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IMPROVED SCALES OF SPACES AND ELLIPTIC BOUNDARY-VALUE PROBLEMS. II

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We study improved scales of functional Hilbert spaces over $\mathbb{R}^n$ and smooth manifolds with boundary. The isotropic Hörmander–Volevich–Paneyakh spaces are elements of these scales. The theory of elliptic boundary-value problems in these spaces is developed.

Introduction

In the present paper, we study an improved scale of Hilbert functional spaces introduced by the authors in [1]. The smoothness properties of the functions in the spaces of this scale are determined not by a family of numbers but by a functional parameter in the form of a regularly varying function of one real variable. This functional parameter enables one to give more precise characteristics of smoothness of a function according to the properties of its Fourier transform at infinity.

The aim of the present paper is to show that the properties of the improved scale and the classical scales of spaces of Bessel potentials are, to a significant extent, similar. This enables one to extend the theory of elliptic boundary-value problems to improved scales. The indicated analogy of the properties is a consequence of the fact that each space of the improved scale can be obtained as a result of interpolation of a couple of spaces of Bessel potentials with proper functional parameter. In the analyzed case, the required parameter should be chosen in the form of a function regularly varying at $+\infty$.

The paper consists of four sections. In Sec. 1, we consider some properties of slowly varying functions necessary for what follows. In Sec. 2, we show that a function of order $\theta,$ where $0 < \theta < 1,$ regularly varying at $+\infty$ can play the role of an interpolation parameter, i.e., it generates an interpolation functor in the category of pairs of Hilbert spaces. On the basis of this result, in Sec. 3, by the method of interpolation, we study improved scales over the space $\mathbb{R}^n,$ half-space $\mathbb{R}^n_+,$ and a compact differentiable manifold of the class $C^\infty.$ In Sec. 4, also by the method of interpolation, we establish a theorem on the Noether property of the operator of the elliptic boundary-value problem in the improved scale of spaces of differentiable functions on a manifold. Sections 1 and 2 were published in the first part of the paper (see [2]).

It should be noted that spaces in which smoothness is described with the help of functional parameters were, for the first time, introduced and studied in [3, 4]. At present, these spaces are extensively investigated (see, e.g., [5, pp. 381–415], [6], and the bibliography therein). Thus, in particular, regular elliptic boundary-value problems in some spaces of this sort were studied on Euclidean domains by the method of interpolation in [7].

3. Improved Scales of Spaces

First, we consider improved scales of functional spaces over $\mathbb{R}^n,$ where $n \geq 1,$ and over the half-space $\mathbb{R}^n_+ = \{(x', x_n) \colon x' \in \mathbb{R}^{n-1}, x_n > 0\}$ $\big($for $n = 1,$ we have $\mathbb{R}^n_+ = (0; +\infty)\big).$ Then, on the basis of these scales,

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using the standard procedure of local rectification, we construct improved scales over smooth compact manifolds. The spaces that form these scales depend on two parameters (numerical and functional). The functional parameter runs through a certain set $\mathcal{M}$, which we define below.

Let $\mathcal{M}$ denote the collection of all positive functions φ defined on $[1;+\infty)$ and such that (a) φ is a Borel measurable function on $[1;+\infty)$, (b) the functions φ and $\dfrac{1}{\varphi}$ are bounded on every segment $[1;b]$, where $1 < b < +\infty$, and (c) φ is a function slowly varying at $+\infty$.

Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. By $H^{s,\varphi}(\mathbb{R}^n)$ we denote the collection of all distributions u of slow growth defined on $\mathbb{R}^n$ and such that the Fourier transform $\widehat{u}$ of the distribution u is a function locally Lebesgue summable on $\mathbb{R}^n$ and such that

$$\int \langle\xi\rangle^{2s}\varphi^2(\langle\xi\rangle)\,|\widehat{u}(\xi)|^2\,d\xi < \infty. \tag{3.1}$$

Here and below, the integral, unless otherwise stated, is taken over $\mathbb{R}^n$, and $\langle\xi\rangle = (1+\xi_1^2+\ldots+\xi_n^2)^{1/2}$ is the smoothed modulus of the vector $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$. In the space $\mathrm{H}^{s,\varphi}(\mathbb{R}^n)$, as the scalar product of its elements u and v, we take the quantity

$$\int \langle\xi\rangle^{2s}\varphi^2(\langle\xi\rangle)\,\widehat{u}(\xi)\,\overline{\widehat{v}(\xi)}d\xi,$$

which generates the norm equal to the square root of the left-hand side of inequality (3.1).

Remark 3.1. The spaces $H^{s,\varphi}(\mathbb{R}^n)$ are a special case of the Hörmander and Volevich–Paneyakh spaces. Namely, $H^{s,\varphi}(\mathbb{R}^n) = \mathcal{B}_{2,k} = H^{\mu}$, where $k(\xi) = \mu(\xi) = \langle\xi\rangle^s\varphi(\langle\xi\rangle)$, $\mathcal{B}_{2,k}$ is the space introduced by Hörmander in [3, p. 54], and H^{μ} is the space introduced by Volevich and Paneyakh in [4, p. 14]. Note that the spaces $\mathcal{B}_{2,k}$ and H^{μ} are defined in the mentioned papers for an arbitrary positive *weight* function $k(\xi) = \mu(\xi)$ of $\xi \in \mathbb{R}^n$. According to Volevich and Paneyakh, the last statement means the continuity of μ and the validity of the estimate $\dfrac{\mu(\xi)}{\mu(\eta)} \le c(1+|\xi-\eta|^l)$, $\xi,\eta \in \mathbb{R}^n$, where the constants c and l are independent of ξ and η. (According to Hörmander, the inequality $\dfrac{k(\xi)}{k(\eta)} \le (1+c|\xi-\eta|)^l$ must be true, but, as follows from the remark given in [3, p. 54], the functions k lead to the same class of spaces as the functions μ.) For arbitrary $\varphi \in \mathcal{M}$, according to Proposition 1.3(a) and the definition of the set $\mathcal{M}$, there exists a function $\varphi_1 \in \mathcal{M}$ continuous on $[1;+\infty)$ and such that $c_1\varphi_1(t) \le \varphi(t) \le c_2\varphi_1(t)$ for $t \ge 1$, where c_1 and c_2 are finite positive constants independent of t. Therefore, $H^{s,\varphi}(\mathbb{R}^n) = H^{s,\varphi_1}(\mathbb{R}^n)$ with equivalence of norms. Furthermore, by virtue of Lemma 1.1, $\mu_1(\xi) = \langle\xi\rangle^s\varphi_1(\langle\xi\rangle)$ is a weight function:

$$\mu_1(\xi)/\mu_1(\eta) = (\langle\xi\rangle/\langle\eta\rangle)^s\varphi_1(\langle\xi\rangle)/\varphi_1(\langle\eta\rangle) \le c(1+|\langle\xi\rangle-\langle\eta\rangle|^{|s|+1}) \le c(1+|\xi-\eta|^{|s|+1}).$$

Thus, all facts established by Hörmander in [3, pp. 54–67] for the space $\mathcal{B}_{2,k}$ and by Volevich and Paneyakh in [4, pp. 14–54] for the space H^{μ} are also true for the spaces $H^{s,\varphi}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$. We are mainly interested in the specific properties of the spaces $H^{s,\varphi}(\mathbb{R}^n)$ caused by the fact that $\varphi \in \mathrm{SV}$.

For $\varphi \equiv 1$, we denote the space $H^{s,\varphi}(\mathbb{R}^n)$ also by $H^s(\mathbb{R}^n)$. This is the well-known space of Bessel potentials of order s over $\mathbb{R}^n$.

Lemma 3.1. *For any $s \in \mathbb{R}$ and $\varphi, \varphi_1 \in \mathcal{M}$, the following imbeddings are true:*

$$H^{s+\varepsilon}(\mathbb{R}^n) \hookrightarrow H^{s,\varphi}(\mathbb{R}^n) \hookrightarrow H^{s-\varepsilon}(\mathbb{R}^n), \quad \varepsilon > 0, \tag{3.2}$$

$$H^{s+\varepsilon,\varphi_1}(\mathbb{R}^n) \hookrightarrow H^{s,\varphi}(\mathbb{R}^n), \quad \varepsilon > 0. \tag{3.3}$$

Proof. Let $\varepsilon > 0$. Since $\varphi \in \mathcal{M} \subset \mathrm{SV}$, by virtue of Proposition 1.3(b) we have $t^{-\varepsilon} \le \varphi(t) \le t^{\varepsilon}$ for $t \gg 1$. Using this result and condition (b) in the definition of the class $\mathcal{M}$, we establish that there exist positive constants c_0 and c_1 such that $c_0 t^{-\varepsilon} \le \varphi(t) \le c_1 t^{\varepsilon}$ for all $t \ge 1$. Setting here $t = \langle\xi\rangle$, $\xi \in \mathbb{R}^n$, we immediately obtain the continuous imbeddings (3.2), which obviously lead to (3.3).

Lemma 3.1 is proved.

Consider the family

$$\{H^{s,\varphi}(\mathbb{R}^n)\colon s \in \mathbb{R}, \varphi \in \mathcal{M}\} \tag{3.4}$$

of spaces of distributions on $\mathbb{R}^n$. In this family, according to Lemma 3.1, the numerical parameter s defines the *main* smoothness of the space, and the functional parameter φ defines the *additional* smoothness subordinate to the main one. Briefly speaking, φ improves the main s-smoothness. For this reason, family (3.4) is called an improved scale over $\mathbb{R}^n$ (with respect to the scale $\{H^s(\mathbb{R}^n)\colon s \in \mathbb{R}\}$ of spaces of Bessel potentials).

There is a close relationship between these scales, as a result of which their properties are analogous in many respects. According to this relationship, every space $H^{s,\varphi}(\mathbb{R}^n)$ can be obtained by interpolation with functional parameter in the scale of spaces of Bessel potentials. Namely, the following theorem is true:

Theorem 3.1. *Suppose that a function $\varphi \in \mathcal{M}$ and positive numbers ε and δ are given. We set $\psi(t) = t^{\varepsilon/(\varepsilon+\delta)}\,\varphi(t^{1/(\varepsilon+\delta)})$ for $t \ge 1$ and $\psi(t) = \varphi(1)$ for $0 < t < 1$. Then the following assertions are true:*

(a) the function ψ satisfies all conditions of Theorem 2.1 and, hence, is an interpolation parameter;

(b) for arbitrary $s \in \mathbb{R}$, the following equality of spaces with equivalence of norms in them is true:

$$\Big[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\Big]_{\psi} = H^{s,\varphi}(\mathbb{R}^n).$$

Proof. Since $\varphi \in \mathcal{M}$, it is obvious that ψ satisfies conditions (a) and (b) of Theorem 2.1. Further, by virtue of the condition $\varphi \in \mathcal{M} \subset \mathrm{SV}$ and Proposition 1.3(d), the function $\varphi(t^{1/(\varepsilon+\delta)})$ of $t \ge 1$ is slowly varying at $+\infty$. Therefore, ψ is a function of order $\theta = \dfrac{\varepsilon}{\varepsilon+\delta} \in (0;1)$ regularly varying at $+\infty$. Thus, ψ satisfies all conditions of Theorem 2.1 and is an interpolation parameter by virtue of this theorem. Assertion (a) is proved.

