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ORIGINAL RESEARCH

Split up, but stay together: Collaboration and cooperation in mathematical problem solving

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Abstract
Conditions under which group work leads to learning have been studied in collaborative settings. Little is known, however, about whether and how the interplay between collaboration and cooperation impinges on group learning. In this paper, we study this interplay in the context of mathematical problem-solving. We focus on how training students to learn together influences this interplay, and on the relations of this interplay with mathematical problem-solving. Five groups of Grade 8 students participated in a course aimed at fostering learning to solve mathematical problems in small groups. Before and after the course, they solved a mathematical problem. An increase in the ratio of cooperation episodes out of total group work time was observed, as well as advancements in mathematical problem-solving. In addition, we found a mid-high correlation between instances of cooperation and mathematical activity: up to a certain threshold, cooperating more in a group yielded an increase in the individual generation of mathematical claims and arguments. We identified the critical role of coordination: for group learning to be productive, students should continuously negotiate and adjust their goals through communication before or while they cooperate on different tasks. We conclude that teachers aiming at fostering group work should encourage the diversification of modes of group work, for the advancement of mathematical problem-solving or of any case in which individual settings are too challenging.

Keywords Group learning · Collaboration · Cooperation · Learning to learn together · Mathematical problem-solving

Introduction

Group work is often beneficial for learning (for a meta-analysis, see for example, Lou et al. 1996). Researchers identified characteristics of productive group work, such as positive interdependence, individual accountability and promotive interaction (Johnson and Johnson 1989, 2002). Others focused on design, and studied classroom formations, how they promote group learning, as well as on the role of attitudes among teachers or students

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towards such change in classroom formations (Sharan and Shachar 2012). However, in the early years of research on group learning, researchers hardly asked *why* group work can be (in-) effective. In the 1990s, scientists in psychology, cognitive development, and computer sciences began tackling this issue. Roschelle and Teasley (1995) discerned between collaboration and cooperation, as two modes of group work: Collaboration refers to cases in which students work on the same task, join their attention and coordinate actions. Cooperation refers to cases in which students work on the same task but do not share attention towards each other's work. While this distinction may seem important to research on group learning, studies on the interplay between collaboration and cooperation in relation to learning are rare.

The current paper attempts to fill this research gap using the domain of mathematical problem-solving as a context for group work when students are free to choose the mode that fits their needs. We look into two aspects of this interplay: how training students to learn together influences this interplay, and the relations between this interplay and how groups solve problems in mathematics.

Theoretical background

Group learning and group work

Group learning is a learning situation in which two or more people learn or attempt to learn something together (Dillenbourg 1999). Some behaviors are productive for group learning, such as explaining one's ideas (Webb et al. 2014), supporting claims with data to form full arguments (Schwarz et al. 2010; Vogel et al. 2016), making active attempts to incorporate ideas proposed by peers (Barron 2003), taking the perspective of the other into account (Wegerif 2006, 2015), maintaining coordination between the actions of group members (Oner 2013), regulating the group's learning process (Iiskala et al. 2011; Smith and Mancy 2018) and distributing the responsibility for learning among group members (Kontorovich et al. 2012). Other behaviors are not productive, such as not balancing workload among group members or disacknowledging soft voices (Kontorovich et al. 2012) and engaging in disputations or over-agreement (Dawes et al. 2000; Wegerif and Mercer 1997). Barron (2003) showed that students in successful groups tended to accept or discuss correct proposals while students in unsuccessful groups tended to reject or ignore correct proposals. As a result, students who participated in successful interactions manifested greater improvement in mastery and transfer problems. Poor communication, on the other hand, inhibited learning by individuals—for some students, individual learning resulted in better scores than when they worked in groups in which communication was poor or not productive (Barron 2003). The issue of the productivity of group work is then complex. We decided to handle this complexity through the lenses of the relations between two general modes of group work—collaboration and cooperation.

Collaboration and cooperation

Collaboration and cooperation are two modes of group work. For both, students share a mutual learning goal. Collaboration happens in episodes during which people share the same goal and communicate through words, gestures or the use of shared artifacts. Cooperation refers to episodes in which group members learn as individuals while keeping mutual learning goals with each other, but do not communicate (Dillenbourg 1999; Roschelle and Teasley 1995). In a single session of group learning, people may alternate several times between collaboration and cooperation. Despite this theoretical distinction, in most of the research on group learning, collaboration became synonymous with group learning, while cooperation as a mode of group learning was neglected. This negligence, however, is unjustified.

Individual thinking within a social context such as group learning or whole-class discussions is, of course, not a new idea. For example, in whole-class discussions teachers may give students some time to think individually for "more than 3 sec" on their questions before letting students express their answers in front of the class, and wait a little bit more after these answers are posed (i.e., wait time, Rowe 1986). Another example of individual thinking within a social context is *productive failure*, which happens when instructors ask students to solve a problem before they offer formal instruction (Kapur 2014). Individual thinking in social contexts could happen when students learn in small groups. Teachers may ask students to think individually and then share their thinking with peers (i.e., *think-pair-share*, Kaddoura 2013), or to learn according to a Jigsaw setting, meaning learning individually before being grouped with other students to learn together (Hernández-Leo et al. 2005; Johnson and Johnson 1989, 2002). In all cases, empirical data suggests that the number and quality of ideas expressed by students increase when given time to think alone within a social context. These two phenomena invite us to consider the interplay between individual and group activity in group learning.

The interplay between collaboration and cooperation in group learning

Hermann et al. (2001), Rummel and Spada (2005) undertook studies that provide some insights on the issue of modes of group learning, in the context of remote problem-solving between experts in different domains. Hermann et al. (2001) showed that when remotely located, dyads that solved a problem using asynchronous and synchronous technology such as emails and telephones, outperformed dyads that used synchronous videoconferencing and shared text-editors. Further analyses revealed that dyads that used videoconferencing and shared text-editors tended to maintain collaboration all through the solution process, while dyads that used emails and telephones blended collaboration and cooperation. In another study, Rummel and Spada (2005) integrated cooperation to group learning scripts in remote problem-solving, for dyads from different domains—psychology and medicine. They showed that successful dyads were mostly ones that spent over-the-average time in cooperation mode, implying the importance of individual sense making within the context of group learning.

