ is even by virtue of the fact that f is real-valued, we conclude that the following relation holds for any function $f \in L_2^r$:

$$A_\sigma^2(f) \;=\; 2\int_\sigma^\infty |F(f;\omega)|^2\,d\omega \;=\; \sum_{\mu=0}^{\infty}\int_{a^\mu\sigma}^{a^{\mu+1}\sigma} 2|F(f;\omega)|^2 d\omega$$

$$=\; \sum_{\mu=0}^{\infty}\int_{a^\mu\sigma}^{a^{\mu+1}\sigma} 2|F(f;\omega)|^2\,\frac{2^p\omega^{2r}\int_0^b\left(1-\cos\frac{\omega}{a^\mu\sigma}t\right)^p\theta(t)\,dt}{2^p\left(a^\mu\sigma\right)^{2r}\left(\frac{\omega}{a^\mu\sigma}\right)^{2r}\int_0^b\left(1-\cos\frac{\omega}{a^\mu\sigma}t\right)^p\theta(t)\,dt}\,d\omega$$

$$\leq \sum_{\mu=0}^{\infty} \frac{\int_0^b \left\{ \int_{a^\mu\sigma}^{a^{\mu+1}\sigma} 2|F(f;\omega)|^2 2^p \omega^{2r} \left(1-\cos\frac{\omega}{a^\mu\sigma}t\right)^p d\omega \right\} \theta(t)\,dt}{\left(a^{2r}\right)^\mu \sigma^{2r} 2^p \inf\limits_{1\le y\le a} y^{2r} \int_0^b (1-\cos yt)^p\,\theta(t)\,dt}$$

$$\leq \sum_{\mu=0}^{\infty} \frac{\int_0^b \left\{ \int_0^{\infty} 2|F(f;\omega)|^2 2^p \omega^{2r} \left(1-\cos\frac{\omega}{a^\mu\sigma}t\right)^p d\omega \right\} \theta(t)\,dt}{\left(a^{2r}\right)^\mu \sigma^{2r} 2^p \inf\limits_{1\le y\le a} \Phi_{b,r,p}(\theta; y)}. \tag{9}$$

Using the fundamental properties of the Fourier transformation, we obtain

$$F\left(\Delta_\eta^p f^{(r)};\omega\right) = (i\omega)^r \left(e^{i\eta\omega}-1\right)^p F(f;\omega). \tag{10}$$

According to the Plancherel theorem, since $\Delta_\eta^p f^{(r)} \in L_2$, the functions $F\left(\Delta_\eta^p f^{(r)}\right)$ belong to L_2 and have equal norms. Taking (10) into account, we get

$$\left\|\Delta_\eta^p f^{(r)}(\cdot)\right\|^2 = 2\int_0^\infty |F(f;\omega)|^2 2^p \omega^{2r} (1-\cos\eta\omega)^p\,d\omega. \tag{11}$$

With regard for the definition of modulus of smoothness [relation (2)], we obtain

$$\int_0^\infty 2^{p+1} |F(f;\omega)|^2 \omega^{2r} (1-\cos x\omega)^p\,d\omega \le \omega_p^2\left(f^{(r)};x\right). \tag{12}$$

Applying this estimate to (9), we obtain the following relation for any function $f \in L_2^r$:

$$A_\sigma^2(f) \le \sum_{\mu=0}^{\infty} \frac{\int_0^b \omega_p^2\left(f^{(r)}; t/a^\mu\sigma\right)\theta(t)\,dt}{(a^{2r})^\mu \sigma^{2r} 2^p \inf\limits_{1\le y\le a} \Phi_{b,r,p}(\theta;y)} \le \frac{\int_0^b \omega_p^2\left(f^{(r)}; t/\sigma\right)\theta(t)\,dt}{\sigma^{2r} 2^p \inf\limits_{1\le y\le a} \Phi_{b,r,p}(\theta;y)} \sum_{\mu=0}^{\infty} \left(\frac{1}{a^{2r}}\right)^\mu$$

$$= \frac{a^{2r}}{a^{2r}-1} \left\{ 2^p \inf_{1\le y\le a} \Phi_{b,r,p}(\theta;y) \right\}^{-1} \frac{1}{\sigma^{2r}} \int_0^b \omega_p^2\left(f^{(r)}; t/\sigma\right)\theta(t)\,dt. \tag{13}$$

Finally, passing in the inequality obtained to the supremum over $f \in L_2^r$, $f \ne \text{const}$, we get inequality (6).

Theorem 1 is proved.

Let $h > 0$ and $\alpha_k \ge 0$. Consider the functions

$$\delta_h(t) = \{1/h\ (|t| < h/2);\ 0\ (|t| \ge h/2)\}$$

and

$$\theta_h(t) = \sum_{k=1}^{n} \alpha_k \delta_h(t-\xi_k),$$

where $0<\xi_1<\xi_2<\ldots<\xi_n$ and $b \geq \xi_n + h/2$.