Let us prove assertion (b). Assume that $s \in \mathbb{R}$. By virtue of the properties of the Hilbert scale of spaces of Bessel potentials [8, pp. 250–253; 9, pp. 211–216], the pair $[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)]$ is admissible, and, furthermore, a pseudodifferential operator with symbol $\langle\xi\rangle^{\varepsilon+\delta}$ is the generating operator A for this pair. Using the Fourier transformation $\mathcal{F}\colon H^{s-\varepsilon}(\mathbb{R}^n) \leftrightarrow L_2(\mathbb{R}^n, \langle\xi\rangle^{2(s-\varepsilon)}d\xi)$, we reduce the operator A to the form of multiplication by the function $\langle\xi\rangle^{\varepsilon+\delta}$ of $\xi \in \mathbb{R}^n$. Since the operator $\psi(A)$ is reduced to the form of multiplication by the function $\psi(\langle\xi\rangle^{\varepsilon+\delta}) = \langle\xi\rangle^{\varepsilon}\varphi(\langle\xi\rangle)$, it has the following domain of definition:

$$\Big[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\Big]_{\psi} = \Big\{u \in H^{s-\varepsilon}(\mathbb{R}^n)\colon \langle\xi\rangle^{\varepsilon}\varphi(\langle\xi\rangle)\,\widehat{u}(\xi) \in L_2(\mathbb{R}^n, \langle\xi\rangle^{2(s-\varepsilon)}d\xi)\Big\}$$

$$= \Big\{u \in H^{s-\varepsilon}(\mathbb{R}^n)\colon \int \langle\xi\rangle^{2s}\varphi^2(\langle\xi\rangle)\;|\widehat{u}(\xi)|^2\,d\xi < \infty\Big\}$$

$$= H^{s-\varepsilon}(\mathbb{R}^n) \cap H^{s,\varphi}(\mathbb{R}^n) = H^{s,\varphi}(\mathbb{R}^n);$$

the last equality here holds by virtue of the right imbedding in (3.2). Furthermore, for the square of the norm of the distribution u in the space $[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)]_\psi,$ we have

$$\|u\|^2_{H^{s-\varepsilon}(\mathbb{R}^n)} + \|\psi(A)u\|^2_{H^{s-\varepsilon}(\mathbb{R}^n)} = \|u\|^2_{H^{s-\varepsilon}(\mathbb{R}^n)} + \int |\langle\xi\rangle^{\varepsilon}\varphi(\langle\xi\rangle)\,\widehat{u}(\xi)|^2\langle\xi\rangle^{2(s-\varepsilon)}\,d\xi$$

$$= \|u\|^2_{H^{s-\varepsilon}(\mathbb{R}^n)} + \|u\|^2_{H^{s,\varphi}(\mathbb{R}^n)}.$$

By virtue of the right continuous imbedding in (3.2), we obtain the equivalence of norms formulated in assertion (b) of the theorem, which completes the proof of Theorem 3.1.

Remark 3.2. In the context of the last theorem, we note the work of Shlenzak [7, p. 54], where the interpolation with functional parameter was applied to the scale of spaces of Bessel potentials, as a result of which several Hörmander–Volevich–Paneyakh Hilbert spaces were obtained. Though the scale of these spaces is called improved, the main (power) and improved (functional) smoothnesses of spaces cannot be selected in it, in contrast to our scale (3.4).

We now establish several properties of the improved scale (3.4) over $\mathbb{R}^n$. Recall that, as usual, $C_0^\infty(\mathbb{R}^n)$ is the set of all functions infinitely differentiable on $\mathbb{R}^n$ and having compact support. Let $C^\rho(\mathbb{R}^n),\ \rho \geq 0,$ denote Hölder spaces over $\mathbb{R}^n$ (see, e.g., [9, p. 242]). It is clear that, for integer $\rho \geq 0,$ a function $u \in C^\rho(\mathbb{R}^n)$ is continuous and bounded on $\mathbb{R}^n$ together with its partial derivatives up to the order ρ inclusive.

Theorem 3.2. *Let $s \in \mathbb{R}$ and $\varphi, \varphi_1 \in \mathcal{M}$. Then the following assertions are true:*

(a) the space $H^{s,\varphi}(\mathbb{R}^n)$ is complete;

(b) the continuous imbeddings (3.2) and (3.3) are dense;

(c) the set $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{s,\varphi}(\mathbb{R}^n)$;

(d) if there exists a constant $c > 0$ such that $\varphi(t) \leq c\varphi_1(t)$ for $t \gg 1,$ then the continuous dense imbedding $H^{s,\varphi_1}(\mathbb{R}^n) \hookrightarrow H^{s,\varphi}(\mathbb{R}^n)$ is true;

(e) if

$$\int\limits_1^{+\infty} \frac{dt}{t\,\varphi^2(t)} < +\infty, \tag{3.5}$$

then the following continuous imbedding is true:

$$H^{\rho+n/2,\,\varphi}(\mathbb{R}^n) \hookrightarrow C^\rho(\mathbb{R}^n) \quad \text{for} \quad \rho \geq 0; \tag{3.6}$$

(f) the spaces $H^{s,\varphi}(\mathbb{R}^n)$ and $H^{-s,1/\varphi}(\mathbb{R}^n)$ are mutually dual with respect to the extension of the scalar product in $L_2(\mathbb{R}^n) = H^0(\mathbb{R}^n)$ by continuity.

Proof. It is well known that the spaces of Bessel potentials are complete. Therefore, by virtue of Theorem 3.1, the space $H^{s,\varphi}(\mathbb{R}^n)$ is compete as a result (up to equivalence of norms) of interpolation of two Hilbert spaces. By virtue of Theorem 3.1 and Lemma 2.1, the continuous imbedding (3.2) is dense. Therefore, (3.3) is also dense. Assertions (a) and (b) are proved. The left imbedding (3.2), together with the already known denseness of the set

$C_0^\infty(\mathbb{R}^n)$ in $H^{s+\varepsilon}(\mathbb{R}^n)$, yields assertion (c). Assertion (d) becomes obvious if one takes into account that, for the functions $\varphi, \varphi_1 \in \mathcal{M}$, it means the following: $\varphi(t) \le c_1\varphi_1(t)$ for $t \ge 1$ and a certain constant $c_1 > 0$. Let us prove assertion (e). Passing from Cartesian coordinates to spherical ones and performing the change of variables $t = (1+r^2)^{1/2}$, we obtain

$$\int \langle\xi\rangle^{-n}\varphi^{-2}(\langle\xi\rangle)\,d\xi = c_2 \int\limits_0^{+\infty} (1+r^2)^{-n/2}\,\varphi^{-2}((1+r^2)^{1/2})\,r^{n-1}dr$$

$$= c_2 \int\limits_1^{+\infty} t^{-n}\varphi^{-2}(t)\,(t^2-1)^{(n-1)/2}\,\frac{t\,dt}{(t^2-1)^{1/2}} \le c_2 \int\limits_1^{+\infty} \frac{dt}{\varphi^2(t)\,(t^2-1)^{1/2}},$$

where c_2 is a certain positive constant. Since the function $\dfrac{1}{\varphi^2(t)}$ is bounded in the neighborhood of the point $t = 1$, the last integral is finite by virtue of (3.5). Thus,

$$J = \int \langle\xi\rangle^{-n}\varphi^{-2}(\langle\xi\rangle)\,d\xi < \infty.$$

Further reasonings are analogous to the proof of Theorem 9.1 in [4, pp. 52, 53]. (This theorem cannot be directly used because it contains anisotropic Hölder spaces such that $C^\rho(\mathbb{R}^n)$ is not their special case.) We represent the number $\rho \ge 0$ in the form $\rho = \rho_0 + \rho_1$, where ρ_0 is the integer part of ρ and $0 \le \rho_1 < 1$. Assume that the nonnegative integers $r_1, \dots, r_n$ satisfy the inequality $r_1 + \ldots + r_n \le \rho_0$. Then, for arbitrary $u \in C^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$\left|\,\partial_{x_1}^{r_1}\dots\partial_{x_n}^{r_n}\,u(x)\right| = \frac{1}{(2\pi)^n}\left|\int \xi_1^{r_1}\dots\xi_n^{r_n}\,\widehat{u}(\xi)\,e^{-ix\xi}\,d\xi\right| \le \frac{1}{(2\pi)^n}\int \langle\xi\rangle^\rho\,|\widehat{u}(\xi)|d\xi \le \frac{J^{1/2}}{(2\pi)^n}\|u\|_H.$$

In this proof, $\|u\|_H$ and $\|u\|_C$ denote the norms of the distribution u in $H^{\rho+n/2,\varphi}(\mathbb{R}^n)$ and $C^\rho(\mathbb{R}^n)$, respectively. Furthermore, for arbitrary $h \in \mathbb{R}^n, h \ne 0$, we write

$$|h|^{-\rho_1}\left|\partial_{x_1}^{r_1}\dots\partial_{x_n}^{r_n}\,(u(x+h)-u(x))\right| = |h|^{-\rho_1}(2\pi)^{-n}\left|\int \xi_1^{r_1}\dots\xi_n^{r_n}(\widehat{u}(\xi)e^{-ih\xi}-\widehat{u}(\xi))\,e^{-ix\xi}\,d\,\xi\right|$$

$$\le |h|^{-\rho_1}(2\pi)^{-n}\int \langle\xi\rangle^{\rho_0}\,|\widehat{u}(\xi)|\,\left|e^{-ih\xi}-1\right|\,d\xi$$

$$\le |h|^{-\rho_1}\,(2\pi)^{-n}\,\|u\|_H\left(\int \frac{\langle\xi\rangle^{2\rho_0}\,\left|e^{-ih\xi}-1\right|^2}{\langle\xi\rangle^{2\rho+n}\,\varphi^2(\langle\xi\rangle)}\,d\xi\right)^{1/2}$$

$$= (2\pi)^{-n}\,\|u\|_H\,\left(\int \langle\xi\rangle^{-n}\,\varphi^{-2}(\langle\xi\rangle)\,\frac{4\sin^2\left(\frac{1}{2}\,\xi\,\eta\right)}{(\langle\xi\rangle\,|h|)^{2\rho_1}}\,d\xi\right)^{1/2}$$

$$\le 2(2\pi)^{-n}\|u\|_H J^{1/2}.$$

The last inequality here follows from the fact that, since $0 \le \rho_1 < 1,$ the fraction under the last integration sign does not exceed $4.$ Thus, $\|u\|_C \le \mathrm{const}\,\|u\|_H,\;\; u \in C_0^\infty(\mathbb{R}^n).$ By virtue of assertion (c), this yields the continuous imbedding (3.6). Assertion (e) is proved. Finally, assertion (f) is a special case of the statement on a dual space in [3, p. 61] (Theorem 2.2.9) and [4, p. 15] [relation (2.3)]. Note that, by virtue of Proposition 1.3(c), we have $\varphi \in \mathcal{M} \Leftrightarrow \dfrac{1}{\varphi} \in \mathcal{M}.$ Therefore, the space $H^{-s,1/\varphi}(\mathbb{R}^n)$ is defined.