Studies focusing on brainstorming during group interaction suggest that cooperation may enhance group learning. According to Wit (2006), individuals that collaborate tend to conform to the group's line of thought and often fear to raise criticism. Similarly, Barron (2003) found that when individuals "speak softly" when learning in groups good ideas might not be heard or developed. According to Wit (2006), individual sense-making enhances groups of professionals' ability to solve problems. The joint solution in these cases is enhanced due to individual efforts to open up new paths out of the conformity of the group. Individuals may sometimes produce more and/or better ideas than when they learn within a group (Wit 2006). Notably, the ideas brought by Wit (2006) were essentially theoretical, and do not rely on systematic empirical grounds.

Some CSCL scientists approached cooperation as well by stating the necessity to create personal interfaces-spaces, together with joint problem spaces, in technologies for remote learning (Baghaei et al. 2007; Hammond 2017; Stahl 2009). Stahl (2009) explained that along with the role of this individual space in the creation of an intimate problem space for the self-generation of ideas, such a personal space also affords individual meaning-making external to the group learning. This process is iterative: when an individual makes meaning out of the joint group solution, she or he is likely to be able to generate an additional idea while cooperation takes place; when the group returns to collaboration, this idea could be presented in the joint problem-space, discussed, evaluated and later on understood by the individuals in the group, and so on. Here also, like for the claims by Wit (2006), the statements of these scientists are working hypotheses that lack empirical support (but see Schneider 2019 for some evidence using eye-tracking equipment).

Learning to learn together and mathematical problem-solving

The positive results reported in studies such as the one by Rummel and Spada (2005) point to the context of professionals, who must learn to be part of a team, to join forces with people who are different and have different expertise. These different types of expertise together help in solving the task. Learning to work in groups in this context is crucial. However, we suggest that schools—that tend to favor either equity amongst students or competition (to receive the best grades) provide another context for learning modes of group-work.

The first effort in this direction was the R&D European-Commission funded Metafora project (*Learning to Learn Together: A visual language for social orchestration of educational activities*). The Metafora project aimed at fostering *Learning to Learn Together* (L2L2) skills. The L2L2 approach extends *Learning to Learn* (L2L)—a set of skills and meta-strategies that help learners become competent when facing challenging learning situations (e.g., Claxton 2004; Fredriksson and Hoskins 2007; Higgins et. al. 2006). L2L2 conveys a twenty-first century skill in learning to become productive group members, at work, school or any other organization, as well as to learning to become a citizen. The Metafora project included the design of a platform for supporting L2L2 in the context of solving STEM problems. The implementation of the platform in an educational program was shown to be productive (Abdu et al. 2015; Schwarz et al. 2015; Wegerif 2015).

In the current study, we focus on the role of cooperation in group learning. We hypothesize that learning to intertwine collaboration and cooperation may be productive for group learning and as such is an important L2L2 behavior. The context we chose for learning this skill is mathematical problem-solving.

A mathematical problem is a task for which a solver does not know what are the means she or he needs to apply or develop, to reach a solution (Schoenfeld 1985). A priori, mathematical problem-solving was conceived and studied as a solipsistic activity. For Pólya (1945), problem-solving is an iterative process done by an individual who uses strategies and heuristics to cope with the unknown. To do so, the solver first understands/identifies the problem, she or he then plans a solution strategy, implements that plan, and follows in looking back at the solution and make sure it answers the problem. Mathematical problems are designed to be challenging for individual solvers (e.g., Lester 1994; Lester and Cai 2016; Liljedahl and Santos-Trigo 2019; Kirschner et al. 2006; Schoenfeld 2007). Solving problems often requires being in a state of uncertainty, for which remaining alone may leave the solver distraught (Schoenfeld 2007). Solving mathematical problems in a group may then become a necessity; it may lessen the burden on the shoulders of the individual (Kontorovich et al. 2012). Therefore, mathematical problem-solving provides a context in which modes of group work (collaboration or cooperation) can be choices made by solvers rather than structures imposed upon them. In other words, mathematical problem-solving can be a suitable context for learning to learn together—for learning to choose appropriate modes of group work.

Research questions

In the present study, we observe patterns of group work of students participating in a course on L2L2 and mathematical problem-solving (i.e., learning to solve mathematical problems together). Before and after the course students were asked to work in groups and solve a mathematical problem, while the organization of group work was not imposed or scripted. Rather, the members of the groups could decide at any moment to work in collaboration or cooperation, according to their needs. Our research questions are: (1) Can a course focusing on L2L2 change students' patterns of group work in solving mathematical problems? (2) Which pattern of group work is more beneficial for mathematical problem-solving?

Methodology

The study took place in the context of a course called *Learning to solve mathematical problems together with technology* (described below; see also, Abdu et al. 2015). Before and after that course, students solved a mathematical problem, and we analyzed the solution processes to answer our research questions.

Participants

Fifteen 14–15 years-old students from a school in a large city in Israel, were enrolled in an enrichment course on learning to solve mathematical problems together. Participation in the course was voluntary. All students participated in a pre-test and thirteen of them participated in the post-test (two students, from groups 3 and 4, did not show up for the post-test). The teacher (the first author) had 2 years of experience as a mathematics instructor.

Setting: A course for fostering L2L2 in the context of mathematical problem-solving

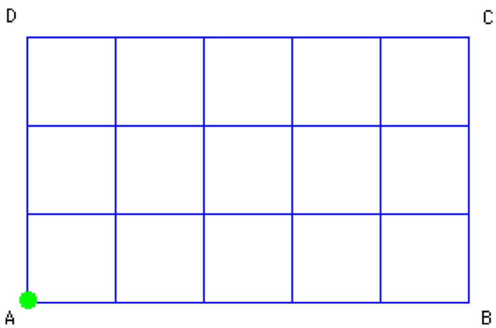
The course lasted seven months and included 19 meetings, 90 minutes long each. The meetings took place at the students' school, in computer rooms, with a computer for every student. The course was aimed at solving mathematical problems that require the use of certain mathematical problem-solving strategies. In each lesson, students solved mathematical problems that encouraged the enactment of specific mathematical problem-solving strategies and specific group learning behaviors with increased difficulty. The teacher helped the students in identifying these strategies and using them in further activities.

