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EXISTENCE PRINCIPLES FOR HIGHER-ORDER NONLOCAL BOUNDARY-VALUE PROBLEMS AND THEIR APPLICATIONS TO SINGULAR STURM–LIOUVILLE PROBLEMS

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We present existence principles for the nonlocal boundary-value problem

$$(\phi(u^{(p-1)}))' = g(t, u, \ldots, u^{(p-1)}),$$

$$\alpha_k(u) = 0, \quad 1 \le k \le p-1,$$

where $p \ge 2$, $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism, g is a Carathéodory function that is either regular or has singularities in its space variables, and $\alpha_k : C^{p-1}[0,T] \to \mathbb{R}$ is a continuous functional. An application of the existence principles to singular Sturm–Liouville problems

$$(-1)^n(\phi(u^{(2n-1)}))' = f(t, u, \ldots, u^{(2n-1)}),$$

$$u^{(2k)}(0) = 0, \quad a_k u^{(2k)}(T) + b_k u^{(2k+1)}(T) = 0, \quad 0 \le k \le n-1,$$

is given.

1. Introduction

Let $T > 0$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$, and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. As usual, $C^j[0,T]$ denotes the set of functions having the jth derivative continuous on $[0,T]$. $AC[0,T]$ and $L_1[0,T]$ are the sets of absolutely continuous functions on $[0,T]$ and Lebesgue integrable functions on $[0,T]$, respectively. $C^0[0,T]$ and $L_1[0,T]$ are equipped with the norms

$$\|x\| = \max\big\{|x(t)| : t \in [0,T]\big\} \qquad \text{and} \qquad \|x\|_L = \int_0^T |x(t)|\,dt,$$

respectively.

Assume that $G \subset \mathbb{R}^p$, $p \ge 2$. Let $\mathrm{Car}\big([0,T] \times G\big)$ denote the set of functions $f : [0,T] \times G \to \mathbb{R}$ satisfying the local Carathéodory conditions on $[0,T] \times G$, i.e.,

(i) for every $(x_0, \ldots, x_{p-1}) \in G$, the function $f(\cdot, x_0, \ldots, x_{p-1}) : [0,T] \to \mathbb{R}$ is measurable,

(ii) for a.e. $t \in [0,T]$, the function $f(t, \cdot, \ldots, \cdot) : G \to \mathbb{R}$ is continuous, and

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(iii) for every compact set $K \subset G$, one has $\sup\{|f(t, x_0, \dots, x_{p-1})| \colon (x_0, \dots, x_{p-1}) \in K\} \in L_1[0, T]$.

Let $p \in \mathbb{N}$, $p \geq 2$. Denote by $\mathcal{A}$ the set of functionals $\alpha \colon C^{p-1}[0, T] \to \mathbb{R}$ that are

(a) continuous and

(b) bounded, i.e., $\alpha(\Omega)$ is bounded for any bounded $\Omega \subset C^{p-1}[0, T]$.

Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be an increasing odd homeomorphism. Assume that either $g \in \operatorname{Car}([0, T] \times \mathbb{R}^p)$ or $g \in \operatorname{Car}([0, T] \times \mathcal{D}_*)$, $\mathcal{D}_* \subset \mathbb{R}^p$, and that it has singularities only at the value 0 of its space variables. Consider the nonlocal boundary-value problem

$$\big(\phi(u^{(p-1)})\big)' = g(t, u, \dots, u^{(p-1)}), \tag{1.1}$$

$$\alpha_k(u) = 0, \qquad \alpha_k \in \mathcal{A}, \quad 0 \leq k \leq p-1, \tag{1.2}$$

where α_k satisfy the following *compatibility condition:* For every $\mu \in [0, 1]$, there exists a solution of the problem

$$(\phi(u^{(p-1)}))' = 0, \qquad \alpha_k(u) - \mu\alpha_k(-u) = 0, \quad 0 \leq k \leq p-1.$$

This problem is equivalent to the fact that the system

$$\alpha_k\left(\sum_{i=0}^{p-1} A_i t^i\right) - \mu\alpha_k\left(-\sum_{i=0}^{p-1} A_i t^i\right) = 0, \quad 0 \leq k \leq p-1, \tag{1.3}$$

has a solution $(A_0, \dots, A_{p-1}) \in \mathbb{R}^p$ for every $\mu \in [0, 1]$.

We say that $u \in C^{p-1}[0, T]$ is a *solution of problem (1.1), (1.2)* if $\phi(u^{(p-1)}) \in AC[0, T]$, u satisfies (1.2), and the relation $\big(\phi(u^{(p-1)}(t))\big)' = g\big(t, u(t), \dots, u^{(p-1)}(t)\big)$ holds for a.e. $t \in [0, T]$.

The aim of this paper is

(1) to present existence principles for problem (1.1), (1.2) in the regular and singular cases and

(2) to give an application of these existence principles to singular Sturm–Liouville boundary-value problems.

Note that our existence principles are a generalization of those obtained for second-order differential equations with ϕ-Laplacian in [1, 2].

Our Sturm–Liouville problem consists of the differential equation

$$(-1)^n\big(\phi(u^{(2n-1)})\big)' = f(t, u, \dots, u^{(2n-1)}) \tag{1.4}$$

and the boundary conditions

$$u^{(2k)}(0) = 0, \qquad a_k u^{(2k)}(T) + b_k u^{(2k+1)}(T) = 0, \quad 0 \leq k \leq n-1. \tag{1.5}$$

Here, $n \geq 2$, $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $f \in \mathrm{Car}([0,T] \times \mathcal{D})$ is positive,

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \ldots \times \mathbb{R}_+ \times \mathbb{R}_0}_{4\ell-2} & \text{if} \quad n = 2\ell - 1, \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \ldots \times \mathbb{R}_- \times \mathbb{R}_0}_{4\ell} & \text{if} \quad n = 2\ell, \end{cases}$$

f may be singular at the value 0 of all its space variables, and

$$a_k > 0, \qquad b_k > 0, \qquad a_k T + b_k = 1 \quad \text{for} \quad 0 \leq k \leq n-1. \tag{1.6}$$

We say that a function $u \in C^{2n-1}[0,T]$ is a *solution of problem (1.4), (1.5)* if $\phi(u^{(2n-1)}) \in AC[0,T]$, u satisfies the boundary conditions (1.5), and $(-1)^n\big(\phi(u^{(2n-1)}(t))\big)' = f\big(t, u(t), \ldots, u^{(2n-1)}(t)\big)$ for a.e. $t \in [0,T]$.

Singular problems of the Sturm–Liouville type for higher-order differential equations were considered in [3–5]. In [3], the authors discuss the differential equation $u^{(n)} + h_1(t, u, \ldots, u^{(n-2)}) = 0$ together with the boundary conditions

$$\begin{gathered} u^{(j)}(0) = 0, \quad 0 \leq j \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \qquad \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{gathered} \tag{1.7}$$

where $\alpha\gamma + \alpha\delta + \beta\gamma > 0$, $\beta, \delta \geq 0$, $\beta + \alpha > 0$, $\delta + \gamma > 0$, and $h_1 \in C^0\big((0,1) \times \mathbb{R}_+^{n-1}\big)$ is positive. The existence of a positive solution $u \in C^{n-1}[0,1] \cap C^n(0,1)$ is proved by a fixed-point theorem for mappings that are decreasing with respect to a cone in a Banach space. Paper [4] deals with the problem $u^{(n)} + h_2(t, u, \ldots, u^{(n-1)}) = 0$, (1.7), where $h_2 \in \mathrm{Car}\big([0,T] \times \mathcal{D}_*\big)$, $\mathcal{D}_* = \mathbb{R}_+^{n-1} \times \mathbb{R}_0$, is positive. The existence of a positive solution $u \in AC^{n-1}[0,T]$ is proved by a combination of regularization and sequential techniques with a Fredholm-type existence theorem. In [5], by constructing some special cones and using a Krasnosel'skii fixed point on a cone, the existence of a positive solution $u \in C^{4n-2}[0,1] \cap C^{4n}(0,1)$ is proved for the problem

$$u^{(4n)} = h_3(t, u, u^{(4n-2)}),$$

$$u(0) = u(1) = 0, \quad au^{(2k)}(0) - bu^{(2k+1)}(0) = 0,$$

$$cu^{(2k)}(1) + du^{(2k+1)}(1) = 0, \quad 1 \leq k \leq 2n-1,$$

where $h_3 \in C\big([0,1] \times \mathbb{R}_+ \times \mathbb{R}_-\big)$ is nonnegative, a, b, c, and d are nonnegative constants, and $ac + ad + bc > 0$.

To our best knowledge, there is no paper considering singular problems of the Sturm–Liouville type in our generalization (1.4), (1.5). In addition, any solution u of problem (1.4), (1.5) has the maximal smoothness, u and its even derivatives $(\leq 2n-2)$ "start" at the singular points of f, and its odd derivatives $(\leq 2n-1)$ "go through" singularities of f somewhere inside $[0,T]$.