Theorem 3.2 is proved.

Remark 3.3. Let ρ be an integer such that $\rho \ge 0.$ By virtue of the known Sobolev imbedding theorem, we have $H^s(\mathbb{R}^n) \hookrightarrow C^\rho(\mathbb{R}^n)$ for $s > \rho + \dfrac{n}{2}.$ However, $H^{\rho+n/2}(\mathbb{R}^n) \not\subseteq C^\rho(\mathbb{R}^n).$ Theorem 3.2(e) enables one to improve, using the parameter $\varphi,$ the main smoothness of the space so that imbedding (3.6) is true. Thus, the scale of the spaces $H^{s,\varphi}(\mathbb{R}^n)$ enables one to characterize the smoothness of the distribution more precisely on the basis of properties of its Fourier transform. Note that statements analogous to assertion (e) of Theorem 3.2 were established for the Hörmander and Volevich–Paneyakh spaces in [3, p. 59] and [4, pp. 33, 52]. The results obtained in these papers imply, in particular, that condition (3.5) is necessary and sufficient for inclusion (3.6) for integer $\rho \ge 0.$

Further, we define an improved scale over the half-space $\mathbb{R}^n_+.$ Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}.$ Let $H^{s,\varphi}(\mathbb{R}^n_+)$ denote the factor space of the Hilbert space $H^{s,\varphi}(\mathbb{R}^n)$ with respect to the subspace

$$\left\{ w \in H^{s,\varphi}(\mathbb{R}^n) \colon \operatorname{supp} w \subseteq \mathbb{R}^n \setminus \mathbb{R}^n_+ \right\}. \tag{3.7}$$

This subspace is closed because it is continuously imbedded into the topological space $\mathcal{D}'(\mathbb{R}^n)$ of distributions on $\mathbb{R}^n.$ [The last fact follows from relation (3.2) and the known continuous imbedding $H^{s-\varepsilon}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$.] Therefore, $H^{s,\varphi}(\mathbb{R}^n_+)$ is a Hilbert space. In this space, the scalar product of cosets of distributions $u_1, u_2 \in H^{s,\varphi}(\mathbb{R}^n)$ is equal to the scalar product in $H^{s,\varphi}(\mathbb{R}^n)$ of the distributions $u_1 - \Pi u_1$ and $u_2 - \Pi u_2,$ where Π is the orthoprojector onto subspace (3.7) in $H^{s,\varphi}(\mathbb{R}^n).$ Note that it is quite natural to interpret $H^{s,\varphi}(\mathbb{R}^n_+)$ as the space of restrictions of all distributions from $H^{s,\varphi}(\mathbb{R}^n)$ to $\mathbb{R}^n_+.$ The norm of such a restriction v in $H^{s,\varphi}(\mathbb{R}^n_+)$ is equal to

$$\inf\left\{ \|u\|_{H^{s,\varphi}(\mathbb{R}^n)} \colon\; u \in H^{s,\varphi}(\mathbb{R}^n),\;\; u = v \;\text{ on }\; \mathbb{R}^n_+ \right\}.$$

In the special case $\varphi \equiv 1,$ we also denote the space $H^{s,\varphi}(\mathbb{R}^n_+)$ by $H^s(\mathbb{R}^n_+).$ This is the known space of Bessel potentials on $\mathbb{R}^n_+$ (see, e.g., [9, p. 265]).

The family $\left\{ H^{s,\varphi}(\mathbb{R}^n_+) \colon s \in \mathbb{R}, \varphi \in \mathcal{M} \right\}$ is called an *improved scale over* $\mathbb{R}^n_+$. For this scale, the following analogs of Theorems 3.1(b) and 3.2 are true:

Theorem 3.3. *Suppose that a function* $\varphi \in \mathcal{M}$ *and positive numbers* ε *and* δ *are given. Then, for any* $s \in \mathbb{R},$ *the following equality of spaces with equivalence of norms in them is true:*

$$\Big[H^{s-\varepsilon}(\mathbb{R}^n_+), H^{s+\delta}(\mathbb{R}^n_+) \Big]_\psi = H^{s,\varphi}(\mathbb{R}^n_+). \tag{3.8}$$

Here, ψ *is the interpolation parameter from Theorem 3.1.*

Proof. The pair of spaces on the left-hand side of (3.8) is obviously admissible. Consider the operator R_+ of restriction of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ to $\mathbb{R}^n_+.$ We have the following linear bounded surjective operators:

$$R_+ \colon H^{s-\varepsilon}(\mathbb{R}^n) \to H^{s-\varepsilon}(\mathbb{R}^n_+), \qquad R_+ \colon H^{s+\delta}(\mathbb{R}^n) \to H^{s+\delta}(\mathbb{R}^n_+),$$

$$R_+ \colon H^{s,\varphi}(\mathbb{R}^n) \to H^{s,\varphi}(\mathbb{R}^n_+). \tag{3.9}$$

According to Theorem 3.1(a), ψ is an interpolation parameter. Therefore, the first two operators yield the boundedness of the operator

$$R_+ \colon \left[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\right]_\psi \to \left[H^{s-\varepsilon}(\mathbb{R}^n_+), H^{s+\delta}(\mathbb{R}^n_+)\right]_\psi,$$

which, by virtue of Theorem 3.1(b), takes the form

$$R_+ \colon H^{s,\varphi}(\mathbb{R}^n) \to \left[H^{s-\varepsilon}(\mathbb{R}^n_+), H^{s+\delta}(\mathbb{R}^n_+)\right]_\psi.$$

By virtue of the surjectivity of operator (3.9), this yields

$$H^{s,\varphi}(\mathbb{R}^n_+) \subseteq \left[H^{s-\varepsilon}(\mathbb{R}^n_+), H^{s+\delta}(\mathbb{R}^n_+)\right]_\psi. \tag{3.10}$$

Let us prove the inverse continuous imbedding. In [9, pp. 265, 266], for an arbitrary number $k,$ a linear bounded operator

$$T_k \colon H^{\sigma}(\mathbb{R}^n_+) \to H^{\sigma}(\mathbb{R}^n), \quad |\sigma| < k, \tag{3.11}$$

that extends a distribution from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ was constructed. This means that R_+T_k is the identity operator. We take a number k such that $|s-\varepsilon| < k$ and $|s+\delta| < k$ and consider the bounded operators (3.11) for $\sigma = s-\varepsilon$ and $\sigma = s+\delta.$ Since ψ is an interpolation parameter, they yield the boundedness of the operator

$$T_k \colon \left[H^{s-\varepsilon}(\mathbb{R}^n_+), H^{s+\delta}(\mathbb{R}^n_+)\right]_\psi \to \left[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\right]_\psi,$$

whence, according to Theorem 3.1(b),

$$T_k \colon \left[H^{s-\varepsilon}(\mathbb{R}^n_+), H^{s+\delta}(\mathbb{R}^n_+)\right]_\psi \to H^{s,\varphi}(\mathbb{R}^n).$$

This and (3.9) yield the boundedness of the identity operator

$$I = R_+T_k \colon \left[H^{s-\varepsilon}(\mathbb{R}^n_+), H^{s+\delta}(\mathbb{R}^n_+)\right]_\psi \to H^{s,\varphi}(\mathbb{R}^n_+).$$

Thus, parallel with inclusion (3.10), its inverse continuous imbedding holds. Therefore, the equality of spaces (3.8) is true, and, furthermore, by virtue of the Banach theorem on an inverse operator, the norms in these spaces are equivalent.

Theorem 3.3 is proved.

Let $\overline{\mathbb{R}}^n_+$ be the closure of the space $\mathbb{R}^n_+$. Let $C_0^\infty(\overline{\mathbb{R}}^n_+)$ and $C^\rho(\overline{\mathbb{R}}^n_+)$, $\rho \geq 0$, denote the spaces of restrictions of all functions from $C_0^\infty(\mathbb{R}^n)$ and $C^\rho(\mathbb{R}^n)$, respectively, to $\overline{\mathbb{R}}^n_+$. The space $C^\rho(\overline{\mathbb{R}}^n_+)$ is a Banach space with respect to the norm

$$\|v\|_{C^\rho(\overline{\mathbb{R}}^n_+)} = \inf\left\{\|u\|_{C^\rho(\mathbb{R}^n)}:\ \ u \in C^\rho(\mathbb{R}^n),\ \ u = v \ \text{ on } \ \overline{\mathbb{R}}^n_+\right\}.$$

Theorem 3.4. *Assertions (a)–(e) of Theorem 3.2 remain true if, in its formulation and in relations (3.2) and (3.3),* $\mathbb{R}^n$ *is replaced by* $\mathbb{R}^n_+$ *in the notation of the spaces of the improved scale,* $C_0^\infty(\mathbb{R}^n)$ *is replaced by* $C_0^\infty(\overline{\mathbb{R}}^n_+)$, *and* $C^\rho(\mathbb{R}^n)$ *is replaced by* $C^\rho(\overline{\mathbb{R}}^n_+)$.

Theorem 3.4 obviously follows from Theorem 3.2 and the definition of the improved scale over $\mathbb{R}^n_+$.

We now proceed to the construction of an improved scale over a manifold. Let $\overline{M}$ be an infinitely smooth compact manifold of dimension $n \geq 1$ with boundary $\partial\overline{M}$. We set $M = \overline{M} \setminus \partial\overline{M}$. Note that we admit the case where $\partial\overline{M} = \varnothing$, i.e., where $\overline{M} = M$ is a closed manifold. Following [10, p. 636], we denote the space of distributions extendable in M by $\overline{\mathcal{D}}\,'(M)$. (If $\overline{M}$ is closed, then $\overline{\mathcal{D}}\,'(M)$ is the space $\mathcal{D}'(\overline{M})$ of all distributions on $\overline{M}$.)