To facilitate L2L2, in every lesson the teacher held explicit discussions about one group learning behavior—such as making sure that all students had the chance to speak their mind or prompting students to ask each other questions in cases where it is clear that they were not on the same page—in several points in the lesson (see Abdu et al. 2015). At the beginning of the lesson, after an initial presentation of the problem, he asked the students to pay attention to the emergence of this behavior in their group learning process. While the students solved problems within groups, the teacher walked between them and prompted for productive group learning behaviors. Each lesson ended in a whole class reflective session during which the teacher prompted students to share descriptions of ways they worked in groups with emphasis on the group learning behavior at the focus of the lesson.

During the course, students also used a visual digital tool developed in the Metafora project, to plan and reflect on group-learning processes, and other tools to foster mathematical exploration, such as Excel sheets and GeoGebra. We report elsewhere on the effects of learning with these tools on group learning (e.g., Abdu et al. 2015; Schwarz et al. 2015; Wegerif 2015). While the current study does not focus on technology, we make some remarks about the potential effects of using these digital tools in the conclusions section.

Procedure

Before the beginning of the course (*pre-test* hereafter), we arranged the students in five groups of three students. The five groups solved the billiard problem (Fig. 1) in a private room. The students were given a case full of pens in various colors and a notebook; the students were allowed to tear up pages from this notebook. The problem-solving session was limited to 40 minutes. Group work was video-recorded and transcribed. At the end of the course (*post-test* hereafter), students were again grouped in the same five groups.



A ball is hit from a pocket of a billiard table (pocket A in the figure) at an angle of 45° with the sides of the table. The ball then bounces on the sides of the table, still at an angle of 45°, until it drops into one of the four pockets (A, B, C, and D) at the corners of the table.

1. *In case the dimensions of the table are 5X3, how many times the ball hits the sides of the table until it drops into one of the pockets?*
2. *Answer the same question for tables with other dimensions. You may investigate this question for as many tables as you wish.*
3. *What should be the size of the billiard table, if the ball falls into one of the pockets after eight hits (not including the start and end points)? Is there more than one possible table? What are they?*
4. *What are the relationships between the dimensions of the table and the number of hits required for the ball to drop into one of the pockets?*
5. *If you came up with such a relationship, why does it work this way?*

Fig. 1 The billiard problem

In three of them, the groups were the same; in two of them (groups 3 and 4), two students remained. The five groups were invited to solve the billiard problem a second time.

The billiard problem

Different versions of *the billiard problem* have been used in various mathematical problem-solving contexts (e.g. Cifarelli and Cai 2005; Kontorovich et al. 2012). The current version of the problem was designed to be gradual, in terms of problem-solving capabilities that are needed to solve it. Firstly, the students need only to simply follow a procedure. Later on, they are invited to produce examples of the problem that require no mathematizing or deductive thinking. Then, they are invited to find a billiard table for which the number of hits is eight. Thereafter the students are expected to create tables and undertake some inquiry on them. Students have to provide a general rule for the connection between the size of a table and the number of hits. In this part, students should look for patterns and should deduct rules. Finally, they need to explain the patterns they found, to express rules, test their validity, and prove their correctness.

The billiard problem is not straightforward and is challenging. Its solution is that the number of hits is $length + width - 2$ when the length and the width have no common denominator. When the width and height have a common denominator D, the formula becomes $(length + width - 2)$/D. Reaching this solution is quite difficult for 9th-grade students; but some intermediate rules may come up along the solution, such as ‘‘*when the length equals the width, there are no hits*’’ or “*when one side is double the size of the other, the ball hits the sides once”*. Alternating collaboration and cooperation may help in solving the problem: labor division may help investigate diverse cases (billiard tables in various sizes), and participants may meet to compare their work to elaborate together strategies or provide a solution based on the cases investigated. It appeared that none of the groups reached the fully justified solution, even in the post-tests, but many partial solutions were produced.

Segmentation of the data

We segmented the data from the ten tests along in three subsequent layers (Derry et al. 2010): sub-tasks of the problem, mathematical problem-solving phases and actions (according to Activity Theory). (1) Section of the problem: The billiard problem had five sections that are related to five questions (see Fig. 1). Accordingly, we allocated the assessments to five initial segments. The beginning of a new segment is an inference in which one of the students reads a particular question. (2) Mathematical problem-solving phase: We identified four kinds of cases. *Understanding the problem* is where students identify the problem’s requirements. Understanding the problem was identified as cases in which students read the problem statements or try to understand how to illustrate the billiard table and the course of the ball (e.g., how the 45 looks like when the ball bounces from the side of the table). These segments were pruned from the analysis. We focused on three other kinds of episodes: *Planning* is a phase in which the students engage in talking about future activities they will take, like envisioning more than one step ahead or allocating tasks/roles. Mere talking about an action to be taken in the short future (e.g. "try a 4X5 table and see what happens"), was not considered as planning. *Execution* is a series of actions that are taken to provide data that will help in solving the problem. *Reflecting* is a phase in which students engage in talking about previous actions that were taken. We used these phases to create a second layer of segmentation. (3) Activity theory: The third line of segmentation that was based on our identification of a single group's goal at a given time, in light of the learning task they need to accomplish (Leont'ev 1974). For example, drawing a billiard table, calculating, coming up with a generalization and debating how to proceed. At the end of this process we came up with learning episodes, each containing at least two turns but not longer than eight turns.

Coding

Coding for collaboration and cooperation as modes of group-learning

For every episode, we used a dichotomous distinction between collaboration and cooperation. An episode segment was coded as *Collaborative* when all group members were involved and focused on the same semiotic means and symbols (discourse, mathematical symbols, diagrams, etc.) while solving the same sub-task. A *Cooperation* episode was an episode in which collaboration was not met: at least two of the group members focused on more than one object (e.g., two students interact with two different billiard tables, or, one of the students was absent from the scene), or, the students engaged in more than one sub-task (e.g., one student created a billiard table and the other created a list that summarizes the results).

Coding for mathematical problem-solving

Measuring progression in mathematical problem-solving is a challenge. Several tools have been proposed: The elaboration of *Heuristics* (Koichu et al. 2007), of epistemic actions such as constructing and building with, towards abstraction (Dreyfus et al. 2015) and of mathematical arguments (Schwarz et al. 2010; Vogel et al. 2016). We chose to code the data along with Toulmin's theory of arguments (Toulmin 1969), because the claims and arguments emerge frequently when solving a problem in small groups, allowing fine-grained scrutiny over problem-solving. A fundamental construct of the argumentative framework is the claim: a set of inferences about a certain mathematical idea. For example, claims could be in the form of conjectures, attempts to generalize to come up with a rule, and attempt to refute or strengthen other's claims. Arguments are claims that are supported by data and/or warrants (Schwarz et al. 2010; Vogel et al. 2016).

