Ukrainian Mathematical Journal, Vol. 59, No. 12, 2007

# ON THE SOLUTION OF THE BASIC INTEGRAL EQUATION OF ACTUARIAL MATHEMATICS BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

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UDC 519.21

We study the basic integral equation of actuarial mathematics for the probability of (non)ruin of an insurance company regarded as a function of the initial capital. We establish necessary and sufficient conditions for the existence of a solution of this equation, general sufficient conditions for its existence and uniqueness, and conditions for the uniform convergence of the method of successive approximations for finding the solution.

Consider a random risk process (with random claims and deterministic and random premiums) $\xi_t$ that describes the evolution of the capital of an insurance company in time $t$ and satisfies the stochastic equation [1]

$$\xi_t = u + \int_0^t c(\xi_s)\,ds - S_t, \qquad t \geq 0, \tag{1}$$

where $u \geq 0$ is the initial capital, $c(\cdot)$ is a nonnegative piecewise-continuous function that describes the intensity of arrival of deterministic premiums as a function of the current capital, $S_t = \sum_{k=1}^{N_t} z_k$ are aggregated random insurance claims and premiums, $z_k$ are independent random variables (claims in the case $z_k \geq 0$ or random premiums in the case $z_k \leq 0$) with common distribution function $F(z)$, and $N_t$ is the number of random claims and premiums arrived by time $t$ (an ordinary renewal process with time distribution function between successive events $K(t)$). We consider the ruin probability of the insurance company $\psi(u) = \mathrm{P}\{\exists t \geq 0 : \xi_t < 0\}$ on an infinite time interval $t \in [0, +\infty)$ and the corresponding nonruin probability $\varphi(u) = 1 - \psi(u)$ as functions of the initial capital $u \geq 0$.

We define the function of growth of capital in the absence of insurance claims $U(u, t)$ as the solution of the following Cauchy problem for an ordinary differential equation:

$$\frac{dU}{dt} = c(U), \qquad U(u, 0) = u.$$

For example, if $c(\cdot) \equiv a$, then $U(u, t) = u + at$. If the capital of the insurance company is deposited with continuous interest rate $\delta$ and $c(\xi_t) \equiv a + \delta\xi_t$, $\delta > 0$, then

$$\frac{dU}{dt} = a + \delta U, \qquad U(u, 0) = u,$$

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 59, No. 12, pp. 1689 – 1698, December, 2007. Original article submitted March 30, 2006.

 0041–5995/07/5912–1902 

and, thus,

$$U(u, t) = ue^{\delta t} + \frac{a}{\delta}\left(e^{\delta t} - 1\right) \geq u + (a + \delta u)t. \tag{2}$$

**Assumption 1.** *The function* $c(\cdot)$ *satisfies the condition* $c(\cdot) \geq 0$, *and, hence,* $U(u, t)$ *does not decrease with respect to its variables.*

It was shown in [2, 3] that the function of nonruin probability $\varphi(u)$ satisfies the following integral equation [the basic equation of actuarial mathematics (see also Eq. (3.74) in [4] with $U(u, t) = u + at$ and $F(0) = 0$)]:

$$\varphi(u) = A\varphi(u), \tag{3}$$

where the integral operator $A$ is defined by the expression

$$A\varphi(u) := \int_0^{\infty} \int_{-\infty}^{U(u,t)} \varphi(U(u, t) - z)\, dF(z)\, dK(t), \qquad u \geq 0. \tag{4}$$

Here, the functions $\varphi(\cdot)$ and $U(\cdot,\cdot)$ are monotone with respect to their variables and the integrals are understood in the Lebesgue–Stieltjes sense. This is a linear homogeneous integral equation with operator $A$ with unbounded domain of integration and nonnegative kernel, and the operator $A$ is defined on bounded nondecreasing functions $\varphi(u)$, $u \geq 0$. Equation (3), (4) always has the trivial (zero) solution. We are interested in a solution $\varphi(u)$, $0 \leq \varphi(u) \leq 1$, that does not decrease with respect to $u$ and satisfies the following boundary condition at infinity:

$$\varphi(+\infty) = \lim_{u \to +\infty} \varphi(u) = 1. \tag{5}$$

This condition means that the insurance company is not ruined in the case of unbounded initial capital.