We take a finite atlas $\alpha_j\colon \overline{\Pi}_j \leftrightarrow U_j$, $j = 1, \ldots, r$, from the C^∞-structure on $\overline{M}$. Here, U_j, $j = 1, \ldots, r$, are open (in the topology of the space $\overline{M}$) sets that form a finite covering of the manifold $\overline{M}$, Π_j denotes either $\mathbb{R}^n$ or $\mathbb{R}^n_+$, and $\overline{\Pi}_j$ is the closure of the set Π_j in $\mathbb{R}^n$ (i.e., $\overline{\Pi}_j$ is either $\mathbb{R}^n$ or $\overline{\mathbb{R}^n_+}$, respectively). (For a closed manifold $\overline{M}$, we have $\Pi_j = \overline{\Pi}_j = \mathbb{R}^n$ for all j.) In addition, we take a partition of unity $\chi_j \in C^\infty(\overline{M})$, $j = 1, \ldots, r$, on $\overline{M}$ that satisfies the condition $\operatorname{supp}\chi_j \subseteq U_j$. Let $\mathcal{A}$ denote the pair that consists of the atlas and the partition of unity thus chosen.

As above, let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. We denote by $H^{s,\varphi}(M, \mathcal{A})$ the space of all $f \in \overline{\mathcal{D}}'(M)$ such that $(\chi_j f) \circ \alpha_j \in H^{s,\varphi}(\Pi_j)$ for each $j = 1, \ldots, r$. Here, $(\chi_j f) \circ \alpha_j$ is the representation of the distribution $\chi_j f$ in the local map α_j. In $H^{s,\varphi}(M, \mathcal{A})$, we introduce the scalar product by the relation

$$(f, g)_{H^{s,\varphi}(M,\mathcal{A})} = \sum_{j=1}^{r} ((\chi_j f) \circ \alpha_j,\ (\chi_j\, g) \circ \alpha_j)_{H^{s,\varphi}(\Pi_j)}.$$

It generates the norm

$$\|f\|_{H^{s,\varphi}(M,\mathcal{A})} = \left(\sum_{j=1}^{r}\ \|(\chi_j f) \circ \alpha_j\|^2_{H^{s,\varphi}(\Pi_j)}\right)^{1/2}.$$

The family $\{H^{s,\varphi}(M, \mathcal{A})\colon s \in \mathbb{R},\ \varphi \in \mathcal{M}\}$ is called an *improved scale over* M *corresponding to the pair* $\mathcal{A}$.

For $\varphi \equiv 1$, we also denote the space $H^{s,\varphi}(M, \mathcal{A})$ by $H^s(M, \mathcal{A})$. The space $H^s(M, \mathcal{A})$ is the space of Bessel potentials of order s on M. As is known, it is a Hilbert space independent (up to equivalence of norms) of the choice of the pair $\mathcal{A}$.

Let us show that any space $H^{s,\varphi}(M, \mathcal{A})$, $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$, can be obtained by interpolation in the scale of spaces of Bessel potentials on M. This will imply that $H^{s,\varphi}(M, \mathcal{A})$ is also independent of $\mathcal{A}$.

Theorem 3.5. *Suppose that a function* $\varphi \in \mathcal{M}$ *and positive numbers* ε *and* δ *are given. Then, for any* $s \in \mathbb{R}$, *the following equality of spaces with equivalence of norms in them is true:*

$$\Big[H^{s-\varepsilon}(M,\mathcal{A}),\ H^{s+\delta}(M,\mathcal{A})\Big]_{\psi}=H^{s,\varphi}(M,\mathcal{A}). \tag{3.12}$$

Here, ψ is the interpolation parameter from Theorem 3.1.

Proof. It is known that the pair of spaces of Bessel potentials on the left-hand side of (3.12) is admissible. We deduce equality (3.12) from Theorems 3.1 and 3.3 by using the standard method of "rectification of the manifold $\overline{M}$." By the definition of improved scale over M, the linear mapping of "rectification"

$$T\colon f\mapsto((\chi_1 f)\circ\alpha_1,\ldots,(\chi_r f)\circ\alpha_r\,,\quad f\in\overline{\mathcal{D}}\,'(M),$$

defines the isometric operators

$$T\colon H^{\sigma}(M,\mathcal{A})\ \to\ \prod_{j=1}^{r}\,H^{\sigma}(\Pi_j),\quad \sigma\in\mathbb{R}, \tag{3.13}$$

$$T\colon H^{s,\varphi}(M,\mathcal{A})\ \to\ \prod_{j=1}^{r}\,H^{s,\varphi}(\Pi_j). \tag{3.14}$$

Since ψ is the interpolation parameter, using (3.13) for $\sigma=s-\varepsilon$ and $\sigma=s+\delta$ we obtain the bounded operator

$$T\colon\ \Big[H^{s-\varepsilon}(M,\mathcal{A}),\ H^{s+\delta}(M,\mathcal{A})\Big]_{\psi}\ \to\ \left[\prod_{j=1}^{r}\,H^{s-\varepsilon}(\Pi_j),\ \prod_{j=1}^{r}\,H^{s+\delta}(\Pi_j)\right]_{\psi}.$$

By virtue of Proposition 2.1 and Theorems 3.1 (for $\Pi_j=\mathbb{R}^n$) and 3.3 (for $\Pi_j=\mathbb{R}^n_+$), we have

$$\left[\prod_{j=1}^{r}\,H^{s-\varepsilon}(\Pi_j),\ \prod_{j=1}^{r}\,H^{s+\delta}(\Pi_j)\right]_{\psi}=\prod_{j=1}^{r}\,\Big[\,H^{s-\varepsilon}(\Pi_j),\ H^{s+\delta}(\Pi_j)\,\Big]_{\psi}=\prod_{j=1}^{r}\,H^{s,\varphi}(\Pi_j) \tag{3.15}$$

with equivalence of norms. Thus, the last bounded operator takes the form

$$T\colon\ \Big[H^{s-\varepsilon}(M,\mathcal{A}),\ H^{s+\delta}(M,\mathcal{A})\Big]_{\psi}\ \to\ \prod_{j=1}^{r}\,H^{s,\varphi}(\Pi_j). \tag{3.16}$$

For $T,$ we construct the left inverse operator $K.$ For each $j=1,\ldots,r,$ we take a function $\eta_j\in C_0^{\infty}(\overline{\Pi}_j)$ such that $\eta_j=1$ on the set $\alpha_j^{-1}(\operatorname{supp}\chi_j).$ Consider the linear mapping

$$K\colon(h_1,\ldots,h_r)\ \mapsto\ \sum_{j=1}^{r}\,\Theta_j\,\Big(\,(\eta_j h_j)\circ\alpha_j^{-1}\Big)$$

defined on the vectors $(h_1,\ldots,h_r)$ whose components h_j are distributions on $\Pi_j.$ Here, $(\eta_j h_j)\circ\alpha_j^{-1}$ is a distribution on $U_j\cap M$ such that its representative in the local map α_j has the form $\eta_j h_j,$ and Θ_j is the operator of extension by zero from $U_j \cap M$ to M. It is obvious that Θ_j is well defined on distributions of the form $(\eta_j h_j) \circ \alpha_j^{-1}$. By virtue of the choice of the functions χ_j and η_j, we have

$$KTf = \sum_{j=1}^{r} \Theta_j \left((\eta_j \left((\chi_j f) \circ \alpha_j \right)) \circ \alpha_j^{-1} \right) = \sum_{j=1}^{r} \Theta_j \left((\chi_j f) \circ \alpha_j \circ \alpha_j^{-1} \right) = \sum_{j=1}^{r} \chi_j f = f,$$

i.e.,

$$KTf = f,' \quad f \in \overline{\mathcal{D}}'(M). \tag{3.17}$$

Let us show that the restriction of the mapping K is a bounded operator

$$K\colon \prod_{j=1}^{r} H^{s,\varphi}(\Pi_j) \;\to\; H^{s,\varphi}(M, \mathcal{A}). \tag{3.18}$$

For an arbitrary vector $(h_1, \ldots, h_r)$ from the left space in (3.18), we write

$$\begin{aligned}
\left\| K(h_1, \ldots, h_r) \right\|^2_{H^{s,\varphi}(M,\mathcal{A})} &= \sum_{l=1}^{r} \left\| (\chi_l \, K(h_1, \ldots, h_r)) \circ \alpha_l \right\|^2_{H^{s,\varphi}(\Pi_l)} \\
&= \sum_{l=1}^{r} \left\| \left(\chi_l \sum_{j=1}^{r} \Theta_j \left((\eta_j h_j) \circ \alpha_j^{-1} \right) \right) \circ \alpha_l \right\|^2_{H^{s,\varphi}(\Pi_l)} \\
&= \sum_{l=1}^{r} \left\| \sum_{j=1}^{r} (\eta_{j,l} \, h_j) \circ \beta_{j,l} \right\|^2_{H^{s,\varphi}(\Pi_l)} \\
&\le \sum_{l=1}^{r} \left(\sum_{j=1}^{r} \left\| (\eta_{j,l} \, h_j) \circ \beta_{j,l} \right\|_{H^{s,\varphi}(\Pi_l)} \right)^2 .
\end{aligned} \tag{3.19}$$

Here, $\eta_{j,l} = (\chi_l \circ \alpha_j) \, \eta_j \in C_0^\infty(\overline{\Pi}_j)$ and, furthermore, if $\operatorname{supp} \eta_{j,l} \subseteq \mathbb{R}^n_+ = \Pi_j$, then the function $\eta_{j,l}$ is extended by zero to $\mathbb{R}^n$ and then $\eta_{j,l} \in C_0^\infty(\mathbb{R}^n)$; $\beta_{j,l}\colon \mathbb{R}^n \leftrightarrow \mathbb{R}^n$ is a C^∞-diffeomorphism such that $\beta_{j,l} = \alpha_j^{-1} \circ \alpha_l$ in the neighborhood (in the topology of the space $\overline{\Pi}_j$) of the set $\operatorname{supp} \eta_{j,l}$ and, moreover, $\beta_{j,l}(x) = x$ for all $x \in \mathbb{R}^n$ sufficiently large in modulus. As is known [5, p. 247; 11, p. 46], the operator of multiplication by a function of the class $C_0^\infty(\mathbb{R}^n)$ and the operator of change of variables $u \mapsto u \circ \beta_{j,l}$ are bounded in every space $H^\sigma(\mathbb{R}^n)$, where $\sigma \in \mathbb{R}$. Therefore, the linear operator $u \mapsto (\eta_{j,l} \, u) \circ \beta_{j,l}$ is bounded as an operator from $H^\sigma(\Pi_j)$ into $H^\sigma(\Pi_l)$. Taking $\sigma = s - \varepsilon$ and then $\sigma = s + \delta$ and using the interpolation theorems (Theorems 3.1 and 3.3), we establish that the mapping $h_j \mapsto (\eta_{j,l} \, h_j) \circ \beta_{j,l}$ is a bounded operator acting from $H^{s,\varphi}(\Pi_j)$ into $H^{s,\varphi}(\Pi_l)$. Therefore, relations (3.19) yield

$$\left\| K(h_1, \ldots, h_r) \right\|^2_{H^{s,\varphi}(M,\mathcal{A})} \;\le\; c \sum_{j=1}^{r} \left\| h_j \right\|^2_{H^{s,\varphi}(\Pi_j)},$$

where the constant c is independent of $(h_1,\dots,h_r)$. This means that operator (3.18) is bounded for any $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. In particular, this yields the boundedness of the operators

$$K\colon \prod_{j=1}^{r} H^{\sigma}(\Pi_j)\;\to\; H^{\sigma}(M,\mathcal{A}),\quad \sigma\in\mathbb{R}.$$

We take them for $\sigma=s-\varepsilon$ and $\sigma=s+\delta$ and use interpolation with parameter φ. By virtue of (3.15), we obtain the bounded operator

$$K\colon \prod_{j=1}^{r} H^{s,\varphi}(\Pi_j)\;\to\;\Bigl[H^{s-\varepsilon}(M,\mathcal{A}),\,H^{s+\delta}(M,\mathcal{A})\Bigr]_{\psi}. \tag{3.20}$$

Using (3.14), (3.20), and (3.17), we establish the continuity of the imbedding

$$I=KT\colon H^{s,\varphi}(M,\mathcal{A})\;\to\;\Bigl[H^{s-\varepsilon}(M,\mathcal{A}),\,H^{s+\delta}(M,\mathcal{A})\Bigr]_{\psi}.$$

The inverse continuous imbedding follows from (3.16)–(3.18). This proves the equality of spaces (3.12) with equivalence of norms in them.