Throughout the paper, we work with the following conditions on the functions ϕ and f in Eq. (1.4):

(H_1) $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing odd homomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$,

(H_2) $f \in \mathrm{Car}([0,T] \times \mathcal{D})$ and there exists $a > 0$ such that

$$a \le f(t, x_0, \dots, x_{2n-1})$$

for a.e. $t \in [0,T]$ and all $(x_0, \dots, x_{2n-1}) \in \mathcal{D}$,

(H_3) the following relation holds for a.e. $t \in [0,T]$ and all $(x_0, \dots, x_{2n-1}) \in \mathcal{D}$:

$$f(t, x_0, \dots, x_{2n-1}) \le h\left(t, \sum_{j=0}^{2n-1} |x_j|\right) + \sum_{j=0}^{2n-1} \omega_j(|x_j|),$$

where $h \in \mathrm{Car}([0,T] \times [0,\infty))$ is positive and nondecreasing in the second variable, $\omega_j \colon \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing,

$$\limsup_{v \to \infty} \frac{1}{\phi(v)} \int_0^T h(t, 2n + Kv)\, dt < 1 \tag{1.8}$$

with

$$K = \begin{cases} 2n & \text{if} \quad T = 1, \\ \dfrac{T^{2n} - 1}{T - 1} & \text{if} \quad T \ne 1, \end{cases} \tag{1.9}$$

and

$$\int_0^1 \omega_{2n-1}(\phi^{-1}(s))\, ds < \infty, \qquad \int_0^1 \omega_{2j}(s)\, ds < \infty \quad \text{for} \quad 0 \le j \le n-1,$$

$$\int_0^1 \omega_{2j+1}(s^2)\, ds < \infty \quad \text{for} \quad 0 \le j \le n-2.$$

Remark 1.1. If ϕ satisfies (H_1), then $\phi(0) = 0$. Under assumption (H_3), the functions $\omega_{2n-1}(\phi^{-1}(s))$, $\omega_{2j}(s)$, $0 \le j \le n-1$, and $\omega_{2i+1}(s^2)$, $0 \le i \le n-2$, are locally Lebesgue integrable on $[0,\infty)$ because ω_k, $0 \le k \le 2n-1$, is nonincreasing and positive on $\mathbb{R}_+$.

The rest of the paper is organized as follows: In Sec. 2, we present existence principles for a regular and a singular problem (1.1), (1.2). The regular existence principle is proved by the Leray–Schauder degree (see, e.g., [6]). An application of both principles to the Sturm–Liouville problem (1.4), (1.5) is given in Sec. 3.

2. Existence Principles

The following result states conditions for the solvability of problem (1.1), (1.2) in the case where g in Eq. (1.1) is regular.

Theorem 2.1. *Let (H_1) hold. Let $g \in \operatorname{Car}([0,T]\times\mathbb{R}^p)$ and $\varphi \in L_1[0,T]$. Suppose that there exists a positive constant L independent of λ and such that*

$$\|u^{(j)}\| < L, \quad 0 \le j \le p-1,$$

for all solutions u of the differential equations

$$(\phi(u^{(p-1)}))' = (1-\lambda)\varphi(t), \quad \lambda \in [0,1], \tag{2.1}$$

$$(\phi(u^{(p-1)}))' = \lambda g(t,u,\ldots,u^{(p-1)}) + (1-\lambda)\varphi(t), \quad \lambda \in [0,1], \tag{2.2}$$

satisfying the boundary conditions (1.2). Also assume that there exists a positive constant Λ such that

$$|A_j| < \Lambda, \quad 0 \le j \le p-1, \tag{2.3}$$

for all solutions $(A_0,\ldots,A_{p-1}) \in \mathbb{R}^p$ of system (1.3) with $\mu \in [0,1]$.

Then problem (1.1), (1.2) has a solution $u \in C^{p-1}[0,T]$, $\phi(u^{(p-1)}) \in AC[0,T]$.

Proof. Let

$$\Omega = \Big\{x \in C^{p-1}[0,T]\colon \|x^{(j)}\| < \max\{L, \Lambda K_1\} \text{ for } 0 \le j \le p-1\Big\},$$

where

$$K_1 = \begin{cases} p & \text{if } T = 1, \\ \dfrac{T^p-1}{T-1} & \text{if } T \ne 1. \end{cases}$$

Then Ω is an open subset of the Banach space $C^{p-1}[0,T]$ symmetric with respect to $0 \in C^{p-1}[0,T]$. Define an operator $\mathcal{P}\colon [0,1]\times\overline{\Omega} \to C^{p-1}[0,T]$ by the formula

$$\mathcal{P}(\rho,x)(t) = \int_0^t \frac{(t-s)^{p-2}}{(p-2)!}\,\phi^{-1}\left(\phi(x^{(p-1)}(0) + \alpha_{p-1}(x)) + \int_0^s V(\rho,x)(v)\,dv\right)ds$$
$$+ \sum_{j=0}^{p-2} \frac{x^{(j)}(0) + \alpha_j(x)}{j!} t^j \tag{2.4}$$

where

$$V(\rho,x)(t) = \rho g(t,x(t),\ldots,x^{(p-1)}(t)) + (1-\rho)\varphi(t).$$

It follows from the continuity of ϕ and α_j, $0 \le j \le p-1$, the inclusion $g \in \operatorname{Car}([0,T]\times\mathbb{R}^p)$, and the Lebesgue dominated-convergence theorem that $\mathcal{P}$ is a continuous operator. We now prove that $\mathcal{P}\big([0,T]\times\overline{\Omega}\big)$ is relatively compact in $C^{p-1}[0,T]$. Note that the boundedness of $\overline{\Omega}$ in $C^{p-1}[0,T]$ guarantees the existence of a positive constant r and $\psi \in L_1[0,T]$ such that

$$|\alpha_k(x)| \le r \quad \text{and} \quad \bigl|g(t,x(t),\dots,x^{(p-1)}(t))\bigr| \le \psi(t)$$

for a.e. $t \in [0,T]$ and all $x \in \overline{\Omega}$ and $0 \le k \le p-1$. Then

$$\bigl|(\mathcal{P}(\rho,x))^{(j)}(t)\bigr| \le \bigl(r + \max\{L, \Lambda K_1\}\bigr) \sum_{i=0}^{p-j-2} \frac{T^i}{i!} + \frac{T^{p-j-1}}{(p-j-2)!}\phi^{-1}\bigl(\phi(r+\max\{L,\Lambda K_1\}\bigr) + \|\psi\|_L + \|\varphi\|_L),$$

$$\bigl|(\mathcal{P}(\rho,x))^{(p-1)}(t)\bigr| \le \phi^{-1}\bigl(\phi\bigl(r+\max\{L,\Lambda K_1\}\bigr) + \|\psi\|_L + \|\varphi\|_L\bigr),$$

$$\Bigl|\phi((\mathcal{P}(\rho,x))^{(p-1)}(t_2)) - \phi((\mathcal{P}(\rho,x))^{(p-1)}(t_1))\Bigr| \le \left|\int_{t_1}^{t_2} (\psi(s) + |\varphi(s)|)\, ds\right|$$

for $t, t_1, t_2 \in [0,T]$, $(\rho,x) \in [0,T] \times \overline{\Omega}$, and $0 \le j \le n-2$. Hence, $\mathcal{P}\bigl([0,T]\times\overline{\Omega}\bigr)$ is bounded in $C^{p-1}[0,T]$, and the set $\{\phi((\mathcal{P}(\rho,x))^{(p-1)})\colon (\rho,x) \in [0,1]\times\overline{\Omega}\}$ is equicontinuous on $[0,T]$. Since $\phi\colon \mathbb{R}\to\mathbb{R}$ is increasing and continuous, the set $\Bigl\{(\mathcal{P}(\rho,x))^{(p-1)}\colon (\rho,x) \in [0,1]\times\overline{\Omega}\Bigr\}$ is equicontinuous on $[0,T]$ too. By the Arzelà–Ascoli theorem, $\mathcal{P}([0,1]\times\overline{\Omega})$ is relatively compact in $C^{p-1}[0,T]$. We have proved that $\mathcal{P}$ is a compact operator.