In the classical case of the so-called compound Poisson process (Cramér–Lundberg model), in which $U(u, t) = u + at$, $K(t) = 1 - e^{-\alpha t}$, and $F(z) = 0$ for $z \leq 0$, problem (3)–(5) for the (non)ruin probability reduces to the solution of an integral renewal equation (a Volterra-type equation with kernel dependent on the difference of arguments, see [1, 4]). An extensive literature is devoted to the investigation of this case in the theory of random walks and actuarial mathematics (see [5] and the bibliography therein). However, in the general case, problem (3)–(5) does not reduce to a Volterra equation and should be studied independently.

In [2, 3], sufficient conditions for the existence and uniqueness of a solution of problem (3)–(5) were established and the method of successive approximations for the solution of this problem was justified, namely,

$$\varphi^{k+1}(u) = \int_0^{\infty} \int_{-\infty}^{U(u,t)} \varphi^k(U(u, t) - z)\, dF(z)\, dK(t), \qquad k = 0, 1, \ldots, \tag{6}$$

where $k$ is the number of iteration and $0 \leq \varphi^0(u) \leq 1$. Results of numerical experiments were also presented in [2, 3]. The case of $K(t) = 1 - e^{-\alpha t}$ (a Poisson flow of insurance claims with intensity $\alpha > 0$) was considered in [2]. The case of the general distribution $K(t)$ was considered in [3]. Model (1) admits both positive payments (claims) and negative payments (premiums) at random times. Therefore, for $K(t) = 1 - e^{-\alpha t}$, it comprises the models with random premiums considered in [6, 7], where it was assumed that $c(\cdot) \equiv 0$.

In the present paper, we generalize the results given in [2, 3], namely,

the more general model (1) of a risk process, which admits both random claims and stochastic premiums, is considered;

the problem of the existence of a solution of problem (3) – (5) and its determination is considered from the general operator point of view, and the properties of monotonicity (Lemma 1) and contraction (Corollary 3) are established for the operator $A$ from (4);

general necessary and sufficient conditions for the existence of a solution of problem (3)–(5) are established (Theorem 2);

new general sufficient conditions for the existence and uniqueness of a solution of problem (3) – (5) are established (Corollary 5);

we prove not only the pointwise convergence but also the uniform convergence of the method of successive approximations (6) (Theorem 4).

Let $\Phi$ denote a (metric) space of functions $\varphi(u)$ nonincreasing with respect to $u \in [0, +\infty)$ and such that $0 \leq \varphi(u) \leq 1$ with the following distance between functions $\varphi_1, \varphi_2 \in \Phi$:

$$\rho(\varphi_1, \varphi_2) := \sup_{u \geq 0} |\varphi_1(u) - \varphi_2(u)|.$$

We introduce a partial order on $\Phi$ as follows: $\varphi_1 \leq \varphi_2$ if $\varphi_1(u) \leq \varphi_2(u)$ for any $u \geq 0$.

**Lemma 1.** *The linear integral operator $A$ acts from $\Phi$ into $\Phi$ and is monotone, i.e., for any $\varphi_1 \leq \varphi_2$, one has $A\varphi_1 \leq A\varphi_2$, and nonexpanding (and, hence, continuous with respect to the metric $\rho(\cdot,\cdot)$):*

$$\rho(A\varphi_1, A\varphi_2) \leq \rho(\varphi_1, \varphi_2) \qquad \forall\, \varphi_1, \varphi_2 \in \Phi.$$

***Proof.*** Since the function $\phi_{u,t}(z) = \varphi(U(u,t) - z)$ is monotone and bounded, the integral

$$\int_{-\infty}^{U(u,t)} \varphi(U(u,t) - z)\, dF(z) = \psi_u(t)$$

is defined. In turn, the function $\psi_u(t)$ is also monotone in $t$ and $0 \leq \psi_u(t) \leq 1$. Therefore, integral (4) exists. For any $\varphi(u)$, $0 \leq \varphi(u) \leq 1$, it is obvious that $A\varphi(u) \geq 0$ and

$$A\varphi(u) \leq \int_0^{\infty} \int_{-\infty}^{U(u,t)} dF(z)\, dK(t) \leq \int_0^{\infty} F(U(u,t))\, dK(t) \leq 1.$$