Theorem 3.5 is proved.

Corollary 3.1. *For any $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$, the space $H^{s,\varphi}(M,\mathcal{A})$ is independent (up to equivalence of norms) of the choice of the pair $\mathcal{A}$.*

Proof. It is known [3, p. 82] that the space of Bessel potentials on M is independent (up to equivalence of norms) of the choice of the pair $\mathcal{A}$. Taking, in addition to $\mathcal{A}$, a pair $\mathcal{A}_1$ (of the same type as $\mathcal{A}$), we establish that the identity operator I realizes the topological isomorphisms $I\colon H^{s\mp\varepsilon}(M,\mathcal{A})\;\leftrightarrow\;H^{s\mp\varepsilon}(M,\mathcal{A}_1)$, $\varepsilon>0$. Now let ψ be the interpolation parameter defined in Theorem 3.1 for $\varepsilon=\delta>0$. Applying the interpolation with parameter ψ to these isomorphisms, we obtain the topological isomorphism

$$I\colon\;\Bigl[H^{s-\varepsilon}(M,\mathcal{A}),\,H^{s+\delta}(M,\mathcal{A})\Bigr]_{\psi}\;\leftrightarrow\;\Bigl[H^{s-\varepsilon}(M,\mathcal{A}_1),\,H^{s+\delta}(M,\mathcal{A}_1)\Bigr]_{\psi},$$

which, by virtue of Theorem 3.5 is such that $I\colon H^{s,\varphi}(M,\mathcal{A})\leftrightarrow H^{s,\varphi}(M,\mathcal{A}_1)$, which was to be proved.

In what follows, according to Corollary 3.1, we can denote the space $H^{s,\varphi}(M,\mathcal{A})$ by $H^{s,\varphi}(M)$. In this case, the scalar product in $H^{s,\varphi}(M)$ is calculated with the use of some fixed pair $\mathcal{A}$.

Properties of the improved scales over M and $\mathbb{R}^n$ (or $\mathbb{R}^n_+$) are analogous. Furthermore, since the manifold $\overline{M}$ is compact, certain imbeddings of spaces are compact for M.

Theorem 3.6. *Suppose that $s\in\mathbb{R}$ and $\varphi,\varphi_1\in\mathcal{M}$. Then the following assertions are true:*

(a) the space $H^{s,\varphi}(M)$ is complete;

(b) the following compact dense imbeddings are true:

$$H^{s+\varepsilon}(M)\hookrightarrow H^{s,\varphi}(M)\hookrightarrow H^{s-\varepsilon}(M),\quad \varepsilon>0, \tag{3.21}$$

$$H^{s+\varepsilon,\varphi_1}(M)\hookrightarrow H^{s,\varphi}(M),\quad \varepsilon>0; \tag{3.22}$$

(c) *the set* $C^\infty(\overline{M})$ *is dense in* $H^{s,\varphi}(M)$;

(d) *if there exists a constant* $c > 0$ *such that* $\varphi(t) \le c\,\varphi_1(t)$ *for* $t \gg 1$, *then the following continuous dense imbedding is true:*

$$H^{s,\varphi_1}(M) \hookrightarrow H^{s,\varphi}(M); \tag{3.23}$$

this imbedding is compact if $\varphi(t)/\varphi_1(t) \to 0$ *as* $t \to +\infty$;

(e) *if relation (3.5) holds, then the following compact imbedding is true:*

$$H^{\rho+n/2,\varphi}(M) \hookrightarrow C^\rho(\,\overline{M}\,) \quad \textit{for} \quad \rho \ge 0, \tag{3.24}$$

where $C^\rho(\,\overline{M}\,)$ *is the Hölder space of order* ρ *on* $\overline{M}$;

(f) *if the manifold* $\overline{M}$ *is closed, then the spaces* $H^{s,\varphi}(M)$ *and* $H^{-s,1/\varphi}(M)$ *are mutually dual with respect to the extension of the scalar product in* $H^0(M)$ *by continuity.*

Proof. By virtue of Theorem 3.5, the space $H^{s,\varphi}(M)$ is complete as the result of interpolation of two Hilbert spaces of Bessel potentials. The continuous imbeddings (3.21) – (3.24) are obvious corollaries of Theorem 3.4 and assertions (b), (d), and (e) of Theorem 3.2. By virtue of Theorem 3.5 and Lemma 2.1, imbeddings (3.21) are dense. This and the known fact that the set $C^\infty(\,\overline{M}\,)$ is dense in $H^{s+\varepsilon}(M)$ imply that $C^\infty(\,\overline{M}\,)$ is dense in $H^{s,\varphi}(M)$. Therefore, imbeddings (3.22) and (3.23) are also dense. Let us establish the compactness of imbeddings. We begin with (3.23). Assume that $\dfrac{\varphi(t)}{\varphi_1(t)} \to 0$ as $t \to +\infty$. Then, according to Theorem 2.2.3 in [3, p. 56] or Theorem 8.1 in [4, p. 48], for an arbitrary compact set $E \subseteq \mathbb{R}^n$ the following compact imbedding is true:

$$\{u \in H^{s,\varphi_1}(\mathbb{R}^n)\colon\ \operatorname{supp} u \subseteq E\,\} \;\hookrightarrow\; H^{s,\varphi}(\mathbb{R}^n). \tag{3.25}$$

We use the operator of "rectification" T and its left inverse K from the proof of Theorem 3.5. These operators are bounded and have the forms

$$T\colon H^{s,\varphi_1}(M) \;\to\; \prod_{j=1}^{r} \{u \in H^{s,\varphi_1}(\Pi_j)\colon\ \operatorname{supp} u \subseteq E_j\}$$

and (3.18), respectively. Here, $E_j = \alpha^{-1}(\operatorname{supp} \chi_j)$ is a compact set in $\mathbb{R}^n$. The compact imbedding (3.25) involves the compact imbedding operator

$$I\colon \prod_{j=1}^{r} \{u \in H^{s,\varphi_1}(\Pi_j)\colon\ \operatorname{supp} u \subseteq E_j\} \;\to\; \prod_{j=1}^{r} H^{s,\varphi}(\Pi_j).$$

Therefore, the imbedding operator (3.23) is compact because it is equal to KIT. This yields the compactness of imbedding (3.22) for any $\varphi, \varphi_1 \in \mathcal{M}$ because it is a composition of the compact and continuous imbeddings $H^{s+\varepsilon,\,\varphi_1}(M) \hookrightarrow H^{s+\varepsilon,\,\varphi_2}(M) \hookrightarrow H^{s,\,\varphi}(M)$, where the function $\varphi_2 \in \mathcal{M}$ is chosen, e.g., so that $\varphi_2(t) = \dfrac{\varphi_1(t)}{\ln\, t}$ for $t \gg 1$. In this case, imbeddings (3.21) are also compact as special cases of (3.22). Let us prove the compactness of the last imbedding (3.24). Assume that condition (3.5) is satisfied. As indicated in Remark 3.1, we can assume, without loss of generality, that the function $\varphi \in \mathcal{M}$ is continuous. Then, by virtue of Proposition 1.3(d), the function $\psi_1 = \varphi^2$ satisfies the conditions of Lemma 1.2. Let ψ_0 be the function from the formulation of this lemma. Then $\varphi_0 = \sqrt{\psi_0} \in \mathcal{M}$ satisfies the condition $\dfrac{\varphi_0(t)}{\varphi(t)} \to 0$ as $t \to +\infty$ and inequality (3.5) with φ_0 instead of φ. Hence, according to the results obtained above, we get

$$H^{\rho+n/2,\,\varphi}(M) \hookrightarrow H^{\rho+n/2,\,\varphi_0}(M) \hookrightarrow C^{\rho}(\,\overline{M}\,), \quad \rho \geq 0,$$

where the first imbedding is compact and the second is continuous. Thus, we have established the compactness of imbedding (3.24). Assertions (a) – (e) of the theorem are proved. Assertion (f) is deduced from Theorem 3.2(f) by analogy with the special case $\varphi \equiv 1$ of spaces of Bessel potentials.

The theorem is proved.

Note the important special case where M is an open set in $\mathbb{R}^n$. Then $H^{s,\varphi}(M)$ can be determined with the use of global coordinates in $\mathbb{R}^n$ by analogy with the space $H^{s,\varphi}(\mathbb{R}^n_+)$. Namely, the following theorem is true:

Theorem 3.7. *Suppose that a compact manifold $\overline{M}$ of the class C^∞ with nonempty boundary $\partial\overline{M}$ is such that $M = \overline{M}\setminus\partial\overline{M}$ is an open set in $\mathbb{R}^n$. Then $H^{s,\varphi}(M)$, $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$, consists of the restrictions of all distributions from $H^{s,\varphi}(\mathbb{R}^n)$ to M. Moreover, the norm of the distribution g in $H^{s,\varphi}(M)$ is equivalent to the norm*

$$\inf\Big\{\|f\|_{H^{s,\varphi}(\mathbb{R}^n)}\colon\ f \in H^{s,\varphi}(\mathbb{R}^n),\ \ f = g \ \text{ on } \ M\Big\}.$$

Proof. For $\varphi \equiv 1$, this theorem is well known (see, e.g., [5, pp. 273–275], Proposition 3.2.3). For arbitrary $\varphi \in \mathcal{M}$, it is proved by analogy. However, this theorem can easily be obtained from the case $\varphi \equiv 1$ by interpolation. Indeed, in this case, there exists the linear bounded operator $R_M\colon H^\sigma(\mathbb{R}^n) \to H^\sigma(M)$, $\sigma \in \mathbb{R}$, of the restriction of a distribution from $\mathbb{R}^n$ to M. It is known [9, p. 386] that, for any integer $k > 0$, the operator R_M has the linear bounded right inverse $T_{M,k}\colon H^\sigma(M) \to H^\sigma(\mathbb{R}^n)$, $|\sigma| < k$, which extends the distribution from M to $\mathbb{R}^n$. Now assume that $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$, and $\varepsilon > 0$. We take an integer k such that $|s \mp \varepsilon| < k$. Let ψ be the interpolation parameter defined in Theorems 3.1 and 3.5 for $\varepsilon = \delta$. Using these theorems for the spaces where the operators R_M and $T_{M,k}$ considered for $\sigma = s \mp \varepsilon$ act, we obtain the bounded operators $R_M\colon H^{s,\varphi}(\mathbb{R}^n) \to H^{s,\varphi}(M)$ and $T_{M,k}\colon H^{s,\varphi}(M) \to H^{s,\varphi}(\mathbb{R}^n)$. This immediately yields the statement of the theorem.