Anderson and colleagues (Anderson et al. 1997) explain that discursive moves that are made by students are often *incomplete arguments*. However, this does not mean that what students intend to express is not valid or incomplete, but that their talk is informal. As Anderson and colleagues clarify, many times the researcher must infer or interpret what was the exact claim that was given by the student. The researcher can: (1) Use terms such as "this" and "it" and infer what they refer to. For example, if the student says "this" and points at 8 X 2 table, we can say that he refers to this "8 X 2 table"; (2) Use context and former discursive moves to derive meanings—for example when the student develops an idea that was offered earlier on; and (3) Take into account implicit assumptions, whether specific to the problem (e.g. the ball has to drop in one of the pockets) or general (a ball moves in a straight line until it hits a side of the table).

We used the methods offered by Anderson and colleagues (1997) to identify *claims* in our data set. If a claim appeared more than once, it was counted once only. Among these claims, we identified *valid claims* (e.g., a square table will yield no hits on the sides). We then looked for *data* and *warrants* that were provided in support of claims. We then used these data and attached them to claims they supported, from the students' perspective. One data/table could support more than one claim. *Warrants* were the explicit use of an earlier (not necessarily true) claim in support of a current claim. An *argument*, thus, is a claim that was based on data, and/or attached with a warrant to a previous claim. A *valid argument* was an argument that was based on a valid claim.

Coding for collaborative mathematical argumentation

We defined two additional argumentative concepts at the group level: *collaborative claims* and *collaborative valid arguments*. In congruence with Van den Bossche et al. (2011), a claim was counted as *a collaborative claim* if it fulfilled at least one of the following requirements: (1) Every group member expressed this claim or a new claim in an attempt to refute the first claim, and (2) the group was in a collaboration mode at the time of the enunciation of this claim by one of the group members. A *collaborative valid argument* in this coding scheme is a collaborative claim that is valid and backed by data or a warrant.

Transcriptions of a video recording, as detailed as they may be, cannot capture the full nature of the talk (Derry et al. 2010), and thus, all of the coding done for this work was a simultaneous process of watching the movies while reading the transcripts. The first author coded all of the assessments according to these dimensions. We then checked the reliability of the coding scheme with additional raters. Twenty percent of the assessments were coded by the second coder—a researcher in the Learning Sciences, for group learning, yielding an agreement of 86%. The other 20% of the assessments were coded for mathematical argumentation by a third coder with a Ph.D. in Learning and Instruction and yielded an agreement of 96%. Disagreements were negotiated among coders.

Findings

The findings section includes three sets of analyses. The first and the second answer the research questions based on quantitative analyses of the pre-and post-tests. The third analysis illustrates the findings with detailed excerpts of three different groups who solve mathematical problems.

Changes in group work

We first assessed the change in group work before and after the course. Table 1 shows that the average amount of cooperation episodes increased from 4.4 in the pre-test to 19.6 in the post-test. A paired one-tailed t-test shows that this change was significant ($p < 0.5$). To operationalize "the interplay between collaboration and cooperation" we collapsed the two modes into one variable, which is the ratio between the number of cooperation episodes and the total—collaboration *and* cooperation—episodes: Cooperation-to-Total Ratio, (CTR). On average, CTR increased as well, from 0.092 in the pre-test to 0.258 in the post-test. This change was significant as well ($p < 0.5$). Note, however, that while Groups 1, 2 and 5 show a notable increase in CTR, Groups 3 and 4 do not. Thus, three of the five groups adopted more cooperation to their group learning, in the post-test. We will return later to this finding.

Table 1 Cooperation episodes—total and relative

Group	Total episodes		Total cooperation episodes		Cooperation to total ratio (CTR)	
	Pre	Post	Pre	Post	Pre	Post
1	61	83	8	27	0.13	0.33
2	38	80	4	44	0.11	0.55
3	26	60	0	0	0	0
4	30	49	1	4	0.03	0.08
5	48	69	9	23	0.19	0.33
Average			4.4	19.6	0.092	0.258
Sig. of change			.049*		.048*	

* Statistically significant changes ($p < .05$)

These changes in group work patterns could be considered as expressing an aspect of L2L2 only if mathematical problem-solving—the context for group work during the course—is promoted. We, consequently, performed a paired one-tailed t-test between the pre-test and post-test, on four variables: the number of claims, arguments, valid claims, and valid arguments. As can be seen in Table 2, there is an increase for all variables, with a significant increase in the number of mathematical claims ($p < 0.5$), and arguments ($p < 0.5$) in the post-test. Despite the small sample of groups, these findings suggest that, in general, students were more engaged in mathematical problem-solving after the course. A closer look at Table 2 reveals that while there was an overall increase in claims and arguments, the change in the valid claims arguments was rather mild for Groups 2, 3 and 4 and more profound for Groups 1 and 5.

As can be seen in Table 3, the number of collaborative claims and collaborative valid arguments increased, none of which significantly, though. A closer look at Table 3 reveals that in Groups 1 and 5, there is an increase in the number of collaborative claims and collaborative valid arguments but not in Groups 2, 3 and 4. Moreover, Group 2 even shows a decrease in the number of collaborative claims and collaborative valid arguments.

To sum up, we identified three patterns in terms of the interplay between collaboration and cooperation, and mathematical problem-solving.