The fact that the function $A\varphi(u)$ does not decrease with respect to $u$ follows from the monotonicity of $U(\cdot, t)$ and $\varphi(\cdot)$. Thus, $A:\Phi \to \Phi$. The monotonicity of the integral operator $A$ follows from the linearity and non-negativity of the kernel. Finally, for any $\varphi_1$ and $\varphi_2$, we have

$$\rho(A\varphi_1, A\varphi_2) \le \int_0^{\infty} \int_{-\infty}^{U(u,t)} \sup_{u\ge 0} |\varphi_1(u) - \varphi_2(u)| dF(z) dK(t) \le \int_0^{\infty} F(U(u,t)) dK(t) \rho(\varphi_1, \varphi_2) \le \rho(\varphi_1, \varphi_2).$$

Thus, the operator $A$ is nonexpanding.

The lemma is proved.

**Assumption 2.** *There exists a nondecreasing function* $\varphi_*(u)$ *such that* $0 \le \varphi_*(u) \le 1$, $\lim_{u\to+\infty} \varphi_*(u) = 1$, *and* $A\varphi_*(u) \ge \varphi_*(u)$.

The theorem presented below gives sufficient conditions for the existence of a function $\varphi_*(u)$ that satisfies Assumption 2.

**Assumption 3.** *There exist constants* $u_* \ge 0$, $c_* \ge 0$, *and* $L > 0$ *such that*

*(a)* $U(u,t) \ge u + c_* t$ *for all* $u \ge u_*$;

*(b)* $\int_{-\infty}^{+\infty} e^{Lz} dF(z) \int_0^{+\infty} e^{-c_* Lt} dK(t) \le 1$;

*(c)* $\lim_{z\to+\infty} e^{Lz}(1 - F(z)) = 0$.

**Theorem 1.** *Suppose that Assumptions 1 and 3 are satisfied. Then Assumption 2 with the function*

$$\varphi_*(u) = \max\{0, 1 - e^{-L(u-u_*)}\}$$

*is satisfied.*

***Proof.*** It suffices to verify that $A\varphi_*(u) \ge \varphi_*(u)$. Consider

$$A\varphi_*(u) = \int_0^{+\infty} \int_{-\infty}^{U(u,t)} \varphi_*(U(u,t) - z) dF(z) dK(t)$$

$$= \int_0^{+\infty} \int_{-\infty}^{U(u,t)} \max\{0, 1 - e^{-L(U(u,t)-u_*-z)}\} dF(z) dK(t)$$

$$= \int_0^{+\infty} \int_{-\infty}^{U(u,t)-u_*} (1 - e^{-L(U(u,t)-u_*-z)}) dF(z) dK(t).$$

Integrating the inner integral by parts and using the fact that $F(-\infty) = 0$, we get

$$\int_{-\infty}^{U(u,t)-u_*} (1 - e^{-L(U(u,t)-u_*-z)})\, dF(z)$$

$$= (1 - e^{-L(U(u,t)-u_*-z)})\, F(z)\Big|_{-\infty}^{U(u,t)-u_*} + e^{-L(U(u,t)-u_*)} L \int_{-\infty}^{U(u,t)-u_*} e^{Lz} F(z)\, dz$$

$$= e^{-L(U(u,t)-u_*)} L \int_{-\infty}^{U(u,t)-u_*} e^{Lz} F(z)\, dz.$$

We transform

$$L \int_{-\infty}^{U(u,t)-u_*} e^{Lz} F(z)\, dz = L \int_{-\infty}^{U(u,t)-u_*} e^{Lz} (1 - (1 - F(z)))\, dz$$

$$= e^{L(U(u,t)-u_*)} - L \int_{-\infty}^{U(u,t)-u_*} e^{Lz} (1 - F(z))\, dz$$

$$\geq e^{L(U(u,t)-u_*)} - L \int_{-\infty}^{+\infty} e^{Lz} (1 - F(z))\, dz = e^{L(U(u,t)-u_*)} - \int_{-\infty}^{+\infty} e^{Lz}\, dF(z).$$