Suppose that x_* is a fixed point of the operator $\mathcal{P}(1,\cdot)$. Then

$$x_*(t) = \sum_{j=0}^{p-2} \frac{x_*^{(j)}(0) + \alpha_j(x_*)}{j!} t^j + \int_0^t \frac{(t-s)^{p-2}}{(p-2)!}\phi^{-1}\left(\phi(x_*^{(p-1)}(0) + \alpha_{p-1}(x_*)) + \int_0^s g(v, x_*(v), \dots, x_*^{(p-1)}(v))dv\right) ds$$

for $t \in [0,T]$. Hence, $\alpha_k(x_*) = 0$ for $0 \le k \le p-1$, and x_* is a solution of Eq. (1.1). Consequently, x_* is a solution of problem (1.1), (1.2). In order to prove the assertion of our theorem it suffices to show that

$$\deg\,(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega, 0) \ne 0 \tag{2.5}$$

where "deg" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identical operator on $C^{p-1}[0,T]$. To show this, let the compact operator $\mathcal{K}\colon [0,2]\times\overline{\Omega} \to C^{p-1}[0,T]$ be defined by

$$\mathcal{K}(\mu,x)(t)=\begin{cases}\displaystyle\sum_{j=0}^{p-1}\Big[x^{(j)}(0)+\alpha_{j+1}(x)-(1-\mu)\alpha_j(-x)\Big]\frac{t^j}{j!} & \text{if } \mu\in[0,1],\\[2ex] \displaystyle\int_0^t\frac{(t-s)^{p-2}}{(p-2)!}\phi^{-1}\bigg(\phi(x^{(p-1)}(0)+\alpha_{p-1}(x)) & \\ \displaystyle\qquad+(\mu-1)\int_0^s\varphi(v)\,dv\bigg)ds+\sum_{j=0}^{p-2}\frac{x^{(j)}(0)+\alpha_j(x)}{j!}t^j & \text{if } \mu\in(1,2].\end{cases}$$

Then $\mathcal{K}(0,\cdot)$ is odd $\big($i.e., $\mathcal{K}(0,-x)=-\mathcal{K}(0,x)$ for $x\in\overline{\Omega}\big)$ and

$$\mathcal{K}(2,x)=\mathcal{P}(0,x)\quad\text{for}\quad x\in\overline{\Omega}. \tag{2.6}$$

Assume that $\mathcal{K}(\mu_0,u_0)=u_0$ for some $(\mu_0,u_0)\in[0,1]\times\overline{\Omega}$. Then

$$u_0(t)=\sum_{j=0}^{p-1}\Big[u_0^{(j)}(0)+\alpha_j(u_0)-(1-\mu_0)\alpha_j(-u_0)\Big]\frac{t^j}{j!},\quad t\in[0,T],$$

and, therefore,

$$u_0(t)=\sum_{j=0}^{p-1}\tilde{A}_j\frac{t^j}{j!},$$

where

$$\tilde{A}_j=u_0^{(j)}(0)+\alpha_j(u_0)-(1-\mu_0)\alpha_j(-u_0).$$

Consequently, $u_0^{(j)}(0)=\tilde{A}_j$ and, hence,

$$\alpha_j(u_0)-(1-\mu_0)\alpha_j(-u_0)=0\quad\text{for}\quad 0\le j\le p-1,$$

which means that

$$\alpha_k\left(\sum_{j=0}^{p-1}\tilde{A}_j\frac{t^j}{j!}\right)-(1-\mu_0)\,\alpha_k\left(-\sum_{j=0}^{p-1}\tilde{A}_j\frac{t^j}{j!}\right)=0,\quad 0\le k\le p-1.$$

Then, by our assumption,

$$\left|\frac{\tilde{A}_j}{j!}\right|<\Lambda\quad\text{for}\quad 0\le j\le p-1,$$

and we have

$$\big\|u_0^{(j)}\big\|<\Lambda\sum_{j=0}^{p-1}T^j=\Lambda K_1,\quad 0\le j\le p-1.$$

Hence, $u_0 \notin \partial\Omega$ and, therefore, by the Borsuk antipodal theorem and the homotopy property, we get

$$\deg\,(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) \neq 0 \tag{2.7}$$

and

$$\deg\,(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) = \deg\,(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega, 0). \tag{2.8}$$

We come to show that

$$\deg\,(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega, 0) = \deg\,(\mathcal{I} - \mathcal{K}(2,\cdot), \Omega, 0). \tag{2.9}$$

If $\mathcal{K}(\mu_1, u_1) = u_1$ for some $(\mu_1, u_1) \in (1,2] \times \overline{\Omega}$, then

$$u_1(t) = \sum_{j=0}^{p-2} \frac{u_1^{(j)}(0) + \alpha_j(u_1)}{j!} t^j$$

$$+ \int_0^t \frac{(t-s)^{p-2}}{(p-2)!}\, \phi^{-1}\left(\phi(u_1^{(p-1)}(0) + \alpha_{p-1}(u_1)) + (\mu_1 - 1)\int_0^s \varphi(v)\,dv\right) ds$$

for $t \in [0,T]$. Hence, u_1 satisfies the boundary conditions (1.2) and is a solution of the differential equation (2.1) with $\lambda = 2 - \mu_1 \in [0,1)$. By our assumptions, $\|u_1^{(j)}\| < L$ for $0 \le j \le p-1$. Therefore, $u_1 \notin \partial\Omega$ and equality (2.9) follows from the homotopy property. Finally, suppose that $\mathcal{P}(\tilde{\rho}, \tilde{u}) = \tilde{u}$ for some $(\tilde{\rho}, \tilde{u}) \in [0,1] \times \overline{\Omega}$. Then $\tilde{u}$ is a solution of problem (2.2), (1.2) with $\lambda = \tilde{\rho}$, and, therefore, $\|\tilde{u}^{(j)}\| < L$ for $0 \le j \le p-1$. Hence, $\tilde{u} \notin \partial\Omega$ and, by the homotopy property,

$$\deg\,(\mathcal{I} - \mathcal{P}(0,\cdot), \Omega, 0) = \deg\,(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega, 0).$$

This and (2.6)–(2.9) yield (2.5), which completes the proof.

Remark 2.1. If a functional $\alpha_k \in \mathcal{A}$ is linear for $0 \le k \le p-1$, then system (1.3) has the form

$$\sum_{j=0}^{p-1} A_j \alpha_k(t^j) = 0, \quad 0 \le k \le p-1.$$

All of its solutions $(A_0, \ldots, A_{p-1}) \in \mathbb{R}^p$ are bounded exactly if $\det\,(\alpha_k(t^j))_{k,j=0}^{p-1} \neq 0$ $\big($and then $A_j = 0$ for $0 \le j \le p-1\big)$, which is equivalent to the fact that problem $\big(\phi(u^{(p-1)})\big)' = 0$, (1.2) has only the trivial solution.

If the function $g \in \operatorname{Car}([0,T] \times \mathcal{D}_*)$, $\mathcal{D}_* \subset \mathbb{R}^p$, in Eq. (1.1) has singularities only at the value 0 of its space variables, then the following result holds for the solvability of problem (1.1), (1.2):

Theorem 2.2. *Suppose that condition (H_1) is satisfied. Let $g \in \operatorname{Car}([0,T] \times \mathcal{D}_*)$, $\mathcal{D}_* \subset \mathbb{R}^p$, have singularities only at the value 0 of its space variables. Let the function $g_m \in \operatorname{Car}\big([0,T] \times \mathbb{R}^p\big)$ in the differential equation*

$$\big(\phi(u^{(p-1)})\big)' = g_m(t, u, \ldots, u^{(p-1)}) \tag{2.10}$$

satisfy the following condition for a.e. $t \in [0,T]$ *and all* $(x_0,\dots,x_{p-1}) \in \mathbb{R}_0^p$ *and* $m \in \mathbb{N}$:

$$0 \le \nu g_m(t,x_0,\dots,x_{p-1}) \le q\big(t,|x_0|,\dots,|x_{p-1}|\big), \tag{2.11}$$

where $q \in \mathrm{Car}([0,T]\times\mathbb{R}_+^p)$ *and* $\nu \in \{-1,1\}$. *Suppose that, for each* $m \in \mathbb{N}$, *the regular problem (2.10), (1.2) has a solution* u_m *and there exists a subsequence* $\{u_{k_m}\}$ *of* $\{u_m\}$ *converging in* $C^{p-1}[0,T]$ *to some* u.

Then $\phi(u^{(p-1)}) \in AC[0,T]$ *and* u *is a solution of the singular problem (1.1), (1.2) if* $u^{(j)}$ *has a finite number of zeros for* $0 \le j \le p-1$ *and*

$$\lim_{m\to\infty} g_{k_m}\big(t,u_{k_m}(t),\dots,u_{k_m}^{(p-1)}(t)\big) = g\big(t,u(t),\dots,u^{(p-1)}(t)\big) \tag{2.12}$$

for a.e. $t \in [0,T]$.