In the conclusion of this section, we prove a theorem on traces of distributions on the boundary of a manifold for the improved scale. Let $\overline{\Omega}$ be an infinite smooth compact manifold of dimension $n \geq 2$ with nonempty boundary Γ. Since Γ is a closed manifold of dimension $n-1$, the improved scales are defined over Γ as well as over $\Omega = \overline{\Omega}\setminus\Gamma$.

Theorem 3.8. *Consider the linear mapping*

$$f \;\to\; f\restriction\Gamma \quad (\textit{the trace of the function } f \textit{ on } \Gamma\,), \quad f \in C^\infty(\overline{\Omega}). \tag{3.26}$$

Then the following assertions are true:

(a) mapping (3.26) is extended by continuity to the bounded operator

$$R_\Gamma\colon H^{s+1/2,\,\varphi}(\Omega) \;\to\; H^{s,\varphi}(\Gamma), \quad s > 0, \quad \varphi \in \mathcal{M}, \tag{3.27}$$

which has the bounded right inverse

$$S_\Gamma\colon H^{s,\varphi}(\Gamma) \;\to\; H^{s+1/2,\varphi}(\Omega), \quad s>0, \quad \varphi\in\mathcal{M}, \tag{3.28}$$

such that S_Γ *is independent of* s *and* φ;

(b) *if* $\varphi\in\mathcal{M}$ *satisfies condition (3.5), then mapping (3.26) is extended by continuity to the bounded operator*

$$R_\Gamma\colon H^{1/2,\varphi}(\Omega) \;\to\; H^{0,\varphi_0}(\Gamma), \tag{3.29}$$

where the function $\varphi_0\in\mathcal{M}$ *is defined by the relation*

$$\varphi_0(\tau)=\left(\int\limits_{\tau}^{+\infty}\frac{d\,t}{t\,\varphi^2(t)}\right)^{-1/2}, \quad \tau\geq 1; \tag{3.30}$$

this operator has the bounded right inverse

$$S_{\Gamma,\varphi}\colon H^{0,\varphi_0}(\Gamma) \;\to\; H^{1/2,\varphi}(\Omega), \tag{3.31}$$

which depends on φ.

Proof. First, we establish assertion (a). We deduce it from an analogous theorem on traces for the spaces of Bessel potentials on $\mathbb{R}^n_+$. Consider a linear mapping $R_0^+\colon v(x',x_n)\mapsto v(x',0),\ v\in C_0^\infty(\overline{\mathbb{R}}^n_+)$, that associates a function $v(x',x_n)$ of variables $x'\in\mathbb{R}^{n-1}$ and $x_n\in\mathbb{R}$ with its trace $v(x',0)$ on the hyperplane $x_n=0$. It is known [9, p. 267] that this mapping can be extended by continuity to a bounded operator $R_0^+\colon H^{\sigma+1/2}(\mathbb{R}^n_+)\to H^\sigma(\mathbb{R}^{n-1}),\ \sigma>0$, that has the linear bounded right inverse $S_0^+\colon H^\sigma(\mathbb{R}^{n-1})\to H^{\sigma+1/2}(\mathbb{R}^n_+),\ \sigma>0$, independent of σ. Let $s>0$ and $\varphi\in\mathcal{M}$. Applying the interpolation theorems (Theorems 3.1 and 3.3) to the operators R_0^+ and S_0^+ for $\sigma=s\mp\varepsilon$, where $\varepsilon=\dfrac{s}{2}>0$, we obtain the bounded operators

$$R_0^+\colon H^{s+1/2,\varphi}(\mathbb{R}^n_+) \;\to\; H^{s,\varphi}(\mathbb{R}^{n-1}), \tag{3.32}$$

$$S_0^+\colon H^{s,\varphi}(\mathbb{R}^{n-1}) \;\to\; H^{s+1/2,\varphi}(\mathbb{R}^n_+). \tag{3.33}$$

One can easily "glue" these operators to obtain R_Γ and S_Γ by using the operator T and its left inverse K from the proof of Theorem 3.5. Indeed, we set $R_\Gamma f=KR_0^+Tf,\ f\in H^{s+1/2,\varphi}(\Omega)$, and $S_\Gamma g=KS_0^+Tg$, $g\in H^{s,\varphi}(\Gamma)$. Here, the operators R_0^+ and S_0^+ act on the vectors

$$Tf\in\prod_{j=1}^{r}H^{s+1/2,\varphi}(\Pi_j) \quad\text{and}\quad Tg\in\big(H^{s,\varphi}(\mathbb{R}^{n-1})\big)^r$$

componentwise, and, furthermore, if $\Gamma\cap\operatorname{supp}\chi_j=\varnothing$, then we assume that the value of R_0^+ on the jth component of the vector Tf is equal to zero and the jth component of the vector Tg is also equal to zero. The bounded operators (3.32), (3.33), (3.13), and (3.14) (the last two operators are considered for both $M=\Omega$ and $M=\Gamma$)

now yield the boundedness of operators (3.27) and (3.28). It is clear that (3.27) extends mapping (3.26), i.e., R_Γ is a trace operator on Γ. It remains to show that $R_\Gamma\, S_\Gamma = I$ is the identity operator. To this end, we use the equality

$$R_\Gamma\, Kh = KR_0^+\, h, \quad h \in \prod_{j=1}^{r} H^{s+1/2,\,\varphi}(\Pi_j).$$

This equality is obvious on vectors of the class C^∞. Then it is extended by continuity to the indicated vectors h. We have $R_\Gamma\, S_\Gamma = R_\Gamma\, KS_0^+\, T = KR_0^+\, S_0^+\, T = K\, T = I$. Assertion (a) is proved.

Let us prove assertion (b). We deduce it from the theorem on traces for Volevich–Paneyakh spaces [4, pp. 36–39] (Theorems 6.1 and 6.2). Let $\varphi \in \mathcal{M}$ satisfy (3.5). As noted in Remark 3.1, we can assume, without loss of generality, that the function φ is continuous on $[1; +\infty)$. Then $H^{1/2,\varphi}(\mathbb{R}^n)$ coincides with the Volevich–Paneyakh space $H^\mu = H^\mu(\mathbb{R}^n)$, where $\mu(\xi) = \langle\xi\rangle^{1/2}\varphi(\langle\xi\rangle)$ is the weight function of $\xi \in \mathbb{R}^n$. By virtue of the theorems indicated, the linear mapping $R_0\colon u(x', x^n) \mapsto u(x', 0)$, $u \in C_0^\infty(\mathbb{R}^n)$, can be extended by continuity to a bounded operator

$$R_0\colon\; H^\mu(\mathbb{R}^n) \;\to\; H^\nu(\mathbb{R}^{n-1}) \tag{3.34}$$

if and only if

$$\nu^{-2}(\xi') = \int_{-\infty}^{+\infty} \mu^{-2}(\xi', \xi_n)\, d\,\xi_n \; < +\infty, \quad \xi' \in \mathbb{R}^{n-1}. \tag{3.35}$$

Here, $H^\nu(\mathbb{R}^{n-1})$ is the Volevich–Paneyakh space over $\mathbb{R}^{n-1}$. Moreover, if the last condition is satisfied, then operator (3.34) has the linear bounded right inverse

$$S_{0,\,\varphi}\colon\; H^\nu(\mathbb{R}^{n-1}) \;\to\; H^\mu(\mathbb{R}^n), \tag{3.36}$$

and, furthermore, $S_{0,\,\varphi}$ depends on μ, i.e., on φ. We now pass from Volevich–Paneyakh spaces to the corresponding spaces of improved scales. We have

$$\nu^{-2}(\xi'\,) = \int_{-\infty}^{+\infty} \mu^{-2}(\xi', \xi_n)\, d\,\xi_n = 2\int_{0}^{+\infty} \frac{d\,\xi_n}{\langle\xi\rangle\, \varphi^2(\langle\xi\rangle)} = 2\int_{\langle\xi'\rangle}^{+\infty} \frac{d\,t}{(t^2 - \langle\xi'\rangle^2)^{1/2}\, \varphi^2(t)}$$

for any $\xi' \in \mathbb{R}^{n-1}$ $\big($the last equality is obtained by the change of variables $t = \langle\xi\rangle = (\langle\xi'\rangle^2 + \xi_n^2)^{1/2}\big)$. Using this result and Lemma 1.3 for $\psi_1 = \varphi^2$ and $\tau = \langle\xi'\rangle$, we obtain the inequality $\varphi_0^{-2}(\tau) \le (1/2)\,\nu^{-2}(\xi'\,) \le c\,\varphi_0^{-2}(\tau)$, $\xi' \in \mathbb{R}^{n-1}$, $\tau = \langle\xi'\rangle$, where the function φ_0 is defined by (3.30). Therefore, conditions (3.5) and (3.35) are equivalent in $H^\nu(\mathbb{R}^{n-1}) = H^{0,\varphi_0}(\mathbb{R}^{n-1})$ up to equivalence of norms. Note that here $\varphi_0 \in \mathcal{M}$, which follows from the inclusion $\varphi_0^{-2} \in \mathrm{SV}$ established in the proof of Lemma 1.2. Thus, operators (3.34) and (3.36) exist and, furthermore, we have $R_0\colon H^{1/2,\,\varphi}(\mathbb{R}^n) \to H^{0,\varphi_0}(\mathbb{R}^{n-1})$ and $S_{0,\varphi}\colon H^{0,\varphi_0}(\mathbb{R}^{n-1}) \to H^{1/2,\,\varphi}(\mathbb{R}^n)$. We pass from these operators to analogous operators for $\mathbb{R}^n_+$. To this end, we need the operator R_+ of restriction of a distribution from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ and the operator T_{n+1} right inverse to R_+ from the proof of Theorem 3.3. These operators are linear and bounded in the following pairs of spaces:

$$\begin{gathered} R_+\colon H^{1/2,\,\varphi}(\mathbb{R}^n) \;\to\; H^{1/2,\,\varphi}(\mathbb{R}^n_+), \\ T_{n+1}\colon H^\sigma(\mathbb{R}^n_+) \;\to\; H^\sigma(\mathbb{R}^n), \quad |\sigma| < n+1. \end{gathered} \tag{3.37}$$