Thus, for $u \geq u_*$, with regard for Assumption 3a, we obtain

$$A\varphi_*(u) \geq \int_0^{+\infty} dK(t) e^{-L(U(u,t)-u_*)} \left( e^{L(U(u,t)-u_*)} - \int_{-\infty}^{+\infty} e^{Lz}\, dF(z) \right)$$

$$= 1 - \int_{-\infty}^{+\infty} e^{Lz}\, dF(z) \int_0^{+\infty} e^{-L(U(u,t)-u_*)}\, dK(t)$$

$$\geq 1 - e^{-L(u-u_*)} \int_{-\infty}^{+\infty} e^{Lz}\, dF(z) \int_0^{+\infty} e^{-c_* Lt}\, dK(t) \geq 1 - e^{-L(u-u_*)}.$$

Since $A\varphi_* \geq 0$, we get

$$A\varphi_*(u) \geq \max\{0, 1 - e^{-L(u-u_*)}\} = \varphi_*(u) \quad \text{for all} \quad u \geq 0.$$

The theorem is proved.

***Remark 1.*** In [2, 3], the role of $\varphi_*(u)$ was played by the Cramér–Lundberg limit $\varphi_*(u) = 1 - e^{-Lu}$, where $L$ is a certain positive (Lundberg) constant.

***Remark 2.*** If insurance claims are bounded with probability one, i.e., $F(z) = 1$ for all sufficiently large $z$, then condition 3c is *a fortiori* satisfied and condition 3b is satisfied for any $c_* > \max\{0, \bar{z}/\tau\}$ with $\bar{z} \geq 0$ and for any $c_* \geq 0$ with $\bar{z} < 0$, where

$$\bar{z} = \int_{-\infty}^{+\infty} z\, dF(z)$$

is the average value of payments and

$$\tau = \int_{0}^{+\infty} t\, dK(t)$$

is the average time between payments. For $U(u, t)$ of the form (2), there *a fortiori* exist $u_* \geq 0$ and $c_* > 0$ such that $c_* > \max\{0, \bar{z}/\tau\}$ and $U(u, t) \geq u + c_* t$ for all $u \geq u_*$, and, hence, condition 3a is satisfied.

We define the subset $\Phi^* \subset \Phi$ of nondecreasing functions $\varphi(u)\colon [0, +\infty) \to [0, 1]$ such that $\varphi_*(u) \leq \varphi(u) \leq 1$, where $\varphi_*(u)$ satisfies Assumption 2.

**Lemma 2.** $A\colon \Phi^* \to \Phi^*$.

The statement of the lemma obviously follows from Lemma 1 and Assumption 2.

By virtue of Lemma 1, the operator $A\colon \Phi \to \Phi$ is nonexpanding, but, generally speaking, it is not a contraction operator on the set of functions $\Phi$. Therefore, we cannot use the contracting-mapping principle. Though $A\colon \Phi^* \to \Phi^*$ and $\Phi^*$ is a compact set with respect to the topology of pointwise convergence (by virtue of the second Helly theorem), this is also insufficient for the proof of the existence of a fixed point of $A$ in $\Phi^*$, i.e., of the existence of a solution of problem (3)–(5). The theorem below establishes the existence of a solution of problem (3)–(5) on the basis of the monotonicity of the operator $A\colon \Phi^* \to \Phi^*$.

**Theorem 2** (on necessary and sufficient conditions for the existence of a solution)**.** *Suppose that Assumption 1 is satisfied. For the existence of a solution of problem (3)–(5), it is necessary and sufficient that there exist a function $\varphi_*(u)$ that satisfies Assumption 2.*

***Proof.*** The *necessity* is obvious. As $\varphi_*(u)$, we can take any solution of problem (3)–(5). To prove the *sufficiency*, we construct a sequence of functions that converge pointwise to a solution of the problem. Namely, we consider the sequence of approximations

$$\left\{\varphi^{k+1}(u) = A\varphi^k(u),\ \varphi^0(u) \equiv 1,\ k = 0, 1, \ldots\right\}.$$

By virtue of the monotonicity of $U(\cdot, t)$, all functions $\varphi^k(u)$ do not decrease with respect to $u$. By induction, we prove that the sequence $\{\varphi^k(u),\ k = 0, 1, \ldots\}$ is monotonically decreasing. Indeed,

$$\varphi^1(u) = A\varphi^0(u) = \int_0^{\infty} \int_{-\infty}^{U(u,t)} dF(z)\, dK(t) = \int_0^{\infty} F(U(u,t))\, dK(t) \le 1 = \varphi^0(u).$$