Proof. Assume that (2.12) holds for a.e. $t \in [0,T]$ and let $0 \le \xi_1 < \ldots < \xi_\ell \le T$ be all zeros of $u^{(j)}$ for $0 \le j \le p-1$. Since $\|u_{k_m}^{(j)}\| \le L$ for each $m \in \mathbb{N}$ and $0 \le j \le p-1$, where L is a positive constant, it follows that

$$\int_0^T \nu g_{k_m}\big(t,u_{k_m}(t),\dots,u_{k_m}^{(p-1)}(t)\big)\,dt = \nu\Big[\phi\big(u_{k_m}^{(p-1)}(T)\big) - \phi\big(u_{k_m}^{(p-1)}(0)\big)\Big] \le 2\phi(L)$$

for $m \in \mathbb{N}$. Relations (2.11) and (2.12) and the Fatou lemma [7, 8] now give

$$\int_0^T \nu g(t,u(t),\dots,u^{(p-1)}(t))\,dt \le 2\phi(L).$$

Hence, $\nu g\big(t,u(t),\dots,u^{(p-1)}(t)\big) \in L_1[0,T]$, and so $g\big(t,u(t),\dots,u^{(p-1)}(t)\big) \in L_1[0,T]$. We set $\xi_0 = 0$ and $\xi_{\ell+1} = T$. Let us show that the equality

$$\phi(u^{(p-1)}(t)) = \phi\left(u^{(p-1)}\left(\frac{\xi_{i+1}+\xi_i}{2}\right)\right) + \int_{(\xi_{i+1}+\xi_i)/2}^{t} g(s,u(s),\dots,u^{(p-1)}(s))\,ds \tag{2.13}$$

is satisfied on $[\xi_i,\xi_{i+1}]$ for each $i \in \{0,\dots,\ell\}$ such that $\xi_i < \xi_{i+1}$. Indeed, let $i \in \{0,\dots,\ell\}$ and $\xi_i < \xi_{i+1}$. We choose an arbitrary

$$\rho \in \left(0, \frac{\xi_{i+1}+\xi_i}{2}\right)$$

and consider the interval $[\xi_i+\rho,\xi_{i+1}-\rho]$. We know that $|u^{(j)}| > 0$ on (ξ_i,ξ_{i+1}) for $0 \le j \le p-1$ and, therefore, $|u^{(j)}(t)| \ge \varepsilon$ for $t \in [\xi_i+\rho,\xi_{i+1}-\rho]$ and $0 \le j \le p-1$, where ε is a positive constant. Hence, there exists $m_0 \in \mathbb{N}$ such that

$$\big|u_{k_m}^{(j)}(t)\big| \ge \frac{\varepsilon}{2} \qquad \text{for} \quad t \in [\xi_i+\rho,\xi_{i+1}-\rho],\ \ 0 \le j \le p-1,\ \ m \ge m_0.$$

This gives [see (2.11)]

$$\begin{aligned}&\left|g_{k_m}(t, u_{k_m}(t), \dots, u_{k_m}^{(p-1)}(t))\right|\\&\qquad\le \sup\left\{q(t, x_0, \dots, x_{p-1})\colon t\in[0,T],\ \ x_j \in \left[\frac{\varepsilon}{2}, L\right] \ \text{ for } \ 0\le j\le p-1\right\} \in L_1[0,T]\end{aligned}$$

for a.e. $t\in[\xi_i+\rho, \xi_{i+1}-\rho]$ and all $m\ge m_0$. Letting $m\to\infty$ in

$$\phi\big(u_{k_m}^{(p-1)}(t)\big) = \phi\left(u_{k_m}^{(p-1)}\left(\frac{\xi_{i+1}+\xi_i}{2}\right)\right) + \int\limits_{(\xi_{i+1}+\xi_i)/2}^{t} g_{k_m}\big(s, u_{k_m}(s), \dots, u_{k_m}^{(p-1)}(s)\big)\,ds,$$

we get (2.13) for $t\in[\xi_i+\rho,\ \xi_{i+1}+\rho]$ by the Lebesgue dominated-convergence theorem. Since

$$\rho\in\left(0, \frac{\xi_{i+1}+\xi_i}{2}\right)$$

is arbitrary, equality (2.13) holds on the interval (ξ_i, ξ_{i+1}), and, using the fact that $g\big(t, u(t), \dots, u^{(p-1)}(t)\big)\in L_1[0,T]$, we conclude that (2.13) is also satisfied at $t=\xi_i$ and ξ_{i+1}. From equality (2.13) on $[\xi_i, \xi_{i+1}]$ $\big($for $0\le i\le \ell\big)$, we deduce that $\phi(u^{(p-1)})\in AC[0,T]$ and u is a solution of Eq. (1.1). Finally, it follows from the fact that $\alpha_j(u_{k_m}) = 0$ for $0\le j\le p-1$ and $m\in\mathbb{N}$ and from the continuity of α_j that $\alpha_j(u)=0$ for $0\le j\le p-1$. Consequently, u is a solution of problem (1.1), (1.2).

The theorem is proved.

3. Sturm–Liouville Problem

3.1. Auxiliary Results. Throughout the next part of this paper, we assume that the numbers a_k and b_k in the boundary conditions (1.5) satisfy condition (1.6). For each $j\in\{0,\dots,n-2\}$, denote by G_j the Green function of the Sturm–Liouville problem

$$-u''=0, \qquad u(0)=0, \qquad a_j u(T) + b_j u'(T) = 0.$$

Then

$$G_j(t,s) = \begin{cases} s(1-a_j t) & \text{for } \ 0\le s\le t\le T,\\[2ex] t(1-a_j s) & \text{for } \ 0\le t< s\le T.\end{cases}$$

Hence, $G_j(t,s)>0$ for $(t,s)\in(0,T]\times(0,T]$, and $G_j(t,s)=G_j(s,t)$ for $(t,s)\in[0,T]\times[0,T]$. We set $G^{[1]}(t,s) = G_{n-2}(t,s)$ for $(t,s)\in[0,T]\times[0,T]$ and define $G^{[j]}$ recursively by the formula

$$G^{[j]}(t,s) = \int\limits_0^T G_{n-j-1}(t,v) G^{[j-1]}(v,s)\,dv, \quad (t,s)\in[0,T]\times[0,T], \tag{3.1}$$

for $2 \le j \le n-1$. It follows from the definition of the function $G^{[j]}$ that the equalities

$$u^{(2n-2j)}(t) = (-1)^{j-1} \int_0^T G^{[j-1]}(t,s) u^{(2n-2)}(s)\, ds, \quad 2 \le j \le n, \tag{3.2}$$

are true on $[0,T]$ for every $u \in C^{2n-2}[0,T]$ satisfying the boundary conditions (1.5).

Lemma 3.1. *For $1 \le j \le n-1$, the following inequality is true:*

$$G^{[j]}(t,s) \ge \frac{T^{2j-3}(1-\alpha T)^j}{3^{j-1}} ts \quad \textit{for} \quad (t,s) \in [0,T] \times [0,T], \tag{3.3}$$

where

$$\alpha = \max\{a_k \colon 0 \le k \le n-2\} \quad \left(< \frac{1}{T}\right). \tag{3.4}$$

Proof. Since

$$G_j(t,s) = \begin{cases} s(1-a_j t) \ge s(1-a_j T) & \text{for } 0 \le s \le t \le T, \\ t(1-a_j s) \ge t(1-a_j T) & \text{for } 0 \le t < s \le T \end{cases}$$

for $0 \le j \le n-2$, we have

$$G_j(t,s) \ge \frac{1-a_j T}{T} st \ge \frac{1-\alpha T}{T} st$$

for $(t,s) \in [0,T] \times [0,T]$ and $0 \le j \le n-2$. Consequently,

$$G^{[1]}(t,s) = G_{n-2}(t,s) \ge \frac{1-\alpha T}{T} st$$

for $(t,s) \in [0,T] \times [0,T]$, and, therefore, inequality (3.3) holds for $j=1$. We now proceed by induction. Assume that (3.3) is true for $j = i \ (< n-1)$. Then

$$G^{[i+1]}(t,s) = \int_0^T G_{n-i-2}(t,v) G^{[i]}(v,s)\, dv \ge \int_0^T \frac{1-\alpha T}{T} tv \frac{T^{2i-3}(1-\alpha T)^i}{3^{i-1}} vs\, dv$$

$$= \frac{T^{2i-4}(1-\alpha T)^{i+1}}{3^{i-1}} ts \int_0^T v^2 ds = \frac{T^{2i-1}(1-\alpha T)^{i+1}}{3^i} ts$$

for $(t,s) \in [0,T] \times [0,T]$. Therefore (3.3) is true with $j = i+1$.

The lemma is proved.