Setting $\theta(t) = \theta_h(t)$ in Theorem 1 and passing to the limit as $h \to 0$, we obtain the following result:

Corollary 1. *For any* $a>1$, $\sigma>0$, $r \geq 0$, $p = 1, 2, \ldots$, $\alpha_k \geq 0$, *and* $0<\xi_1<\xi_2<\ldots<\xi_n$, *the following inequality is true:*

$$\sup_{\substack{f \in L_2^r \\ f \neq \text{const}}} \frac{\sigma^{2r} A_\sigma^2(f)}{\sum_{k=1}^n \alpha_k \omega_p^2\left(f^{(r)}; \xi_k/\sigma\right)} \leq \frac{a^{2r}}{a^{2r}-1}\left\{2^p \inf_{1 \leq y \leq a} y^{2r} \sum_{k=1}^n \alpha_k (1-\cos\xi_k y)^p\right\}^{-1}. \tag{14}$$

Passing to the limit as $a \to \infty$ in relations (6) and (14), we obtain

$$\sup_{\substack{f \in L_2^r \\ f \neq \text{const}}} \frac{\sigma^{2r} A_\sigma^2(f)}{\int_0^b \omega_p^2\left(f^{(r)}; t/\sigma\right)\theta(t)dt} \leq \left\{2^p \inf_{y \geq 1} y^{2r} \Phi_{b,r,p}(\theta; y)\right\}^{-1}, \tag{15}$$

$$\sup_{\substack{f \in L_2^r \\ f \neq \text{const}}} \frac{\sigma^{2r} A_\sigma^2(f)}{\sum_{k=1}^n \alpha_k \omega_p^2\left(f^{(r)}; \xi_k/\sigma\right)} \leq \left\{2^p \inf_{y \geq 1} y^{2r} \sum_{k=1}^n \alpha_k (1-\cos\xi_k y)^p\right\}^{-1}. \tag{16}$$

Note that these results agree well with the results of [5] (the upper bounds in Corollaries 1 and 2).

We set

$$c_{r,p} = \left(4-2^{-2r/p}\right)^{-p/2} \tag{17}$$

and

$$\xi = \xi_{r,p} = \frac{2}{\pi}\arcsin 2^{-(r/p)-1}. \tag{18}$$

The following statement is true:

Theorem 2. *Suppose that* $r \geq p$ *are such that, for every irreducible fraction* l/L, *one has*

$$|\xi - l/L| \geq 4L^{-(r/p)-1}. \tag{19}$$

Then, for any $\sigma>0$ *and* $\delta \geq (1+\xi)\pi$, *the following inequalities are true:*

$$\aleph_{\sigma,r,p}(\delta) \leq c_{r,p}. \tag{20}$$

Proof. We choose

$$a = \xi + \frac{r}{p\pi}\tan(\pi\xi), \tag{21}$$

$$\alpha_1 = (1+\alpha)/2, \qquad \alpha_2 = (1-\alpha)/2, \tag{22}$$

$$\xi_1 = (1-\xi)\pi, \qquad \xi_2 = (1+\xi)\pi. \tag{23}$$

It follows from (14) that, for any function $f \in L_2^r$ and any $a > 1$, one has

$$\sigma^{2r} A_\sigma^2(f) \le \frac{a^{2r}}{a^{2r}-1}\frac{\alpha_1\,\omega_p^2\big(f^{(r)};\xi_1/\sigma\big)+\alpha_2\,\omega_p^2\big(f^{(r)};\xi_2/\sigma\big)}{2^p \inf\limits_{1\le y\le a} y^{2r}\big\{\alpha_1(1-\cos\xi_1 y)^p + \alpha_2(1-\cos\xi_2 y)^p\big\}}$$

$$\le \frac{a^{2r}}{a^{2r}-1}\frac{\alpha_1\,\omega_p^2\big(f^{(r)};(1-\xi)\pi/\sigma\big)+\alpha_2\,\omega_p^2\big(f^{(r)};(1+\xi)\pi/\sigma\big)}{2^p \inf\limits_{1\le y\le a}\theta_{r,p}(y)}$$

$$\le \frac{a^{2r}}{a^{2r}-1}\frac{\omega_p^2\big(f^{(r)};(1+\xi)\pi/\sigma\big)}{2^p \inf\limits_{1\le y\le a}\theta_{r,p}(y)}, \tag{24}$$

where

$$\theta_{r,p}(y) = y^{2r}\big\{\alpha_1(1-\cos(1-\xi)\pi y)^p + \alpha_2(1-\cos(1+\xi)\pi y)^p\big\}. \tag{25}$$