By virtue of the monotonicity of the operator $A$, the assumption that $\varphi^k(u) \le \varphi^{k-1}(u)$ yields

$$\varphi^{k+1}(u) = A\varphi^k(u) \le A\varphi^{k-1}(u) = \varphi^k(u).$$

By analogy, we prove by induction that $\varphi^k(u) \ge \varphi_*(u)$ for all $k$. Indeed, $\varphi^0(u) \equiv 1 \ge \varphi_*(u)$. By virtue of the monotonicity of the operator $A$, the assumption that $\varphi^k(u) \ge \varphi_*(u)$ yields

$$\varphi^{k+1}(u) = A\varphi^k(u) \ge A\varphi_*(u) \ge \varphi_*(u).$$

Thus, the sequence of functions $\{\varphi^k(u)\}$ decreases monotonically and is bounded from below by the function $\varphi_*(u)$. Therefore, there exists the limit function $\varphi(u) = \lim_{k\to+\infty} \varphi^k(u)$, which, together with all $\varphi^k(u)$, does not decrease with respect to $u$, $1 \ge \varphi(u) \ge \varphi_*(u)$, and, hence, $\lim_{u\to+\infty} \varphi(u) = 1$. We now pass to the limit with respect to $k$ in (6). By virtue of the Lebesgue theorem, we can pass to the limit under the sign of the integral operator. Thus, the limit function $\varphi(u)$ satisfies Eq. (3).

The theorem is proved.

***Corollary 1.*** *Under Assumptions 1 and 2, the sequence of approximations* $\{\varphi^k(u)\}$ *constructed according to (6) and beginning with* $\varphi^0(u) \equiv 1$ *decreases monotonically and converges pointwise to a certain solution of problem (3)–(5) from above.*

***Corollary 2.*** *Under Assumptions 1 and 2, the sequence of approximations* $\{\varphi^k(u)\}$ *beginning with* $\varphi^0(u) = \varphi_*(u)$ *increases monotonically and converges pointwise to a certain solution of problem (3)–(5) from below.*

To guarantee the uniqueness of a solution of problem (3)–(5), we make additional assumptions concerning the operator $A$.

**Assumption 4.** *The functions* $U(u,t)$, $F(z)$, *and* $K(t)$ *in the operator* $A$ *satisfy one of the following conditions:*

*(a)* $F(z) < 1\ \forall z$;

*(b)* $F(z) > 0\ \forall z$;

*(c)* $K(t) < 1\ \forall t \ge 0$, $F(z) = 1\ \forall z \ge \bar{z} \ge 0$, *and* $\lim_{t\to+\infty} U(0,t) = +\infty$.

The lemma presented below shows that, under Assumptions 1, 2, and 4, the operator $A$ possesses a certain contracting property on $\Phi^* \subset \Phi$, which implies (Corollary 3) that it is nonuniformly contracting on $\Phi^*$. The property of nonuniform contraction is insufficient for the existence of a solution of problem (3)–(5); for this reason, the existence has been independently proved in Theorem 2. For the proof of the uniqueness of a solution, the property of nonuniform contraction is sufficient (Corollary 4).

**Lemma 3.** *Suppose that Assumptions 1, 2, and 4 are satisfied. Then, for any* $\varepsilon > 0$, *there exists* $q^*(\varepsilon)$, $0 \le q^*(\varepsilon) < 1$, *such that, for any* $\varphi_1, \varphi_2 \in \Phi^*$ *such that* $\rho(A\varphi_1, A\varphi_2) \ge \varepsilon$, *the following relation is true:*

$$\rho(A\varphi_1, A\varphi_2) \le q^*(\varepsilon) \cdot \rho(\varphi_1, \varphi_2).$$