Let ϕ satisfy (H_1). We choose an arbitrary $a > 0$ and put

$$\mathcal{B}_a = \Big\{ u \in C^{2n-1}[0,T] \colon \phi(u^{(2n-1)}) \in AC[0,T],$$

$$(-1)^n \big(\phi(u^{(2n-1)}(t))\big)' \geq a \text{ for a.e. } t \in [0,T], \text{ and } u \text{ satisfies (1.5)} \Big\}. \tag{3.5}$$

The properties of functions belonging to the set $\mathcal{B}_a$ are given in the following lemma:

Lemma 3.2. *Let $u \in \mathcal{B}_a$. Then there exists $\{\xi_{2j+1}\}_{j=0}^{n-1} \subset (0,T)$ such that*

$$u^{(2j+1)}(\xi_{2j+1}) = 0, \quad 0 \leq j \leq n-1, \tag{3.6}$$

and

$$\big|u^{(2n-1)}(t)\big| \geq \phi^{-1}\big(a|t - \xi_{2n-1}|\big), \tag{3.7}$$

$$\big|u^{(2n-2j+1)}(t)\big| \geq \frac{T^{2j-4}S}{2 \cdot 3^{j-2}}(1-\alpha T)^{j-2}(t - \xi_{2n-2j+1})^2, \quad 2 \leq j \leq n, \tag{3.8}$$

$$(-1)^{n+j} u^{(2n-2j)}(t) \geq \frac{T^{2j-2}S}{3^{j-1}}(1-\alpha T)^{j-1} t, \quad 1 \leq j \leq n, \tag{3.9}$$

for $t \in [0,T]$, where

$$S = \frac{1}{T} \min \left\{ b_{n-1} \int_0^{T/2} \phi^{-1}(at)\, dt,\ \frac{b_{n-1}}{a_{n-1}} \phi^{-1}\left(\frac{aT}{2}\right) \right\} \tag{3.10}$$

and α is given in (3.4).

Proof. Since ϕ is increasing and

$$\Big(\phi((-1)^n u^{(2n-1)}(t))\Big)' = (-1)^n \Big(\phi(u^{(2n-1)}(t))\Big)' \geq a \quad \text{for a.e. } t \in [0,T],$$

it follows that $(-1)^n u^{(2n-1)}$ is increasing on $[0,T]$, and $(-1)^{n-1} u^{(2n-2)}$ is concave on this interval. If $u^{(2n-1)}(t) \neq 0$ for $t \in (0,T)$, then

$$\big|a_{n-1} u^{(2n-2)}(T) + b_{n-1} u^{(2n-1)}(T)\big| = \left| a_{n-1} \int_0^T u^{(2n-1)}(t) dt + b_{n-1} u^{(2n-1)}(T) \right| > 0,$$

contrary to the fact that $a_{n-1} u^{(2n-2)}(T) + b_{n-1} u^{(2n-1)}(T) = 0$ by (1.5) with $k = n-1$. Consequently, $u^{(2n-1)}(\xi_{2n-1}) = 0$ for a unique $\xi_{2n-1} \in (0,T)$. The integration of the equality $\big(\phi((-1)^n u^{(2n-1)}(t))\big)' \geq a$ over $[t, \xi_{2n-1}]$ and $[\xi_{2n-1}, t]$ now gives

$$(-1)^{n-1}u^{(2n-1)}(t) \geq \phi^{-1}\big(a(\xi_{2n-1}-t)\big), \quad t \in [0, \xi_{2n-1}], \tag{3.11}$$

$$(-1)^{n}u^{(2n-1)}(t) \geq \phi^{-1}\big(a(t-\xi_{2n-1})\big), \quad t \in [\xi_{2n-1}, T], \tag{3.12}$$

which shows that (3.7) holds. In order to prove inequality (3.9) for $j = 1$ we consider two cases, namely $\xi_{2n-1} < \frac{T}{2}$ and $\xi_{2n-1} \geq \frac{T}{2}$.

Case 1. Let $\xi_{2n-1} < \frac{T}{2}$. Then [see (3.12)]

$$(-1)^{n}u^{(2n-1)}(T) \geq \phi^{-1}(a(T-\xi_{2n-1})) > \phi^{-1}\left(\frac{aT}{2}\right),$$

and, therefore [see (1.5) with $k = n-1$],

$$(-1)^{n-1}u^{(2n-2)}(T) = (-1)^{n}\frac{b_{n-1}}{a_{n-1}}u^{(2n-1)}(T) > \frac{b_{n-1}}{a_{n-1}}\phi^{-1}\left(\frac{aT}{2}\right). \tag{3.13}$$

Case 2. Let $\xi_{2n-1} \geq \frac{T}{2}$. Then (3.11) yields

$$(-1)^{n-1}u^{(2n-2)}\left(\frac{T}{2}\right) = (-1)^{n-1}\int_0^{T/2} u^{(2n-1)}(t)\,dt \geq \int_0^{T/2} \phi^{-1}\big(a(\xi_{2n-1}-t)\big)\,dt$$

$$\geq \int_0^{T/2} \phi^{-1}\left(a\left(\frac{T}{2}-t\right)\right)dt = \int_0^{T/2} \phi^{-1}(at)\,dt =: L.$$

Let $\varepsilon := (-1)^{n}u^{(2n-1)}(T)$. We know that $(-1)^{n}u^{(2n-1)}$ is increasing on $[0,T]$ and $u^{(2n-1)}(\xi_{2n-1}) = 0$. Hence, $\varepsilon > 0$ and

$$(-1)^{n-1}u^{(2n-2)}(t) = (-1)^{n-1}u^{(2n-2)}(\xi_{2n-1}) + (-1)^{n-1}\int_{\xi_{2n-1}}^{t} u^{(2n-1)}(s)\,ds$$

$$> (-1)^{n-1}u^{(2n-2)}(\xi_{2n-1}) - \varepsilon(t-\xi_{2n-1}) \geq (-1)^{n-1}u^{(2n-2)}\left(\frac{T}{2}\right) - \varepsilon(t-\xi_{2n-1})$$

for $t \in (\xi_{2n-1}, T]$. Consequently,

$$(-1)^{n-1}u^{(2n-2)}(T) > L - \varepsilon(T-\xi_{2n-1}) > L - \varepsilon T.$$

Then

$$\frac{b_{n-1}}{a_{n-1}}\varepsilon = (-1)^n \frac{b_{n-1}}{a_{n-1}} u^{(2n-1)}(T) = (-1)^{n-1} u^{(2n-2)}(T) > L - \varepsilon T,$$

and so [see (1.6)]

$$\varepsilon > L\left(\frac{b_{n-1}}{a_{n-1}} + T\right)^{-1} = a_{n-1}L.$$

It follows that

$$(-1)^{n-1}u^{(2n-2)}(T) = (-1)^n \frac{b_{n-1}}{a_{n-1}} u^{(2n-1)}(T) = \frac{b_{n-1}}{a_{n-1}}\varepsilon > b_{n-1}L. \tag{3.14}$$

Relations (3.13) and (3.14) now imply that $(-1)^{n-1}u^{(2n-2)}(T) > ST$, where S is given in (3.10). This, the equality $u^{(2n-2)}(0) = 0$, and the fact that $(-1)^{n-1}u^{(2n-2)}$ is concave on $[0,T]$ guarantee that $(-1)^{n-1}u^{(2n-2)}(t) \geq St$ for $t \in [0,T]$, which proves (3.9) for $j = 1$.

Combining (3.2), (3.3), and (3.9) (with $j = 1$), we get

$$(-1)^{n+j}u^{(2n-2j)}(t) = (-1)^{n-1}\int_0^T G^{[j-1]}(t,s)u^{(2n-2)}(s)\,ds$$

$$\geq \frac{T^{2j-5}S}{3^{j-2}}(1-\alpha T)^{j-1} t \int_0^T s^2\,ds = \frac{T^{2j-2}S}{3^{j-1}}(1-\alpha T)^{j-1} t$$

for $t \in [0,T]$ and $2 \leq j \leq n$. We have proved that (3.9) is true.

Since, by (3.9), $|u^{(2n-2j)}| > 0$ on $(0,T]$ for $1 \leq j \leq n$ and u satisfies (1.5), essentially the same reasoning as in the beginning of this proof shows that $u^{(2j+1)}(\xi_{2j+1}) = 0$ for a unique $\xi_{2j+1} \in (0,T)$, $0 \leq j \leq n-2$. Using (3.9), we obtain

$$\left|u^{(2n-2j+1)}(t)\right| = \left|\int_{\xi_{2n-2j+1}}^t u^{(2n-2j+2)}(s)\,ds\right| \geq \frac{T^{2j-4}S}{3^{j-2}}(1-\alpha T)^{j-2}\left|\int_{\xi_{2n-2j+1}}^t s\,ds\right|$$

$$= \frac{T^{2j-4}S}{2\cdot 3^{j-2}}(1-\alpha T)^{j-2}|t^2 - \xi_{2n-2j+1}^2| \geq \frac{T^{2j-4}S}{2\cdot 3^{j-2}}(1-\alpha T)^{j-2}(t-\xi_{2n-2j+1})^2$$

for $t \in [0,T]$ and $2 \leq j \leq n$. Hence, (3.8) is true, which completes the proof.