Passing to the limit as $a \to \infty$, for any function $f \in L_2^r$ we get

$$\sigma^{2r} A_\sigma^2(f) \le \frac{\omega_p^2\big(f^{(r)};(1+\xi)\pi/\sigma\big)}{2^p \inf\limits_{y\ge 1}\theta_{r,p}(y)}. \tag{26}$$

The quantity

$$\inf_{y\ge 1}\theta_{r,p}(y)$$

was studied in [4], where it was shown that

$$\inf_{y\ge 1}\theta_{r,p}(y) = \theta_{r,p}(1) = 2^p\big(1-2^{-2(r/p)-2}\big)^p. \tag{27}$$

Using this result and inequality (26), we establish that the following relation holds for any function $f \in L_2^r$ and any $\delta \ge (1+\xi)\pi$:

$$\sigma^{2r} A_\sigma^2(f) \le \frac{2^{2r+2p}}{2^{2p}\left(2^{2(r/p)+2}-1\right)^p}\,\omega_p^2\left(f^{(r)}; (1+\xi)\pi/\sigma\right) \le c_{r,p}^2\,\omega_p^2\left(f^{(r)}; \delta/\sigma\right). \tag{28}$$

This yields inequality (20).

Theorem 2 is proved.

Theorem 3. *For any* $\sigma > 0$, $r \ge 0$, $p = 1, 2, \ldots$, *and* $\delta > 0$, *the following inequalities are true:*

$$\aleph_{\sigma,r,p}(\delta) \ge \sup_{\beta_1,\beta_2,\ldots,\beta_m} \frac{\sum_{k=1}^m \beta_k^2}{2^p \max\limits_{|t|\le\delta} \sum_{k=1}^m k^{2r}\beta_k^2(1-\cos kt)^p}. \tag{29}$$

Proof. Under the conditions of the theorem, we choose an arbitrary vector $B = (\beta_1, \beta_2, \ldots, \beta_m)$ and consider a sequence of even functions $f_{n,B} \in L_2$, namely,

$$f_{n,B}(x) = \begin{cases} \sum\limits_{k=1}^{n} \beta_k \cos k(\sigma+\alpha_n)x, & 0 \le x \le 2n\pi, \\ \psi_n(x) \sum\limits_{k=1}^{n} \beta_k \cos k(\sigma+\alpha_n)x, & 2n\pi \le x \le (2n+1)\pi,, \\ 0, & x \ge (2n+1)\pi, \end{cases} \tag{30}$$

where

$$\psi_n(x) = H_r \int_x^{(2n+1)\pi} \sum_{k=1}^{m} \beta_k \cos^r k\left(y - \frac{(2n+1)\pi}{2}\right) dy,$$

the constant H_r is defined by the condition $\psi_n(2n\pi) = 1$, and $\alpha_n = 1/\sqrt{n}$.

By analogy with [1], we first construct the Fourier transform $F(f_{n,B};\, \omega)$ of the sequence $f_{n,B}(x)$ and then use relation (8). As a result, we obtain the asymptotic equality

$$A_\sigma^2(f_{n,B}) = 2n\pi \sum_{k=1}^{m} \beta_k^2 \{1 + o(1)\}, \qquad n \to \infty. \tag{31}$$

In what follows, we use the quantity $\omega_p^2\left(f_{n,B}^{(r)};\, \delta/\sigma\right)$. For this reason, we first construct (step by step with respect to p) the function $\Delta_\eta^p f_{n,B}^{(r)}(x)$ and then establish the following asymptotic equality for each $p = 1, 2, \ldots$ as $n \to \infty$:

$$\left\| \Delta_\eta^p f_{n,B}^{(r)}(\cdot) \right\|^2 = 2^{p+1} n\pi(\sigma+\alpha_n)^{2r}\{1+o(1)\}\left[\sum_{k=1}^{m} k^{2r}\beta_k^2(1-\cos k(\sigma+\alpha_n)\eta)^p + o(1)\right]. \tag{32}$$

This implies that, for any fixed σ, B, r, p, and η, the following relation holds uniformly in σ, $0 \le \sigma \le \pi$:

$$\left\| \Delta_\eta^p f_{n,B}^{(r)} \right\|^2 = 2^{p+1} n\pi\{1 + o(1)\}\sigma^{2r} \sum_{k=1}^{m} k^{2r} \beta_k^2 (1 - \cos k\sigma\eta)^p, \qquad n \to \infty. \tag{33}$$