***Proof.*** First, we prove the lemma under Assumption 4a. We fix $\varepsilon > 0$ and find a number $u^*(\varepsilon) \ge 0$ such that $1 - \varphi_*(u) \le \varepsilon/2$ for all $u \ge u^*(\varepsilon)$. We set

$$q^*(\varepsilon) = \int_0^\infty F(U(u^*(\varepsilon), t))\, dK(t). \tag{7}$$

Since $F(\cdot) < 1$ and

$$\int_0^\infty dK(t) = 1,$$

we have $q^*(\varepsilon) < 1$. Let functions $\varphi_1, \varphi_2 \in \Phi^*$ be such that $\rho(A\varphi_1, A\varphi_2) \ge \varepsilon$. By definition, there exists a sequence $\{u^s\}$ such that

$$\lim_{s \to +\infty} \left| A\varphi_1(u^s) - A\varphi_2(u^s) \right| = \rho(A\varphi_1, A\varphi_2) \ge \varepsilon > 0.$$

Since $\varphi_*(u^s) \le A\varphi_1(u^s) \le 1$ and $\varphi_*(u^s) \le A\varphi_2(u^s) \le 1$, for sufficiently large $s$ we get

$$\frac{\varepsilon}{2} < \left| A\varphi_1(u^s) - A\varphi_2(u^s) \right| \le 1 - \varphi_*(u^s).$$

This implies that $u^s \le u^*(\varepsilon)$ for all sufficiently large $s$. For large $s$, the following estimate is true:

$$\left| A\varphi_1(u^s) - A\varphi_2(u^s) \right| \le \int_0^\infty \int_{-\infty}^{U(u^s,t)} \left| \varphi_1(U(u^s,t) - z) - \varphi_2(U(u^s,t) - z) \right| dF(z)\, dK(t)$$

$$\le \rho(\varphi_1, \varphi_2) \int_0^\infty F(U(u^s, t))\, dK(t) \le \rho(\varphi_1, \varphi_2) \int_0^\infty F(U(u^*(\varepsilon), t))\, dK(t).$$

Passing here to the limit with respect to $s$, we obtain the statement of the lemma.

Let us prove Lemma 3 under Assumption 4b. By virtue of Lemma 1, we have

$$\rho(\varphi_1, \varphi_2) \ \ge\ \rho(A\varphi_1, A\varphi_2) \ \ge\ \varepsilon.$$

Using the fact that $|\varphi_1(u) - \varphi_2(u)| \le 1 - \varphi_*(u)$, we obtain the estimates

$$|A\varphi_1(u) - A\varphi_2(u)| \le \int_0^{+\infty}\int_{-\infty}^{U(u,t)} |\varphi_1(U(u,t)-z) - \varphi_2(U(u,t)-z)|\,dF(z)\,dK(t)$$

$$\le \int_0^{+\infty}\int_{-\infty}^{U(u,t)} \min\{\rho(\varphi_1,\varphi_2),\ 1-\varphi_*(U(u,t)-z)\}\,dF(z)\,dK(t)$$

$$\le \int_0^{+\infty}\int_{-\infty}^{+\infty} \min\{\rho(\varphi_1,\varphi_2),\ 1-\varphi_*(\max\{0,-z\})\}\,dF(z)\,dK(t)$$

$$\le \rho(\varphi_1,\varphi_2)\int_{-\infty}^{+\infty} \min\{1,\ (1-\varphi_*(\max\{0,-z\}))/\varepsilon\}\,dF(z).$$

Under the assumptions made, we have

$$q(\varepsilon) = \int_{-\infty}^{+\infty} \min\{1,(1-\varphi_*(\max\{0,-z\}))/\varepsilon\}\,dF(z) < 1$$

This yields the statement of the lemma.

We now prove the lemma under Assumption 4c. Let $K(\cdot) < 1$ and let $F(z) = 1$ for $z \ge \bar{z} \ge 0$. It is obvious that $|\varphi_1(u) - \varphi_2(u)| \le 1 - \varphi_*(u)$. Then, for any $u \ge 0$, the following estimates are true:

$$|A\varphi_1(u) - A\varphi_2(u)| \le \int_0^{+\infty}\int_{-\infty}^{U(u,t)} |\varphi_1(U(u,t)-z) - \varphi_2(U(u,t)-z)|\,dF(z)\,dK(t)$$

$$\le \int_0^{+\infty}\int_{-\infty}^{U(u,t)} \min\{\rho(\varphi_1,\varphi_2),\ 1-\varphi_*(U(u,t)-z)\}\,dF(z)\,dK(t)$$