3.2. Auxiliary Regular Problems. Let (H_2) and (H_3) hold. For each $m \in \mathbb{N}$, we define $\chi_m, \varphi_m, \tau_m \in C^0(\mathbb{R})$ and $\mathbb{R}_m \subset \mathbb{R}$ by the formulas

$$\chi_m(v) = \begin{cases} v & \text{for } v \geq \dfrac{1}{m}, \\[2mm] \dfrac{1}{m} & \text{for } v < \dfrac{1}{m}, \end{cases} \qquad \varphi_m(v) = \begin{cases} -\dfrac{1}{m} & \text{for } v > -\dfrac{1}{m}, \\[2mm] v & \text{for } v \leq -\dfrac{1}{m}, \end{cases}$$

$$\tau_m = \begin{cases} \chi_m & \text{if } n = 2k-1, \\ \varphi_m & \text{if } n = 2k, \end{cases} \qquad \mathbb{R}_m = \mathbb{R} \setminus \left(-\frac{1}{m}, \frac{1}{m}\right).$$

We choose $m \in \mathbb{N}$ and use the function f to define $f_m \in \mathrm{Car}\left([0,T] \times \mathbb{R}^{2n}\right)$ by the formula

$$f_m(t, x_0, x_1, x_2, x_3, \dots, x_{2n-2}, x_{2n-1})$$

$$= \begin{cases} f(t, \chi_m(x_0), x_1, \varphi_m(x_2), x_3, \dots, \tau_m(x_{2n-2}), x_{2n-1}) \\ \qquad \text{for } (t, x_0, x_1, x_2, x_3, \dots, x_{2n-2}, x_{2n-1}) \in [0,T] \times \mathbb{R} \times \mathbb{R}_m \times \mathbb{R} \times \mathbb{R}_m \times \dots \times \mathbb{R} \times \mathbb{R}_m, \\ \frac{m}{2}\Big[f_m\Big(t, x_0, \frac{1}{m}, x_2, x_3, \dots, x_{2n-2}, x_{2n-1}\Big)\Big(x_1 + \frac{1}{m}\Big) \\ \qquad -f_m\Big(t, x_0, -\frac{1}{m}, x_2, x_3, \dots, x_{2n-2}, x_{2n-1}\Big)\Big(x_1 - \frac{1}{m}\Big)\Big] \\ \qquad \text{for } (t, x_0, x_1, x_2, x_3, \dots, x_{2n-2}, x_{2n-1}) \\ \qquad\qquad \in [0,T] \times \mathbb{R} \times \left[-\frac{1}{m}, \frac{1}{m}\right] \times \mathbb{R} \times \mathbb{R}_m \times \dots \times \mathbb{R} \times \mathbb{R}_m, \\ \frac{m}{2}\Big[f_m\Big(t, x_0, x_1, x_2, \frac{1}{m}, \dots, x_{2n-2}, x_{2n-1}\Big)\Big(x_3 + \frac{1}{m}\Big) \\ \qquad -f_m\Big(t, x_0, x_1, x_2, -\frac{1}{m}, \dots, x_{2n-2}, x_{2n-1}\Big)\Big(x_3 - \frac{1}{m}\Big)\Big] \\ \qquad \text{for } (t, x_0, x_1, x_2, x_3, \dots, x_{2n-2}, x_{2n-1}) \in [0,T] \times \mathbb{R}^3 \times \left[-\frac{1}{m}, \frac{1}{m}\right] \times \dots \times \mathbb{R} \times \mathbb{R}_m, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{m}{2}\Big[f_m\Big(t, x_0, x_1, x_2, \dots, x_{2n-2}, \frac{1}{m}\Big)\Big(x_{2n-1} + \frac{1}{m}\Big) \\ \qquad -f_m\Big(t, x_0, x_1, x_2, \dots, x_{2n-2}, -\frac{1}{m}\Big)\Big(x_{2n-1} - \frac{1}{m}\Big)\Big] \\ \qquad \text{for } (t, x_0, x_1, x_2, \dots, x_{2n-2}, x_{2n-1}) \in [0,T] \times \mathbb{R}^{2n-1} \times \left[-\frac{1}{m}, \frac{1}{m}\right]. \end{cases}$$

Then conditions (H_2) and (H_3) give

$$a \le (1-\lambda)a + \lambda f_m(t, x_0, \dots, x_{2n-1}) \tag{3.15}$$

for a.e. $t \in [0,T]$ and all $(x_0,\dots,x_{2n-1}) \in \mathbb{R}^{2n}$ and $\lambda \in [0,1]$, and

$$(1-\lambda)a + \lambda f_m(t,x_0,\dots,x_{2n-1}) \le h\left(t, 2n + \sum_{j=0}^{2n-1} |x_j|\right) + \sum_{j=0}^{2n-1} \omega_j\big(|x_j|\big) \tag{3.16}$$

for a.e. $t \in [0,T]$ and all $(x_0,\dots,x_{2n-1}) \in \mathbb{R}_0^{2n}$ and $\lambda \in [0,1]$.

Consider the family of approximate regular differential equations

$$(-1)^n\big(\phi(u^{(2n-1)})\big) = \lambda f_m(t,u,\dots,u^{(2n-1)}) + (1-\lambda)a, \quad \lambda \in [0,1]. \tag{3.17}$$

Lemma 3.3. *Let (H_1)–(H_3) hold. Then there exists a positive constant W independent of $m \in \mathbb{N}$ and $\lambda \in [0,1]$ and such that*

$$\|u^{(j)}\| < W, \quad 0 \le j \le 2n-1, \tag{3.18}$$

for all solutions u of problem (3.17), (1.5).

Proof. Let u be a solution of problem (3.17), (1.5). Then $(-1)^n\big(\phi(u^{(2n-1)}(t))\big)' \ge a$ for a.e. $t \in [0,T]$ by (3.15), and, consequently, $u \in \mathcal{B}_a$, where the set $\mathcal{B}_a$ is given in (3.5). Hence, by Lemma 3.2, u satisfies (3.6) and (3.7), where $\xi_{2j+1} \in (0,T)$ is the unique zero of $u^{(2j+1)}$, $0 \le j \le n-1$, and

$$\big|u^{(2n-2j+1)}(t)\big| \ge Q_j(t-\xi_{2n-2j+1})^2, \quad 2 \le j \le n,$$

$$(-1)^{n+i}u^{(2n-2i)}(t) \ge P_i t, \quad 1 \le i \le n,$$

for $t \in [0,T]$, where

$$Q_j = \frac{T^{2j-4}S}{2\cdot 3^{j-2}}(1-\alpha T)^{j-2}, \qquad P_i = \frac{T^{2i-2}S}{3^{i-1}}(1-\alpha T)^{i-1} \tag{3.19}$$

with α and S given in (3.4) and (3.10), respectively. Accordingly,

$$\begin{aligned}
\sum_{j=0}^{2n-1}\int_0^T \omega_j\big(|u^{(j)}(t)|\big)\,dt &\le \sum_{j=1}^{n}\int_0^T \omega_{2n-2j}(P_j t)\,dt \\
&\quad + \sum_{j=2}^{n}\int_0^T \omega_{2n-2j+1}\big(Q_j(t-\xi_{2n-2j+1})^2\big)\,dt + \int_0^T \omega_{2n-1}(\phi^{-1}(a|t-\xi_{2n-1}|))\,dt \\
&< \sum_{j=1}^{n}\frac{1}{P_j}\int_0^{P_jT} \omega_{2n-2j}(s)\,ds + 2\sum_{j=2}^{n}\frac{1}{\sqrt{Q_j}}\int_0^{\sqrt{Q_j}T} \omega_{2n-2j+1}(s^2)\,ds \\
&\quad + \frac{2}{aT}\int_0^{aT} \omega_{2n-1}(\phi^{-1}(s))\,ds =: \Lambda.
\end{aligned} \tag{3.20}$$

By (H_3), we have $\Lambda < \infty$. Since $u^{(2j)}(0) = 0$ and $u^{(2j+1)}(\xi_{2j+1}) = 0$ for $0 \le j \le n-1$, we get

$$\|u^{(j)}\| \le T^{2n-j-1}\|u^{(2n-1)}\|, \quad 0 \le j \le 2n-2. \tag{3.21}$$

Combining (3.16), (3.20), (3.21), and the equality $u^{(2n-1)}(\xi_{2n-1}) = 0$, we obtain

$$\phi\big(|u^{(2n-1)}(t)|\big) = \left| \int\limits_{\xi_{2n-1}}^{t} [(1-\lambda)a + \lambda f_m(s, u(s), \ldots, u^{(2n-1)}(s))]\, ds \right|$$

$$< \int\limits_0^T h\left(t, 2n + \sum_{j=0}^{2n-1} |u^{(j)}(t)|\right) dt + \sum_{j=0}^{2n-1} \int\limits_0^T \omega_j\big(|u^{(j)}(t)|\big)\, dt$$

$$< \int\limits_0^T h\left(t, 2n + \|u^{(2n-1)}\| \sum_{j=0}^{2n-1} T^j\right) dt + \Lambda = \int\limits_0^T h(t, 2n + K\|u^{(2n-1)}\|)\, dt + \Lambda$$

for $t \in [0,T]$, where K is given in (1.9). Hence,

$$\phi\big(\|u^{(2n-1)}\|\big) < \int\limits_0^T h\big(t, 2n + K\|u^{(2n-1)}\|\big)\, dt + \Lambda. \tag{3.22}$$

It follows from condition (1.8) that there exists a positive constant W_* such that

$$\int\limits_0^T h(t, 2n + Kv)\, dt < \phi(v)$$

whenever $v \ge W_*$. This and (3.22) yields $\|u^{(2n-1)}\| < W_*$. Consequently, (3.21) shows that (3.18) is satisfied with $W = W_* \max\big\{1, T^{2n-1}\big\}$.