Using the definition of the exact constant in a Jackson-type inequality [see (5)] and relations (31) and (33), for any vector $B = (\beta_1, \beta_2, \ldots, \beta_m)$ we get

$$\aleph_{\sigma,r,p}(\delta) \ge \frac{\sigma^{2r} A_\sigma^2\left(f_{n,B}^{(r)}\right)}{\omega_p^2\left(f_{n,B}^{(r)}; \delta/\sigma\right)} = \frac{\sigma^{2r}\, 2n\pi \sum_{k=1}^{m} \beta_k^2 \{1 + o(1)\}}{\max\limits_{|\eta| \le \delta/\sigma} \left\| \Delta_\eta^p f_{n,B}^{(r)} \right\|^2}$$

$$= \frac{\sigma^{2r}\, 2n\pi \sum_{k=1}^{m} \beta_k^2 \{1 + o(1)\}}{\max\limits_{|t| \le \delta} \left\| \Delta_{t/\sigma}^p f_{n,B}^{(r)} \right\|^2}$$

$$= \frac{\sigma^{2r}\, 2n\pi \sum_{k=1}^{m} \beta_k^2 \{1 + o(1)\}}{\max\limits_{|t| \le \delta} 2^{p+1} n\pi\{1 + o(1)\}\, \sigma^{2r} \sum_{k=1}^{m} k^{2r} \beta_k^2 (1 - \cos kt)^p}$$

$$= \frac{\sum_{k=1}^{m} \beta_k^2}{2^p \max\limits_{|t| \le \delta} \sum_{k=1}^{m} k^{2r} \beta_k^2 (1 - \cos kt)^p} \{1 + o(1)\}, \qquad n \to \infty. \tag{34}$$

Passing to the upper bound over $B = (\beta_1, \beta_2, \ldots, \beta_m)$, we complete the proof of Theorem 3.

Theorem 4. *For any* $\sigma > 0$, $r \ge 0$, $p = 1, 2, \ldots$, *and* $\delta \ge (1 - \xi)\pi$, *the following inequality is true:*

$$\aleph_{\sigma,r,p}(\delta) \ge c_{r,p}. \tag{35}$$

Proof. In Theorem 3, we set $(\beta_1, \beta_2) = (1, \beta)$. Using (30), we obtain the following relation for any $\delta \ge (1 - \xi)\pi$:

$$\aleph_{\sigma,r,p}(\delta) \ge \sup_\beta \frac{1 + \beta^2}{2^p \max\limits_{|t| \le \delta}\left[(1 - \cos t)^p + 2^{2r} \beta^2 (1 - \cos 2t)^p \right]} = \sup_\beta \frac{1 + \beta^2}{2^p \max\limits_{u \in [u_\delta, 1]} \Psi_\beta(u)}, \tag{36}$$

where $u = \cos t$,

$$u_\delta = \begin{cases} \cos\delta, & (1 - \xi)\pi \le \delta \le \pi, \\ -1, & \delta \ge \pi, \end{cases}$$

and

$$\Psi_\beta(u) = (1-u)^p\left[1+2^{2r+p}\beta^2(1+u)^p\right]. \tag{37}$$

In [4], it was proved, in particular, that, for $\beta = \beta_{r,p}$, where

$$\beta_{r,p}^{-2} = 2\left(2^{2(r/p)+1}-1\right), \tag{38}$$

the function $\Psi_{\beta_{r,p}}(u)$ is equal to zero at the point $u_* \in [u_\delta, 1]$, where

$$u_* = \cos(1-\xi)\pi = 2^{-2(r/p)-1} - 1. \tag{39}$$

Using this result, we get

$$\max_{u\in[u_\delta,1]}\Psi_{\beta_{r,p}}(u) = \Psi_{\beta_{r,p}}(u_*) = \frac{\left(2^{2(r/p)+2}-1\right)}{2^{2r+p+1}\left(2^{2(r/p)+1}-1\right)}. \tag{40}$$

Finally, by virtue of (36) and (40), we obtain

$$\aleph_{\sigma,r,p}(\delta) \ge \frac{1+\beta_{r,p}^2}{2^p\max\limits_{u\in[u_\delta,1]}\Psi_{\beta_{r,p}}(u)} = \frac{1+\beta_{r,p}^2}{2^p\,\Psi_{\beta_{r,p}}(u_*)}$$

$$= \left[1+\left(2\left(2^{2(r/p)+1}-1\right)\right)^{-1}\right]\frac{2^{2r+p+1}\left(2^{2(r/p)+1}-1\right)}{2^p\left(2^{2(r/p)+2}-1\right)^{p+1}} = \frac{2^{2r}}{\left(2^{2(r/p)+2}-1\right)^p} = c_{r,p}^2.$$

Theorem 4 is proved.

Comparing Theorems 2 and 4, we establish the following statement:

Theorem 5. *Suppose that the conditions of Theorem 2 are satisfied. Then the following equality holds for all* $\delta \ge (1+\xi)\pi$:

$$\aleph_{\sigma,r,p}(\delta) = c_{r,p}.$$

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