$$\le \int_0^{+\infty}\int_{-\infty}^{\min\{U(u,t),\bar{z}\}} \min\{\rho(\varphi_1,\varphi_2),\ 1-\varphi_*(U(u,t)-z)\}\,dF(z)\,dK(t)$$

$$\le \int_0^{+\infty}\int_{-\infty}^{\min\{U(u,t),\bar{z}\}} \min\{\rho(\varphi_1,\varphi_2),\ 1-\varphi_*(\max\{0,\ U(u,t)-\bar{z}\})\}\,dF(z)\,dK(t)$$

$$\leq \int_0^{+\infty} \min\{\rho(\varphi_1, \varphi_2),\ 1 - \varphi_*(\max\{0,\ U(u,t) - \bar{z}\})\} F(\min\{U(u,t), \bar{z}\}) dK(t)$$

$$\leq \int_0^{+\infty} \min\{\rho(\varphi_1, \varphi_2),\ 1 - \varphi_*(\max\{0,\ U(u,t) - \bar{z}\})\} dK(t).$$

By virtue of Lemma 1, we have

$$\rho(\varphi_1, \varphi_2) \geq \rho(A\varphi_1, A\varphi_2) \geq \varepsilon.$$

Hence,

$$\rho(A\varphi_1, A\varphi_2) \leq \rho(\varphi_1, \varphi_2) \int_0^{+\infty} \min\{1,\ (1 - \varphi_*(\max\{0, U(0,t) - \bar{z}\}))/\varepsilon\} dK(t).$$

Since, by assumption, $U(0,t) \to +\infty$, we conclude that

$$(1 - \varphi_*(\max\{0, U(0,t) - \bar{z}\})) \to 0 \quad \text{as} \quad t \to +\infty$$

and, hence,

$$\min\{1,\ (1 - \varphi_*(\max\{0, U(0,t) - \bar{z}\}))/\varepsilon\} < 1$$

for all sufficiently large $t$. Since

$$\int_0^{+\infty} dK(t) = 1,$$

this yields

$$q(\varepsilon) = \int_0^{+\infty} \min\{1,\ (1 - \varphi_*(\max\{0, U(0,t) - \bar{z}\}))/\varepsilon\} dK(t) < 1.$$

The lemma is proved.

***Corollary 3*** (on nonuniform contraction)**.** *Under Assumptions 1, 2, and 4, the following inequality holds for any* $\varphi_1, \varphi_2 \in \Phi^*$, $\varphi_1 \neq \varphi_2$:

$$\rho(A\varphi_1, A\varphi_2) < \rho(\varphi_1, \varphi_2).$$

***Corollary 4.*** *Under Assumptions 1, 2, and 4, there exists a unique solution* $\varphi(u)$ *of problem (3)–(5) on the set* $\Phi^*$.

Corollary 4 does not exclude the possibility of the existence of other solutions of problem (3)–(5) on a broader set $\Phi \supset \Phi^*$.

**Theorem 3.** *Under Assumptions 1 and 4, only one solution of problem (3)–(5) can exist.*

***Proof.*** Assume the contrary. Let there exist two solutions $\varphi_1 \neq \varphi_2$ (i.e., $\varphi_1(u) \neq \varphi_2(u)$ for a certain $u \geq 0$) of problem (3)–(5). We set $\varphi^0(u) = \max\{\varphi_1(u), \varphi_2(u)\}$. Consider the sequence $\{\varphi^k(u) := A\varphi^k(u), k = 0, 1, \ldots\}$. It is obvious that $1 \geq \varphi^0(u) \geq \varphi_1(u)$ and $1 \geq \varphi^0(u) \geq \varphi_2(u)$. By virtue of Lemma 1 and the monotonicity of the operator $A$, we get $1 \geq \varphi^1(u) = A\varphi^0(u) \geq A\varphi_1(u) = \varphi_1(u)$ and $1 \geq \varphi^1(u) = A\varphi^0(u) \geq A\varphi_2(u) = \varphi_2(u)$, whence $\varphi^1(u) \geq \max\{\varphi_1(u), \varphi_2(u)\} = \varphi^0(u)$. By virtue of the monotonicity of $A$, we establish by induction that $\varphi^{k+1}(u) \geq \varphi^k(u) \geq \varphi^0(u)$ for all $k \geq 0$. The sequence $\{\varphi^k(\cdot)\}$ increases monotonically and is bounded from above by unity. Therefore, it has the pointwise limit $\varphi(u)$, $1 \geq \varphi(u) \geq \varphi^0(u)$, which is a solution of Eq. (3), (4) by virtue of the Lebesgue theorem on dominated convergence. It is obvious that the obtained function $\varphi(u)$ is a solution of problem (3)–(5) that differs from at least one of the solutions $\varphi_1$ or $\varphi_2$, say, from $\varphi_1$. We now take the function $\varphi_*(u) = \varphi_1(u)$, which satisfies Assumption 2. Then, on the corresponding set $\Phi^*$, we have two different solutions $\varphi(u)$ and $\varphi_1(u)$ of problem (3)–(5), which contradicts Corollary 4.