Table 2 Claims, arguments, valid claims and valid arguments produced by students in group work (collaboration and cooperation) in the pre- and post-tests

Group	Claims		Arguments		Valid claims		Valid arguments	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
1	13	25	10	16	5	12	4	10
2	8	12	6	8	5	6	5	5
3	8	14	6	9	3	5	3	4
4	6	11	3	8	1	2	1	2
5	9	29	8	21	5	11	5	11
Avg	8.8	18.2	6.6	12.4	3.8	7.2	3.6	6.4
Sig. of change	.035*		.04*		.057		.101	

* Statistically significant changes ($p < .05$)

Table 3 Collaborative claims and collaborative valid arguments per group, before and after the course

	Collaborative claims		Collaborative valid arguments	
	Pre	Post	Pre	Post
1	9	20	3	10
2	6	5	5	2
3	8	10	3	3
4	4	9	1	2
5	7	27	5	11
Pre	6.2	14.2	.4	5.6
Sig. of change	0.118		0.307	

1. *Groups 3 and 4:* Almost no increase in Cooperation to Total Ratio (CTR). *Some* increase in claims, arguments, collaborative claims, and collaborative valid arguments.
2. *Groups 1 and 5:* An increase in CTR. A major increase in claims, arguments, collaborative claims, and collaborative valid arguments.
3. *Group 2:* Major increase in CTR (from 11 to 55%). Some increase in claims and arguments, somewhat similar to groups 3 and 4, but a *decrease* in the number of collaborative claims and collaborative valid arguments.

The relationships between modes of group work and problem-solving

To understand the relationship between Cooperation to Total Ratio (CTR) and mathematical problem-solving, in congruence with the second research question, we collapsed the results from the pre- and post-tests (five tests twice) into one data set of ten tests. We then performed a set of Pearson correlation tests between CTR, and the six argumentative variables (claims, arguments, valid claims, valid arguments, collaborative claims, and collaborative valid arguments) over the ten tests. As can be seen in Table 4 and Fig. 2, the number of valid claims produced significantly correlates with cooperation ($p<0.05$). Other argumentative variables mildly correlate with CTR, suggesting that an increase in cooperation may lead to the improvement of individual and group mathematical problem-solving.

Taking into consideration the nature of Group 2′s post-test—in which high cooperation appeared along with a decrease of some of the argumentative variables, we performed six quadratic regression analyses for each of the six mathematical problem-solving variables

Table 4 Correlations between argumentation and percentage of cooperation episodes

	Claims		Arguments		Valid Claims		Valid Arguments		Collaborative claims		Collaborative valid Arguments	
	R^2	Sig	R^2	Sig	R^2	Sig	R^2	Sig	R^2	Sig	R^2	Sig
Linear	.25	.138	.24	.144	.40	.049*	.37	.06	.11	.349	.17	.241
Quadratic	.44	.129	.51	.081	.61	.037*	.63	.03*	.39	.182	.60	.039*

* Statistically significant changes ($p < .05$)

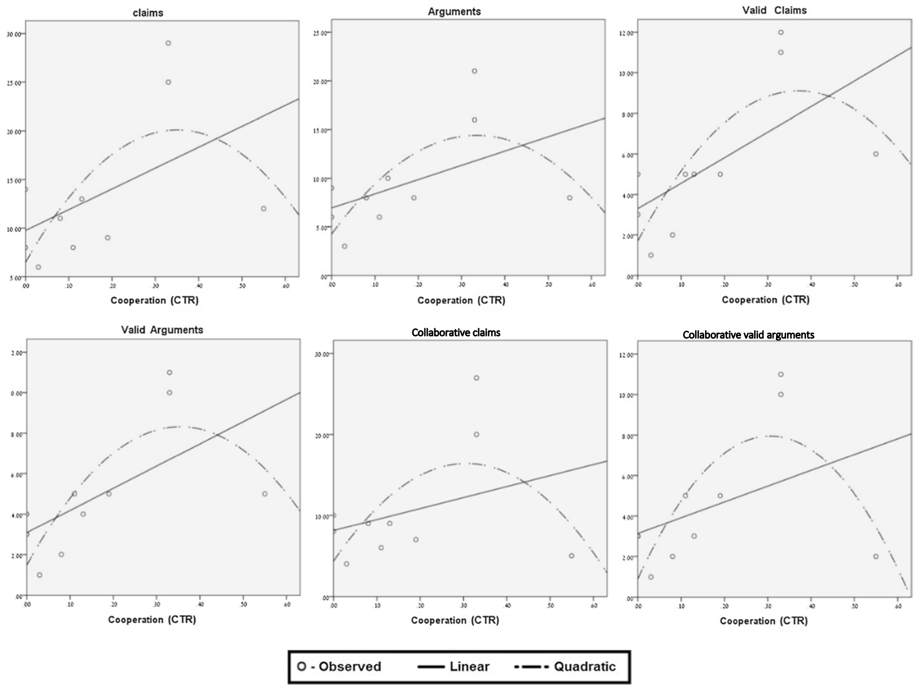


Fig. 2 Linear and quadratic estimations of the relationship between CTR and the six mathematical problem-solving variables

[dependent variables], with CTR as an independent variable. The findings, as can be seen in Table 4 and Fig. 2, show that the best fit is a quadratic rather than a linear one, for all mathematical problem-solving variables. In particular, the model is statistically significant for the cases of valid claims, valid arguments, and collaborative valid arguments.

A sample of ten tests that are interrelated (thus, representing five groups only) is not enough to draw significant conclusions. However, these findings suggest that integrating cooperation was beneficial only if it was not done "too much". Proper balance between collaboration and cooperation is beneficial for group and individual advancement. In the case of this problem and these students, around 35% of the episodes should have been cooperative (see Fig. 2).

Three patterns of group-learning

The quantitative analysis in the former section suggests, in light of the second research question, that cooperation is related to productive mathematical problem-solving by individuals and the group. We will now bring three cases, with examples of three mathematical problem-solving processes by three groups, to illustrate the interplay between collaboration and cooperation in mathematical problem-solving. We chose three cases to demonstrate three distinct patterns of the interplay between collaboration and cooperation: (1) collaboration only, (2) balancing collaboration and cooperation and (3) high cooperation. In Tables 5, 6 and 7, the verbal utterances of the students will be highlighted in italics.