By virtue of the interpolation theorems (Theorems 3.1 and 3.3), the boundedness of (3.37) yields the boundedness of the operator $T_{n+1}\colon H^{1/2,\varphi}(\mathbb{R}^n_+) \to H^{1/2,\varphi}(\mathbb{R}^n)$, which, in turn, yields the boundedness of the operators

$$R_0^+ = R_0 T_{n+1}\colon H^{1/2,\varphi}(\mathbb{R}^n_+) \;\to\; H^{0,\varphi_0}(\mathbb{R}^{n-1}), \tag{3.38}$$

$$S_{0,\varphi}^+ = R_+ S_{0,\varphi}\colon H^{0,\varphi_0}(\mathbb{R}^{n-1}) \;\to\; H^{1/2,\varphi}(\mathbb{R}^n_+). \tag{3.39}$$

Moreover, R_0^+ associates the function $v(x', x_n)$ of the class $C_0^\infty(\overline{\mathbb{R}}^n_+)$ with its trace $v(x', 0)$ on the hyperplane $x_n = 0$. Indeed, by virtue of the Sobolev imbedding theorem, operator (3.37) involves $T_{n+1}\, v \in H^n(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$. Hence, $R_0^+ v = R_0 T_{n+1} v = v(x', 0)$. Furthermore, the operator $S_{0,\varphi}^+$ is the right inverse of R_0^+. Indeed, for any $u \in C_0^\infty(\mathbb{R}^n)$, we have $T_{n+1} R_+ u \in H^n(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$. Therefore, the function $R_0 T_{n+1} R_+ u$ is calculated pointwise and is equal to $R_0 u$. Passing to the limit, we obtain the equality $R_0 T_{n+1} R_+ u = R_0 u$, $u \in H^{1/2,\varphi}(\mathbb{R}^n)$. Setting $u = S_{0,\varphi}\,\omega$, where $\omega \in H^{0,\varphi_0}(\mathbb{R}^{n-1})$, we write $R_0^+\, S_0^+\, \omega = R_0 T_{n+1} R_+\, S_{0,\varphi}\,\omega = R_0\, S_{0,\varphi}\,\omega = \omega$. Thus, we have the trace operator (3.38) and the operator of extension (3.39) right inverse to it. As in the proof of assertion (a), this implies that $R_\Gamma = K R_0^+\, T$ and $S_{\Gamma,\varphi} = K S_{0,\varphi}^+\, T$ are the required operators (3.29) and (3.31). Assertion (b) and, hence, Theorem 3.8 are proved.

Remark 3.4. By virtue of Theorem 6.1 in [4, p. 36], condition (3.5) is necessary and sufficient for mapping (3.26) to be extended to a continuous trace operator $R_0 : H^{1/2,\varphi}(\Omega) \to \mathcal{D}'(\Gamma)$.

In the conclusion of this section, we give the description of some spaces of the improved scale over Γ that follows from Theorem 3.8.

Corollary 3.2. *The following assertions are true:*

(i) for any $s > 0$ and $\varphi \in \mathcal{M}$, one has $H^{s,\varphi}(\Gamma) = \{R_\Gamma f\colon f \in H^{s+1/2,\varphi}(\Omega)\}$, and, furthermore, the norm of the distribution h in $H^{s,\varphi}(\Gamma)$ is equivalent to the norm

$$\inf\Big\{\|f\|_{H^{s+1/2,\,\varphi}(\Omega)}\colon R_\Gamma f = h\Big\}; \tag{3.40}$$

(ii) if $\varphi \in \mathcal{M}$ satisfies condition (3.5), then, for the function $\varphi_0 \in \mathcal{M}$ defined by relation (3.30), one has $H^{0,\varphi_0}(\Gamma) = \big\{R_\Gamma f\colon f \in H^{1/2,\varphi}(\Omega)\big\}$, and, furthermore, the norm of the distribution h in $H^{0,\varphi_0}(\Gamma)$ is equivalent to norm (3.40), where $s = 0$.

This description of some ("positive") spaces of the improved scale over Γ as trace spaces is especially important in the case where Ω is an open set in $\mathbb{R}^n$. In this case, by virtue of Theorem 3.7, such spaces admit the definition with the use of global coordinates in $\mathbb{R}^n$ according to Corollary 3.2. Moreover, in the last statement, Ω can be replaced by $\mathbb{R}^n$.

4. Elliptic Boundary-Value Problem in the Improved Scale of Spaces

As above, assume that $\overline{\Omega}$ is an infinite smooth compact manifold of dimension $n \geq 2$ with nonempty boundary Γ. We set $\Omega = \overline{\Omega} \setminus \Gamma$. According to this assumption, Γ is an infinite smooth closed manifold of dimension $n - 1$. We fix an arbitrary pair $\mathcal{A}$ that consists of a finite atlas from the C^∞-structure on $\overline{\Omega}$ and the C^∞-partition of unity on $\overline{\Omega}$ subordinate to it. Let $\mathcal{A}_\Gamma$ be the pair formed by the restrictions of this atlas and the partition of unity to Γ. On Ω and Γ, we consider the improved scales $\{H^{s,\varphi}(\Omega)\colon s \in \mathbb{R}, \varphi \in \mathcal{M}\}$

and $\{H^{s,\varphi}(\Gamma)\colon s\in\mathbb{R},\varphi\in\mathcal{M}\}$ constructed using the pairs $\mathcal{A}$ and $\mathcal{A}_\Gamma$, respectively. If $\varphi\equiv 1$, then $H^{s,\varphi}(\Omega)=H^{s}(\Omega)$ and $H^{s,\varphi}(\Gamma)=H^{s}(\Gamma)$ are the spaces of Bessel potentials on Ω and Γ. Note that $H^{0}(\Omega)=L_2(\Omega)$ and $H^{0}(\Gamma)=L_2(\Gamma)$ are the Hilbert spaces of functions whose squares are summable on Ω and Γ with respect to the C^∞-densities defined by the pairs $\mathcal{A}$ and $\mathcal{A}_\Gamma$. Let $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ denote the scalar products in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively.

Consider the following boundary-value problem on Ω:

$$Lu=f \quad \text{on} \quad \Omega, \qquad B_j u=g_j \quad \text{on} \quad \Gamma, \quad j=1,\dots,k. \tag{4.1}$$

Here, L is a linear differential operator on $\overline{\Omega}$ with infinitely smooth coefficients, the order of the operator L is even and equal to $2k\geq 2$, B_j, $j=1,\dots,k$, are boundary linear differential operators on Γ with infinitely smooth coefficients, and the order of the operator B_j is equal to $m_j<2k$. We set $m=\max\{m_1,\dots,m_k\}$.

In what follows, we assume that *problem (4.1) is elliptic*. This means that (see, e.g., [12, pp. 6, 7]) the operator L is elliptic on $\overline{\Omega}$ and regularly elliptic on Γ and the system $\{B_1,\dots,B_k\}$ satisfies the Shapiro–Lopatinskii condition with respect to L on Γ.

For elliptic boundary-value problems, solvability theorems and estimates for solutions in various classes of functional spaces are known (see [2, 4, 8–11, 13–15] and the survey [12]). We need the statement on the solvability of the elliptic boundary-value problem (4.1) in the spaces of Bessel potentials presented below (see [14, pp. 128–130]). First, recall that a linear bounded operator $T\colon X\to Y$, where X and Y are Banach spaces, is called Noetherian if its kernel and cokernel (i.e., the kernel of the adjoint operator) are finite-dimensional and the range of values of the operator T is closed in Y.

Proposition 4.1. *Let $\sigma>m+\dfrac{1}{2}$. Then the linear mapping*

$$u\mapsto \Lambda\,u=(L\,u,B_1\,u,\dots,B_k\,u),\quad u\in C^\infty(\,\overline{\Omega}\,), \tag{4.2}$$

can be extended by continuity to the bounded Noetherian operator

$$\Lambda\colon H^{\sigma}(\Omega)\to\mathcal{H}_\sigma=H^{\sigma-2k}(\Omega)\times\prod_{j=1}^{k}H^{\sigma-m_j-1/2}(\Gamma). \tag{4.3}$$

Moreover, the kernel N and the cokernel N_* of this operator are independent of σ and consist of infinitely smooth elements:

$$N\subset C^\infty(\,\overline{\Omega}\,),\qquad N_*\subset C^\infty(\,\overline{\Omega}\,)\times(\,C^\infty(\Gamma)\,)^k. \tag{4.4}$$

Remark 4.1. Let us explain the last inclusion. It means that the functionals from N_* [the kernels of the operator adjoint to (4.3)] have the form $(\cdot,w_0)_\Omega+(\cdot,w_1)_\Gamma+\ldots+(\cdot,w_k)_\Gamma$ for certain functions $w_0\in C^\infty(\,\overline{\Omega}\,)$, $w_1,\dots,w_k\in C^\infty(\Gamma)$. Therefore, the range of values of operator (4.3) consists of all vectors $(f,g_1,\dots,g_k)\in\mathcal{H}_\sigma$ such that $(f,w_0)_\Omega+(g_1,w_1)_\Gamma+\ldots+(g_k,w_k)_\Gamma=0$ for any $(w_0,w_1,\dots,w_k)\in N_*$. Here, one should take the following into account: Since $\sigma>m+\dfrac{1}{2}$, we have $g_j\in L_2(\Gamma)$, and the scalar product $(g_j,w_j)_\Gamma$ is defined for $j=1,\dots,k$. Further, if $\sigma\geq 2k$, then $f\in L_2(\Omega)$, and $(f,w_0)_\Omega$ is also defined. It remains to consider the case where $m+\dfrac{1}{2}<\sigma<2k$. In this case, generally speaking, one has $f\notin L_2(\Omega)$, but the form $(\cdot,w_0)_\Omega$ is extended by continuity to $H^{\sigma-2k}(\Omega)$. Therefore, in this case, $(f,w_0)_\Omega$ denotes the extension of the scalar product in $L_2(\Omega)$ by continuity.

Note that the trace theorem for the spaces of Bessel potentials is extensively used in Proposition 4.1. For example, if $\sigma = m + 1/2,$ then the trace operator B_j is not defined on $H^{\sigma}(\Omega)$ for all j such that $m_j = m$ (see Theorem 3.8 and Remark 3.4 in the case where $\varphi \equiv 1$).