The lemma is proved.

Remark 3.1. Assume that $c > 0$. If follows from the proof of Lemma 3.3 that any solution u of problem $(-1)^n\big(\phi(u^{(2n-1)})\big)' = c$, (1.5) satisfies the inequality

$$\|u^{(j)}\| < \phi^{-1}(cT)\max\{1, T^{2n-1}\}$$

for $0 \le j \le 2n-1$.

We are now in a position to show that, for each $m \in \mathbb{N}$, there exists a solution u_m of the regular differential equation

$$(-1)^n\big(\phi(u^{(2n-1)})\big)' = f_m(t, u, \ldots, u^{(2n-1)}) \tag{3.23}$$

that satisfies the boundary conditions (1.5).

Lemma 3.4. *Let (H_1)–(H_3) hold. Then, for each $m \in \mathbb{N}$, there exists a solution $u_m \in C^{2n-1}[0,T]$, $\phi(u^{(2n-1)}) \in AC[0,T]$ of problem (3.23), (1.5) and*

$$\|u_m^{(j)}\| < W \quad \text{for } m \in \mathbb{N} \text{ and } 0 \le j \le 2n-1, \tag{3.24}$$

where W is a positive constant. In addition, the sequence $\{u_m^{(2n-1)}\}$ is equicontinuous on $[0,T]$.

Proof. Choose an arbitrary $m \in \mathbb{N}$. Let W be the positive constant in Lemma 3.3. In order to prove the existence of a solution of problem (3.23), (1.5) we use Theorem 2.1 with $p = 2n$, $g = (-1)^n f_m$, and $\varphi = (-1)^n a$ in Eqs. (2.1) and (2.2) and with

$$\alpha_{2k}(u) = u^{(2k)}(0), \qquad \alpha_{2k+1}(u) = a_k u^{(2k)}(T) + b_k u^{(2k+1)}(T), \quad 0 \le k \le n-1, \tag{3.25}$$

in the boundary conditions (1.2).

Due to Lemma 3.3 and Remark 3.1, all solutions u of problems (3.17), (1.5) and $(-1)^n\big(\phi(u^{(2n-1)})\big)' = \lambda a$, (1.5) $(0 \le \lambda \le 1)$ satisfy inequality (3.18). Moreover, α_k [defined in (3.25)] belongs to the set $\mathcal{A}$ (with $p = 2n$) for $0 \le k \le 2n-1$. The system [see (1.3)]

$$\alpha_k\left(\sum_{i=0}^{2n-1} A_i t^i\right) - \mu\alpha_k\left(-\sum_{i=0}^{2n-1} A_i t^i\right) = 0, \quad 0 \le k \le 2n-1, \tag{3.26}$$

has the form [see (3.25)]

$$(1+\mu)\left(\sum_{i=0}^{2n-1} A_i t^i\right)^{(2k)}\Bigg|_{t=0} = 0, \quad 0 \le k \le n-1, \tag{3.27}$$

$$(1+\mu)\left[a_k\left(\sum_{i=0}^{2n-1} A_i t^i\right)^{(2k)}\Bigg|_{t=T} + b_k\left(\sum_{i=0}^{2n-1} A_i t^i\right)^{(2k+1)}\Bigg|_{t=T}\right] = 0, \quad 0 \le k \le n-1. \tag{3.28}$$

It follows from (3.27) that $A_{2k} = 0$ for $0 \le k \le n-1$, and then we deduce from (3.28) and the equality $a_k T + b_k = 1$ that $A_{2j+1} = 0$ for $0 \le j \le n-1$. Consequently, $(A_0, \ldots, A_{2n-1}) = (0, \ldots, 0) \in \mathbb{R}^{2n}$ is the unique solution of (3.26) for every $\mu \in [0,1]$. Hence, all assumptions of Theorem 2.1 are satisfied, and, therefore, for each $m \in \mathbb{N}$, there exists a solution $u_m \in C^{2n-1}[0,T]$, $\phi(u^{(2n-1)}) \in AC[0,T]$ of problem (3.23), (1.5) that satisfies inequality (3.24).

It remains to show that the sequence $\{u_m^{(2n-1)}\}$ is equicontinuous on $[0,T]$. Note that $u_m \in \mathcal{B}_a$ for all $m \in \mathbb{N}$, where the set $\mathcal{B}_a$ is given in (3.5). Then, by Lemma 3.2, there exists $\{\xi_{2j+1,m}\}_{j=0}^{n-1} \subset (0,T)$, $m \in \mathbb{N}$, such that

$$u_m^{(2j+1)}(\xi_{2j+1,m}) = 0, \qquad 0 \le j \le n-1, \quad m \in \mathbb{N}, \tag{3.29}$$

and

$$\begin{gathered}
\left|u_m^{(2n-1)}(t)\right| \geq \phi^{-1}\big(a|t-\xi_{2n-1,m}|\big), \\
\left|u_m^{(2n-2j+1)}(t)\right| \geq Q_j(t-\xi_{2n-2j+1,m})^2, \quad 2 \leq j \leq n, \\
(-1)^{n+j} u_m^{(2n-2j)}(t) \geq P_j t, \quad 1 \leq j \leq n,
\end{gathered} \tag{3.30}$$

for $t \in [0,T]$ and $m \in \mathbb{N}$, where Q_j and P_j are given in (3.19). Let $0 \leq t_1 < t_2 \leq T$. Then [see (3.16) with $\lambda = 1$, (3.24), and (3.30)]

$$\begin{aligned}
&\left|\phi\big(u_m^{(2n-1)}(t_2)\big) - \phi\big(u_m^{(2n-1)}(t_1)\big)\right| \\
&\qquad = \int_{t_1}^{t_2} f_m\big(t, u_m(t), \dots, u_m^{(2n-1)}(t)\big)\, dt \\
&\qquad \leq \int_{t_1}^{t_2} h\left(t, 2n + \sum_{j=0}^{2n-1} \|u_m^{(j)}\|\right) dt + \sum_{j=0}^{2n-1} \int_{t_1}^{t_2} \omega_j\big(|u_m^{(j)}(t)|\big)\, dt \\
&\qquad \leq \int_{t_1}^{t_2} h(t, 2n(1+W))\, dt + \int_{t_1}^{t_2} \omega_{2n-1}\big(\phi^{-1}(a|t-\xi_{2n-1,m}|\big)\, dt \\
&\qquad\qquad + \sum_{j=2}^{n} \int_{t_1}^{t_2} \omega_{2n-2j+1}\big(Q_j(t-\xi_{2n-2j+1,m})^2\big)\, dt + \sum_{j=1}^{n} \int_{t_1}^{t_2} \omega_{2n-2j}(P_j t)\, dt
\end{aligned} \tag{3.31}$$

for $m \in \mathbb{N}$. By assumption (H_3),

$$h(t, 2n(1+W)) \in L_1[0,T]$$

and

$$\omega_{2n-1}(\phi^{-1}(s)), \omega_{2j}(s), \ 0 \leq j \leq n-1, \quad \omega_{2i+1}(s^2), \ 0 \leq i \leq n-2,$$

are locally integrable on $[0,\infty)$. These facts, (3.31), and the relations

$$\int_{t_1}^{t_2} \omega_{2n-1}\big(\phi^{-1}(a|t-\xi_{2n-1,m}|)\big)\,dt$$

$$= \begin{cases} \dfrac{1}{a} \displaystyle\int_{a(\xi_{2n-1,m}-t_2)}^{a(\xi_{2n-1,m}-t_1)} \omega_{2n-1}\big(\phi^{-1}(t)\big)\,dt, & \text{if} \quad t_2 \le \xi_{2n-1,m}, \\ \dfrac{1}{a}\left[\displaystyle\int_0^{a(\xi_{2n-1,m}-t_1)} \omega_{2n-1}\big(\phi^{-1}(t)\big)\,dt + \int_0^{a(t_2-\xi_{2n-1,m})} \omega_{2n-1}\big(\phi^{-1}(t)\big)\,dt\right] & \text{if} \quad t_1 < \xi_{2n-1,m} < t_2, \\ \dfrac{1}{a} \displaystyle\int_{a(t_1-\xi_{2n-1,m})}^{a(t_2-\xi_{2n-1,m})} \omega_{2n-1}\big(\phi^{-1}(t)\big)\,dt & \text{if} \quad \xi_{2n-1,m} \le t_1, \end{cases}$$