Table 5 An excerpt from Group 3′s pre-test—high collaboration

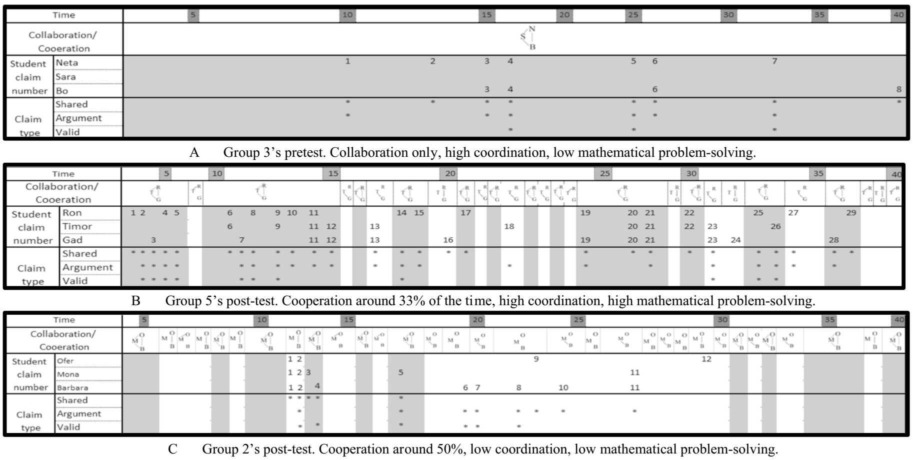
	Neta	Bo	Sara
3.1	[Flips to former page] … *how many hits did we do now*?	[Looks at notebook]	[Looks at notebook]
3.2	[Points to a 5X6 table with a pen] *Hey, here, 5 plus 6 is 11, right*?	[Looks at notebook]	
3.3	[Flips the page back] *And also 4 plus 7 is 11*		
3.4	[Smiles]	It is related	[Smiles]
3.5	[Flips to former page] *I don't know. If we take 4 plus 5 it is9, right*?		
3.6		It was *5* [*hits*]	[Looks at Bo]
3.7	[Looks at BO] *That's ok*		
3.8		*So how is it related*?	
3.9	[Looks at Bo] *If we will create, say, a 2X3 table, then it also yields 3* [*hits*]	[Looks at notebook]	[Looks at notebook]
3.10		*OK*	
3.11	[Smiles] *I think so, I don't know*		[Smiles]
3.12		*But how is it all related? It doesn't give the same score*	
3.13	[Stares at the notebook] *I know. One moment*	[Leans back, frustrated]	
3.14	*Maybe it was like that by a chance*		
3.15		[…] *there is no connection* [*between the table's dimensions and the number of hits*]	
3.16	[Looks at Sara] *I have no idea*		[Looks at Noga]
3.17		*Can we say that there is no solution?*	

Collaboration only

In Fig. 3a we bring the mathematical problem-solving process of Group 3 in the pre-test. In this case, three female students solved the problem. They collaborate throughout the assessment (could be seen as the grey areas). Neta and Bo produced considerably more inferences than Sara, who did not express any “claim”. In total they produced eight claims, all were shared by all of them.

The following interaction was taken from the 25th minute of the solution process. Before this episode, the three girls created three tables and inferred three claims: (A) ‘a 4X5 table yields five hits’, which was a mistake caused by incorrect drawing of the table—the correct answer is seven; (B) ‘a 4X6 table yields three hits’, which is correct: if the two sides have a common denominator, there is a need to divide by the denominator, and then the number of hits is the amount of the sides minus two; (C) ‘a 5X6 table yields nine hits’, which is correct. In the following excerpt, Neta creates a 4X7 table and counts the hits. Bo guides Neta’s drawing. Sara is passive, only looking. They count nine hits.

We claim that this excerpt provides an episode of full collaboration between the three students. Neta and Bo communicate the whole time—every utterance by Neta is regarded by Bo and vice versa. Sara does not talk but remains connected to them by observing their work [3.1; 3.9] and smiles with them in places of puzzlement [3.4; 3.9]. The group identifies a pattern: two tables whose length and width sum up to eleven, yield nine hits each [3.2; 3.3]. This may lead to enunciate the general rule—the amount of hits equals the sum of the two sides minus two. Neta even correctly claims that a 2X3 table would yield three hits [3.9]. However, two other tables that were created contradict this claim. This contradiction leads the fully-collaborating group to abandon Neta’s idea and to move on to claim that there is no solution to the problem [3.15]. Until the end of the session, Neta’s notion or



A Group 3’s pretest. Collaboration only, high coordination, low mathematical problem-solving.

B Group 5’s post-test. Cooperation around 33% of the time, high coordination, high mathematical problem-solving.

C Group 2’s post-test. Cooperation around 50%, low coordination, low mathematical problem-solving.

Fig. 3 Illustrating three assessments by three groups. In the first row an indication of the time from the beginning of the assessment. In the second row an illustration of the connections between group members; when two initials (e.g., ‘O’ for Ofer) are connected it means that these two students collaborate. In columns 3–5 [group claim number] we indicate the first time each claim was expressed. In columns 6–8 we characterize each claim. (see Sect. 4.4.2. for coding) the 2X3 table is not mentioned again. Ultimately, three valid arguments were produced by the group.

Neta's claim could have led the three to a correct solution path. However, Bo provides a counter-example to this claim [3.6], shows her frustration [3.13] and provides an alternative explanation: there is no connection (between the table's dimensions and the number of hits) [3.15]. The three girls remain together. Sara seems alone—she could have cooperated by checking Neta's claim, while her peers progressed but she followed her peers' moves [3.1; 3.6; 3.9] and merely remained passive throughout this episode, as in most parts of the solution. The three students remained in collaboration.

Balancing collaboration and cooperation

In Fig. 3b, we bring the mathematical problem-solving process of group 5 in the post-test. In this case, three male students solved the problem. They spent a third of their time cooperating, as each of the students worked alone (as shown in the white areas of Fig. 3b) and reconnected again. None of the students spent too much time off the group; yet, each of them was alone for some time. Coordination was manifested as twenty-seven of twenty-nine total claims were eventually shared by all group members.

Table 6 shows an excerpt at the 37th minute of the solution process. Before this exchange, the three students stated several arguments, among them, the incorrect argument "*when the two dimensions of the table are odd, the number of hits equals the greater dimension plus one*". This excerpt starts with cooperation. Gad and Timor are together and Ron is alone. Each of then holds a pen in his hand and has paper at disposal.

The excerpt begins in a cooperation mode: Ron is busy creating a 2X3 table [5.1; 5.4; 5.5] and checking for other tables that one of their sides equals three. In parallel, Gad and Timor articulate a claim that was discussed previously during a collaboration phase: If the product of the division between two sides of a table—which they define as X and Y—is an integer, then the amount of hits—which they define as M—equals the divisor, minus one (claim A) [5.1 to 5.5]. This claim is correct. This claim was supported by data brought earlier; for example, if the table's dimensions are 9X3, then the amount of hits equals 9 divided by 3 minus 1. This claim was uttered as a collaborative valid argument.