In what follows, N and N_* denote the kernel and the cokernel of the operator Λ in Proposition 4.1. Since N and N_* are finite-dimensional and infinitely smooth, in improved scales there exist projectors onto subspaces orthogonal to N and $N_*,$ respectively, with respect to the scalar products in $L_2(\Omega)$ and $L_2(\Omega) \times (L_2(\Gamma))^k$. Namely, the following two lemmas are true:

Lemma 4.1. *Suppose that $s > 0$ and $\varphi \in \mathcal{M}$. Then, for any $u \in H^{s,\varphi}(\Omega),$ there exists a unique element $u_0 \in N$ such that $(u - u_0, v)_\Omega = 0$ for any $v \in N$. Moreover, the mapping $P\colon u \mapsto u_1 = u - u_0$ is the linear bounded projection operator of the space $H^{s,\varphi}(\Omega)$ onto its closed subspace*

$$\{u_1 \in H^{s,\varphi}(\Omega)\colon (u_1, v)_\Omega = 0 \quad \textit{for any} \;\; v \in N\}, \tag{4.5}$$

and, furthermore, Pu is independent of s and φ.

Proof. First, note that, since $H^{s,\varphi}(\Omega) \hookrightarrow L_2(\Omega)$ (the condition $s > 0$) and $N \subset C^\infty(\overline{\Omega})$ (Proposition 4.1), the scalar product $(u, v)_\Omega$ is defined for any $u \in H^{s,\varphi}(\Omega)$ and $v \in N$. Therefore, we can identify an element $v \in N$ with a linear functional $(\cdot, v)_\Omega$ on $H^{s,\varphi}(\Omega)$. This functional is bounded:

$$|\,(u,v)_\Omega\,| \le \|\,u\,\|_{L_2(\Omega)} \|\,v\,\|_{L_2(\Omega)} \le \text{const}\,\|\,u\,\|_{H^{s,\varphi}(\Omega)} \|\,v\,\|_{L_2(\Omega)}, \quad u \in H^{s,\varphi}(\Omega).$$

This implies that subspace (4.5) is closed in $H^{s,\varphi}(\Omega)$. Further, according to Proposition 4.1, N is a finite-dimensional subspace in $H^{s,\varphi}(\Omega)$. It is clear that $\dim N$ coincides with the codimension of subspace (4.5), and, furthermore, N and (4.5) have the trivial intersection. Therefore, $H^{s,\varphi}(\Omega)$ decomposes into the direct sum of the closed subspaces N and (4.5) with the bounded projector P onto (4.5), which is obviously independent of s and φ.

Lemma 4.2. *Suppose that $s > m + \dfrac{1}{2}$ and $\varphi \in \mathcal{M}$. We set*

$$\mathcal{H}_{s,\varphi} = H^{s-2k,\,\varphi}(\Omega) \times \prod_{j=1}^{k} H^{s-m_j-1/2,\,\varphi}(\Gamma). \tag{4.6}$$

Let $(\cdot,\cdot)_{\Omega,\Gamma}$ denote the scalar product in $L_2(\Omega) \times (\,L_2(\Gamma)\,)^k$ and its extension by continuity. Then, for any $F \in \mathcal{H}_{s,\varphi}$, there exists a unique vector $F_0 \in N_*$ such that $(F - F_0, W)_{\Omega,\Gamma} = 0$ for any $W \in N_*$. Moreover, the mapping $Q\colon F \mapsto F_1 = F - F_0$ is the linear bounded projection operator of space (4.6) onto its closed subspace

$$\big\{F_1 = (f, g_1, \ldots, g_k) \in \mathcal{H}_{s,\varphi} : (F_1, W)_{\Omega,\Gamma} \equiv (f, w_0)_\Omega + (g_1, w_1)_\Gamma + \ldots + (g_k, w_k)_\Gamma = 0$$
$$\textit{for any} \quad W = (w_0, w_1, \ldots, w_k) \in N_*\big\}, \tag{4.7}$$

and, furthermore, QF is independent of s and φ.

Proof. Let $W \in N_*$. By virtue of Proposition 4.1 (see also Remark 4.1), the form $(\cdot, W)_{\Omega,\Gamma}$ defines a linear bounded functional on the space $\mathcal{H}_\sigma$ for any $\sigma > m + \frac{1}{2}$. By virtue of Theorem 3.5 for $\varepsilon = \delta > 0$ and Proposition 2.1, we have $\mathcal{H}_{s,\varphi} = [\mathcal{H}_{s-\varepsilon}, \mathcal{H}_{s+\varepsilon}]_\psi$ with equivalence of norms. This implies that $(\cdot, W)_{\Omega,\Gamma}$ is a linear bounded functional on $\mathcal{H}_{s,\varphi}$. Therefore, subspace (4.7) is closed in $\mathcal{H}_{s,\varphi}$. Further, we proceed by analogy with the proof of the last lemma. Namely, according to Proposition 4.1, N_* is a finite-dimensional subspace in $\mathcal{H}_{s,\varphi}$. Moreover, $\dim N_*$ is equal to the codimension of subspace (4.7), and, furthermore, N_* and (4.7) have the trivial intersection. Therefore, $\mathcal{H}_{s,\varphi}$ decomposes into the direct sum of the closed subspaces N_* and (4.7) with the bounded projector Q onto (4.7), which is independent of s and φ, which was to be proved.

We now establish the main result of this section, namely, a theorem on properties of the operator of the elliptic boundary-value problem (4.1) in the improved scale of spaces.

Theorem 4.1. *Suppose that* $s > m + \frac{1}{2}$ *and* $\varphi \in \mathcal{M}$. *Then mapping (4.2) can be extended by continuity to the bounded Noetherian operator*

$$\Lambda \colon H^{s,\varphi}(\Omega) \to \mathcal{H}_{s,\varphi} = H^{s-2k,\varphi}(\Omega) \times \prod_{j=1}^{k} H^{s-m_j-1/2,\varphi}(\Gamma) \tag{4.8}$$

with kernel N *and cokernel* N_*, *which, by virtue of Proposition 4.1, are independent of* s *and* φ *and satisfy (4.4). The restriction of operator (4.8) to subspace (4.5) is performed by the topological isomorphism*

$$\Lambda \colon P(\,H^{s,\varphi}(\Omega)\,) \leftrightarrow Q(\,\mathcal{H}_{s,\varphi}\,) \tag{4.9}$$

between spaces (4.5) and (4.7). Furthermore, the following estimate is true:

$$\|\,u\,\|_{H^{s,\varphi}(\Omega)} \le c\left(\|\,\Lambda\,u\,\|_{\mathcal{H}_{s,\varphi}} + \|\,u\,\|_{L_2(\Omega)}\right), \qquad u \in H^{s,\varphi}(\Omega), \tag{4.10}$$

where the constant c *is independent of* u.

We see that the index φ, which improves the main s-smoothness of the space, remains invariant under the action of the operator (4.8) of the elliptic boundary-value problem. Moreover, the properties of the operator are analogous to the special case $\varphi \equiv 1$ of the spaces of Bessel potentials.

We deduce Theorem 4.1 from Proposition 4.1 using interpolation with functional parameter. We use Geymonat's result [16, pp. 280, 281] (Proposition 5.2) on the interpolation of operators with finite index. Below, we formulate this result as applied to the case considered.

Proposition 4.2. *Let two admissible pairs* $[X_0, X_1]$ *and* $[Y_0, Y_1]$ *of Hilbert spaces be given. Suppose that, on* X_0, *a linear mapping* T *is given for which bounded Noetherian operators* $T \colon X_j \to Y_j$, $j = 0, 1$, *with common kernel* $\mathcal{N}$ *and common cokernel* $\mathcal{N}_*$ *are valid. Then, for an arbitrary interpolation parameter* ψ, *the bounded operator* $T \colon [X_0, X_1]_\psi \to [Y_0, Y_1]_\psi$ *is a Noetherian operator with kernel* $\mathcal{N}$ *and cokernel* $\mathcal{N}_*$.

Proof of Theorem 4.1. We take a number $\varepsilon > 0$ such that $s - \varepsilon > m + \frac{1}{2}$. According to Proposition 4.1, the Noetherian operators (4.3) for $\sigma = s \mp \varepsilon$ with common kernel N and common cokernel N_* are valid. We apply the interpolation with parameter ψ from Theorem 3.5, in which we take $\varepsilon = \delta$ and $M = \Omega$ and then $M = \Gamma$, to these operators. By virtue of Proposition 4.2, we obtain a bounded Noetherian operator with kernel N and cokernel N_*, which coincides with (4.8) by virtue of Theorem 3.5 and Proposition 2.1. [Since $C^\infty(\,\overline{\Omega}\,)$

is dense in $H^{s,\varphi}(\Omega)$, this operator is the extension of mapping (4.2) by continuity.] By virtue of Lemmas 4.1 and 4.2, we directly obtain the algebraic isomorphism (4.9). Since operator (4.9) is bounded, this isomorphism is topological by virtue of the Banach theorem on the inverse operator. It remains to prove estimate (4.10). Using Lemma 4.1, we represent a distribution $u \in H^{s,\varphi}(\Omega)$ in the form $u = u_0 + u_1$, where $u_0 = (1-P)\,u \in N$ and $u_1 = P\,u \in P(H^{s,\varphi}(\Omega))$. By virtue of (4.9), we get

$$\|\, u_1 \,\|_{H^{s,\,\varphi}(\Omega)} \le c_1 \|\, \Lambda\, u_1 \,\|_{\mathcal{H}_{s,\,\varphi}} = c_1 \|\, \Lambda\, u \,\|_{\mathcal{H}_{s,\,\varphi}}.$$

Furthermore, since N is finite-dimensional and $1 - P$ is the orthoprojector onto N in $L_2(\Omega)$, we have

$$\|\, u_0 \,\|_{H^{s,\,\varphi}(\Omega)} \le c_0 \,\|\, u_0 \,\|_{L_2(\Omega)} \le c_0 \,\|\, u \,\|_{L_2(\Omega)},$$

where the constants c_0 and c_1 are independent of u. Summing up these inequalities, we get (4.10).

Theorem 4.1 is proved.

Remark 4.2. In connection with the last theorem, we again note Shlenzak's work [7], where, with the use of interpolation with functional parameter, a theorem on isomorphism was proved for an operator of a *regular* elliptic boundary-value problem that acts in certain Hörmander–Volevich–Paneyakh spaces defined in an infinitely smooth domain. These spaces differ from the spaces considered in the present paper (see Remark 3.2).

Remark 4.3. As Theorem 4.1 is compared with assertion (b) of Theorem 3.8, the following question arises: Is it possible to generalize Theorem 4.1 to the *limit case* $s = m + \dfrac{1}{2}$ for $\varphi \in \mathcal{M}$ satisfying condition (3.5)? In this case, in (4.8), for all j such that $m_j = m$, we use the space $H^{s-m_j-1/2,\,\varphi_0}(\Gamma)$ instead of $H^{s-m_j-1/2,\,\varphi}(\Gamma)$, where φ_0 is defined by (3.30). The answer to this question is negative.

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