The theorem is proved.

***Corollary 5*** (sufficient conditions for the existence and uniqueness of a solution)**.** *Under Assumptions 1, 2 (or 3), and 4, there exists a unique solution of problem (3)–(5).*

***Corollary 6*** (convergence of the method of successive appropriations)**.** *Under Assumptions 1, 2, and 4, for any initial approximation $\varphi^0 \in \Phi^*$ the sequence $\{\varphi^k(u), k = 0, 1, \ldots\}$ generated by algorithm (6) converges pointwise to a solution of problem (3)–(5).*

***Remark 3.*** Corollaries 1, 2, 4, and 6 were proved in [2, 3] by using the special function $\varphi_*(u) = 1 - e^{-Lu}$ with certain constant $L > 0$, i.e., the existence and uniqueness of a solution of problem (3)–(5) were proved on a special subset $\Phi^* \subset \Phi$.

It turns out that, under the assumptions made, one has not only the pointwise convergence but also the uniform convergence of the sequence of approximations (6) to a solution of problem (3)–(5). Note that the solution $\varphi(u)$ of problem (3)–(5) can also be a discontinuous function for the discontinuous function $K(t)$ (see [6]).

**Theorem 4** (on the uniform convergence and the rate of convergence of the method of successive approximations)**.** *Under Assumptions 1, 2 (or 3), and 4, the method of successive approximations (6) beginning with an initial approximation $\varphi^0(u)$ such that $\varphi_*(u) \leq \varphi^0(u) \leq 1$ converges uniformly and monotonically to the solution $\varphi(u)$ of problem (3)–(5), i.e., $\rho(\varphi^k, \varphi) = \sup_{u \geq 0} \left|\varphi^k(u) - \varphi(u)\right|$ tends monotonically to zero; moreover, method (6) converges uniformly in any $\varepsilon$-neighborhood of the solution of problem (3)–(5) with the rate of geometric progression with denominator $q(\varepsilon)$ that depends on $\varepsilon$, i.e.,*

$$\rho(\varphi^k, \varphi) \leq (q(\varepsilon))^k \rho(\varphi^0, \varphi), \qquad 0 \leq q(\varepsilon) < 1,$$

*for all $k$ such that $\rho(\varphi^k, \varphi) \geq \varepsilon$.*

***Proof.*** By virtue of Theorem 2 and Corollary 5, a solution $\varphi(u)$ of problem (3)–(5) exists and is unique. It follows from Lemma 1 that the sequence $\rho(\varphi^k, \varphi)$ decreases monotonically. By virtue of Lemma 3, for any $\varepsilon > 0$ there exists $q(\varepsilon)$, $0 \le q(\varepsilon) < 1$, such that, for all $k$ such that $\rho(\varphi^k, \varphi) \ge \varepsilon$, one has

$$\rho(\varphi^k, \varphi) \le q(\varepsilon)\rho(\varphi^{k-1}, \varphi) \le (q(\varepsilon))^k \rho(\varphi^0, \varphi).$$

It remains to show that

$$\lim_{k \to \infty} \rho(\varphi^k, \varphi) = 0.$$

Assume the contrary, i.e., let

$$\lim_{k \to \infty} \rho(\varphi^k, \varphi) = \varepsilon > 0.$$

Then, by virtue of the arguments presented above, we get $\varepsilon \le \rho(\varphi^k, \varphi) \le (q(\varepsilon))^k \rho(\varphi^0, \varphi)$ for all $k$, which is impossible for sufficiently large $k$.

The theorem is proved.

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