Right as Gad and Timor finish articulating claim A, Ron joins them and comes up with a new claim (claim B): *When one of the dimensions equals 3, the number of hits equals the other dimension plus 1* [5.7]. Gad does not consider this claim to be important since the group already articulated an (incorrect) claim (claim C): for the case of two odd dimensions, the number of hits equals the bigger dimension plus one. Consequentially, Timor abandons the group for a moment to re-check this last claim [5.7], with a 5X7 table, and Ron decides to look for another rule/claim [5.13]. Later on, upon Timor's re-check, claim C is refuted and claim B is accepted and written done. To sum up, it appears that episodes of collaboration alternate with episodes of cooperation in this excerpt.

Table 6 An excerpt from Group 5′s post-test—balanced collaboration and cooperation

	Ron	Timor	Gad
5.1	[Creates a 3X2 table and checks how many hits it yields]		[Writes down] *if there is* […] *a common divider* [to the dimensions]
5.2		[Continues Gad's words] *And the product is an integer*	
5.3			*A common divider which is an integer*
5.4	[Sees that the number of hits is three]	*No, which divides*	
5.5	[Looks at other tables drawn before on his paper: 3X7 and 3X4. Writes "8 hits" next to the 3X7 table and "5 hits" next to the 2X4 table]		[Writes down] *A common divider which is an integer, then Y divided by X equals N. M* [the number of hits] *equals N minus 1*
5.6		[To Gad] *Now write down, M equals the number of hits, X and Y are the sides* [of a table]	
5.7	[Looks towards his peers. Now, the group is in collaboration] *I have a new rule for you after* [you finish writing]		
5.8	[To Gad] *When one of the dimensions equals 3, the number of hits equals the other dimension plus 1*		
5.9			[To Ron] *We did it already,* [cites from the notebook] *when the two dimensions of the table are odd* [*the number of hits equals the bigger dimension plus one*]
5.10		[Grabs the paper, starts to learn individually] *Let's check it for the case of two odd* [table] *dimensions. Five and seven.*[Starts drawing a 5X7 table]	
5.11	*We didn't try five and seven*	[Counts] *three four, five six…*	
5.12	[Looks at Timor]		[To Timor] *You'll get six hits. No, you will get eight hits*
5.13	[To gad] *Let's try another thing so we will not waste time. What else can I do?*	[Keeps drawing]	

Table 7 An excerpt from Group 2's post-test—extensive cooperation

	Ofer	Mona	Barbara
2.1	[Inspects a 4X6 table] *What am I doing…oh, this is right. It is ok*	[Creates a 10X7 table]	[Creates a 4X1 table. Next to it on her page are drawings of 2X1 and 3X1 tables]
2.2		*I got a square. Did I get a square or a rectangle?* [Measures the sides of the tables]	
2.3			[Correctly draws three hits in the 4X1 table]
2.4	*Two, three*		
2.5			[drops the pen dramatically] *I have a rule! This is a series*
2.6	[Shows three hits. Correctly]. *Four*		
2.7			[shouts] *Ofer, I have it!*
2.8	*With a four on six* [table*] I got four*		[Looks at Ofer's model] *The final pocket does not count, yes?*
2.9	*Oh…*		
2.10			*So, three*
2.11	*Three*		
2.12			*What is the connection?* [Breathes heavily and looks at the camera] *the prefix, minus 1, is the solution*
2.13	[To the researcher] OK, when it is four on six it [yielded] three [hits]		[…To Mona] *Listen, the prefix, minus 1, is the solution*

Extensive cooperation

In Fig. 3c, we bring the mathematical problem-solving process of Group 2 in the post-test. In this case, two females (Mona and Barbara) and one male (Ofer) solved the problem. Approximately half of their problem-solving was spent in a cooperation mode, with long periods of cooperation—mainly between the 17th and the 30th minute—as each group member solved the problem on her own without communicating with the other members of the group. Table 7 shows an episode that occurred at the 20th minute of the solution process. The three students talk aloud but do not build on each other's ideas. Each one learns with his/her pen and paper.

The three students in this episode do not communicate although they solve the same problem together. Ofer works on a 4X6 model, Mona on a 10X7 model and Barbara works on a generalization for cases in which one of the sides of the table equals 1. Barbara comes up with a valid argument: when one side equals 1, then the number of hits is what she calls "the prefix"—the length of the other dimension—minus one [2.5]. She is ignored by her peers, who continue working on their models [2.6]. She tries again, addressing the researcher in the room who ignores her as well. Then, Barbara tries to get Mona's attention [2.13] with no success. Eventually, to the end of the session, this argument by Barbara was not shared by her peers.

The lack of coordination is manifested by the poor exchange of ideas but also by the fact that both Barbara and Ofer address the researcher in the room—Ofer does receive some feedback from Barbara [2.9], but is not satisfied, asks for validation from the researcher in the room and receives no feedback as well [2.13]. This suggests that both Barbara and Ofer did not find support within their group and had to address an external entity. Claims were hardly expressed, and only five of the 13 expressed claims were done in collaboration. No new claims were shared among the group members after the 17th minute. In the whole problem-solving session, the group produces only three valid solutions.

Discussion

In this paper, we studied the interplay between two modes of group learning–collaboration and cooperation–in the context of mathematical problem-solving, when students are free to choose the mode that fits their needs. This study is in contrast with research trends that stress the importance of scripts to promote collaborative behaviors (Fischer et al. 2013). We developed a course in which students were trained to apply different patterns of group work—in terms of the interplay between collaboration and cooperation—while learning to solve mathematical problems. Five groups of students solved a mathematical problem together without guidance, before and after a course dedicated to learning to solve mathematical problems together with technology. We operationalized "patterns of group work" as combinations of collaboration and cooperation. The interplay between collaboration and cooperation was thus defined as a continuum, ranging from group work that is fully collaborative throughout the solution to the problem by the group (CTR = 0) to hypothetically full cooperative learning throughout the group-learning (CTR = 1). In this study, the range for CTR was from 0.00 to 0.55. We operationalized mathematical problem-solving using the argumentative framework, identifying claims and full arguments expressed by group members (Schwarz et al. 2010). Further, we looked for two indicators of group learning: The extent to which ideas were explicated by group members (i.e., collaborative claims) and ideas that are correct and brought up with supporting data and/or justification (collaborative valid arguments). Correct ideas that were shared by all group members, we maintain in our analysis, are indicators for productive group learning.