$$\int_{t_1}^{t_2} \omega_{2n-2j+1}\big(Q_j(t-\xi_{2n-2j+1,m})^2\big)\,dt$$

$$= \begin{cases} \dfrac{1}{\sqrt{Q_j}} \displaystyle\int_{\sqrt{Q_j}(\xi_{2n-2j+1,m}-t_2)}^{\sqrt{Q_j}(\xi_{2n-2j+1,m}-t_1)} \omega_{2n-2j+1}(t^2)\,dt & \text{if} \quad t_2 \le \xi_{2n-2j+1,m}, \\ \dfrac{1}{\sqrt{Q_j}}\left[\displaystyle\int_0^{\sqrt{Q_j}(\xi_{2n-2j+1,m}-t_1)} \omega_{2n-2j+1}(t^2)\,dt \right. \\ \qquad \left. + \displaystyle\int_0^{\sqrt{Q_j}(t_2-\xi_{2n-2j+1,m})} \omega_{2n-2j+1}(t^2)\,dt\right] & \text{if} \quad t_1 < \xi_{2n-2j+1,m} < t_2, \\ \dfrac{1}{\sqrt{Q_j}} \displaystyle\int_{\sqrt{Q_j}(t_1-\xi_{2n-2j+1,m})}^{\sqrt{Q_j}(t_2-\xi_{2n-2j+1,m})} \omega_{2n-2j+1}(t^2)\,dt & \text{if} \quad \xi_{2n-2j+1,m} \le t_1, \end{cases}$$

imply that $\{\phi(u_m^{(2n-1)})\}$ is equicontinuous on $[0,T]$. We now deduce the equicontinuity of $\{u_m^{(2n-1)}\}$ on $[0,T]$ from the equality

$$\big|u_m^{(2n-1)}(t_2) - u_m^{(2n-1)}(t_1)\big| = \Big|\phi^{-1}\big(\phi(u_m^{(2n-1)}(t_2))\big) - \phi^{-1}\big(\phi(u_m^{(2n-1)}(t_1))\big)\Big|$$

$$\text{for} \quad 0 \le t_1 < t_2 \le T \quad \text{and} \quad m \in \mathbb{N}$$

and the fact that $\{\phi(u_m^{(2n-1)})\}$ is bounded in $C^0[0,T]$ and ϕ^{-1} is continuous and increasing on $\mathbb{R}$.

The lemma is proved.

3.3. Existence Result and Example. The main result is presented in the following theorem:

Theorem 3.1. *Let (H_1)–(H_3) hold. Then problem (1.4), (1.5) has a solution $u \in C^{2n-1}[0,T]$, $\phi(u^{(2n-1)}) \in AC[0,T]$, $(-1)^k u^{(2k)} > 0$ on $(0,T]$, and $u^{(2k+1)}(\xi_{2k+1}) = 0$ for $0 \le k \le n-1$, where $\xi_{2k+1} \in (0,T)$.*

Proof. By Lemma 3.4, for each $m \in \mathbb{N}$ there exists a solution u_m of problem (3.23), (1.5). Consider the sequence $\{u_m\}$. Then inequality (3.24) is satisfied with a positive constant W, and since $u_m \in \mathcal{B}_a$, Lemma 3.2 guarantees the existence of $\{\xi_{2j+1,m}\}_{j=0}^{n-1} \subset (0,T)$ such that (3.29) and (3.30) hold for $t \in [0,T]$ and $m \in \mathbb{N}$, where Q_j and P_j are given in (3.19). Moreover, the sequence $\{u_m^{2n-1}\}$ is equicontinuous on $[0,T]$ by Lemma 3.4. Hence, there exist a subsequence $\{u_{k_m}\}$ converging in $C^{2n-1}[0,T]$ and a subsequence $\{\xi_{2j+1,k_m}\}$, $1 \le j \le n-1$, converging in $\mathbb{R}$. Let

$$\lim_{m\to\infty} u_{k_m} = u \quad \text{and} \quad \lim_{m\to\infty} \xi_{2j+1,k_m} = \xi_{2j+1}, \quad 1 \le j \le n-1.$$

Letting $m \to \infty$ in (3.24), (3.29), and (3.30) (with k_m instead of m), we get $\big($for $t \in [0,T]\big)$

$$\big|u^{(2n-1)}(t)\big| \ge \phi^{-1}\big(a|t-\xi_{2n-1}|\big),$$

$$u^{(2j+1)}(\xi_{2j+1}) = 0 \quad \text{for} \quad 0 \le j \le n-1,$$

$$\big|u^{(2n-2j+1)}(t)\big| \ge Q_j(t-\xi_{2n-2j+1})^2 \quad \text{for} \quad 2 \le j \le n-1,$$

$$\|u^{(j)}\| \le W \quad \text{for} \quad 0 \le j \le 2n-1,$$

and

$$(-1)^{n+j} u^{(2n-2j)}(t) \ge P_j t \quad \text{for} \quad 1 \le j \le n. \tag{3.32}$$

Hence, $u^{(j)}$ has exactly one zero in $[0,T]$ for $0 \le j \le 2n-1$, and

$$\lim_{m\to\infty} f_{k_m}\big(t, u_{k_m}(t), \ldots, u_{k_m}^{(2n-1)}(t)\big) = f\big(t, u(t), \ldots, u^{(2n-1)}(t)\big) \quad \text{for a.e. } t \in [0,T].$$

In addition, by (3.32), we have $(-1)^k u^{(2k)} > 0$ on $(0,T]$ and $(-1)^k u^{(2k+1)}(0) \ge P_{n-k} > 0$ for $0 \le k \le n-1$. Hence, $(-1)^k u^{(2k+1)}(T) < 0$ for $0 \le k \le n-1$ by (1.5), which, combined with $(-1)^k u^{(2k+1)}(0) > 0$, implies that $\xi_{2k+1} \in (0,T)$ for $0 \le k \le n-1$. Finally, having in mind the definition of the function f_m and inequality (3.16), we get

$$0 \le f_m(t, x_0, \ldots, x_{2n-1}) \le q\big(t, |x_0|, \ldots, |x_{2n-1}|\big) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x_0, \ldots, x_{2n-1}) \in \mathbb{R}_0^{2n},$$

where

$$q(t, x_0, \dots, x_{2n-1}) = h\left(t, 2n + \sum_{j=0}^{2n-1} x_j\right) + \sum_{j=0}^{2n-1} \omega_j(x_j) \quad \text{for } t \in [0,T] \text{ and } (x_0, \dots, x_{2n-1}) \in \mathbb{R}_+^{2n}.$$

Clearly, $q \in \mathrm{Car}([0,T] \times \mathbb{R}_+^{2n})$. Hence, problem (1.4), (1.5) satisfies the assumptions of Theorem 2.2 with $p = 2n$, $g = (-1)^n f$, and $g_m = f_m$ [i.e., $\nu = (-1)^n$ in (2.11)] and with the boundary conditions (3.25), which are a special case of the boundary conditions (1.2). Consequently, Theorem 2.2 guarantees that $\phi(u^{(2n-1)}) \in AC[0,T]$ and u is a solution of problem (1.4), (1.5).

The theorem is proved.

Example 3.1. Assume that $p > 1$, $\alpha_{2n-1} \in (0, p-1)$, $\alpha_{2j} \in (0,1)$ for $0 \le j \le n-1$, $\alpha_{2j+1} \in \left(0, \frac{1}{2}\right)$ for $0 \le j \le n-2$, $\beta_k \in (0, p-1)$, $c_k > 0$, $d_k \in L_1[0,T]$ for $0 \le k \le 2n-1$, d_k is nonnegative, $r \in L_1[0,T]$, and $r(t) \ge a > 0$ for a.e. $t \in [0,T]$. Consider the differential equation

$$(-1)^n \left(|u^{(2n-1)}|^{p-2} u^{(2n-1)}\right)' = r(t) + \sum_{k=0}^{2n-1} \left(\frac{c_k}{|u^{(k)}|^{\alpha_k}} + d_k(t) |u^{(k)}|^{\beta_k}\right). \tag{3.33}$$

Equation (3.33) satisfies conditions (H_1)–(H_3) with

$$\phi(v) = |v|^{p-2} v \qquad \text{and} \qquad h(t,v) = r(t) + (2n + v^\gamma) \sum_{j=0}^{2n-1} d_k(t),$$

where

$$\gamma = \max\{\beta_k \colon 0 \le k \le 2n-1\} < p-1 \quad \text{and} \quad \omega_k(v) = \frac{c_k}{v^{\alpha_k}}, \quad 0 \le k \le 2n-1.$$

Hence, Theorem 3.1 guarantees that problem (3.33), (1.5) has a solution $u \in C^{2n-1}[0,T]$, $\phi(u^{(2n-1)}) \in AC[0,T]$, $(-1)^k u^{(2k)} > 0$ on $(0,T]$, and $u^{(2k+1)}(\xi_{2k+1}) = 0$ for $0 \le k \le n-1$, where $\xi_{2k+1} \in (0,T)$.

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