Can a course in L2L2 change students' patterns of group work in solving mathematical problems?

We compared group work—in terms of both modes of group learning and mathematical problem-solving—before and after the course. Before the course, for all groups, the form of work was almost always collaborative. After the course, we observed a significant increase in cooperation, as some students took some time off collaboration. In many cases, as can be seen, for example, in Tables 6 and 7, the time off was used for individual sense-making or production of new mathematical claims. In parallel with these changes in group work, we observed an increase in argumentative activity– in terms of claims, arguments, valid claims, and valid arguments–for all of the five groups (i.e. all twenty cases). The increase in claims and arguments was statistically significant. Students became more prone to express their thinking in the post-test, a behavior that is congruent with competent mathematical problem solvers (Cifarelli and Cai 2005). Enhanced production of claims is also very productive for learning: Kapur (2014) explains that when students produce more solutions—correct or not—they need to compare between the "right" solution and their solutions, which allows them to understand the boundaries of an idea to be learned. Thus, in terms of mathematical problem-solving, participating in the course enhanced the work of the group.

The two changes—significant increases for both cooperation and claims (regardless of their development into valid arguments) suggest that the course yielded new patterns of group work that are productive for solving mathematical problems. These changes convey an important instantiation of Learning to Learn Together. Also, the increase in the production of claims by individual students could be seen as a decrease in the silent voices, as students in the post-test were more prone to express ideas, even if they were half-baked. This evidence–that some students adopted a sort of indifference towards failures throughout the course–is also supported by the evidence we bring in Abdu et al. (2015), in which we describe how Tamir and Gad of Group 5 explicitly talk in one of the class discussions about the fact that "*if the student that is most sure of himself will talk last, everyone will have the chance to talk and we could learn from mistakes*". This is, by all means, a positive group-learning outcome among those groups who adopted cooperation to their group work (Barron 2003; Wit 2006).

Which pattern of group work is more beneficial for mathematical problem-solving?

Among the five groups, the change in CTR and mathematical activity was not consistent. Two groups showed a notable change in cooperation to around a third of the interaction, two groups mainly collaborated also after the course, and a fifth group cooperated in more than half of the episodes. We found that CTR level relates to the productive mathematical problem-solving: CTR was correlated with most of the mathematical problem-solving variables. In the excerpt brought in Table 6, we illustrated how a balance between collaboration and cooperation is productive: When cooperating, students were more inclined to create their own ideas and present them to the group when collaborating.

Balancing collaboration and cooperation in group work seems then effective for group learning. In Group 5′s post-test the three students move back and forth from collaboration, upon need. This "alone time" allows them to think of new claims/rules, and come up with data to support and refute these claims. All three students are engaged, and take the time to break from the group and make sense of ideas that are brought within the group. Therefore, the group members cooperate, and when they come back from cooperation to collaboration they produce an efficient exchange of ideas that helps them share and form new knowledge. This group produced a total of eleven valid solutions in this assessment; which is considerably high.

In the case of Group 2, we observed that "high cooperation" yielded only a small improvement in mathematical problem-solving and a decrease in the number of collaborative claims and collaborative valid arguments. Contrasting the group work in the pre-test of Groups 2 and 5, we identify two main reasons for Group 2′s low performance. First, it seems like Group 2 suffered from poor coordination (Table 7; Fig. 3), which is crucial for productive group learning (Barron 2000; Oner 2013; Van den Bossche et al. 2011; Webb et al. 2014). Cooperation for Groups 2 and 5 are rather dissimilar: for Group 5 cooperation meant usually that one student is taking time off while the two others remain in communication with each other (Fig. 3b), while for Group 2 many times all three group members worked alone (Fig. 3c). Second, cooperation sequences in Group 2′s post-test were quite longer than Group 5 and thus impeded, even more, the information flow between the group members (Fig. 3c). Ideas that were brought by one group member were not always shared by other group members and the group as a whole was showing less competence. In Table 6 we bring an example of how relatively high cooperation negatively affected the number of claims and arguments expressed and shared among group members. Barbara's voice was not a soft voice per se (Barron 2003) but it was still not heard. The quadratic model we elaborated to describe the correlation between mathematical problem-solving and CTR, suggests that productive group work requires students to alternate cooperation with collaboration.

We are aware, however, that the modes we identified to describe group work were coarse. We identified collaboration with "sharing the same goals and communicating" and cooperation with "sharing the same goals without communicating". We identified three patterns of group learning with respect to the interplay between collaboration and cooperation: Collaboration only, balancing collaboration and cooperation and extensive cooperation. However, the nature of communication varied considerably among groups. For example, in Group 3, Neta and Bo coordinated actions but Sara was only attentive at their actions. She was quite passive. In episodes different from the episode shown in Table 5, she simply acquiesced to Neta and Bo's suggestions. Collaboration among the members of this group was then quite limited and definitively did not refer to the kinds of collaborative processes for which all members of the group coordinate actions towards the execution of the same goal. The "high cooperation" between members of Group 2 that are showed in Table 7 simply refers to the fact that the members of the groups solve the same tasks but do not communicate. However, in some cases the processes involved were all but cooperative: they not only worked alone but intentionally ignored what their peers said. In their individual efforts, they wanted to concentrate on difficult stages of the problem at stake on their own. Our study should then be understood at the level of settings rather than processes. Our study shows that learning to alternate being alone with being together is an important group learning skill. Alternating cooperative and collaborative settings is beneficial for problem-solving.

Conclusions

Although research on group learning gained attention in identifying group learning behaviors that are (non-)productive for learning, so far, little was known about the interplay between collaboration and cooperation. In this paper, we put this interplay to the foreground. Our findings indicate that when a balance between collaboration and cooperation is reached, group learning is more beneficial. On the one hand, too much collaboration leads individuals to conform to arguments of the group, with less consideration of individual voices; on the other hand, too much cooperation does not enable the co-elaboration of arguments.

A second finding is that students can learn to apply different patterns of group work: Before the course students were much collaborative, and after the course learned to take some time for cooperation, and thus integrated cooperative episodes with collaborative ones. This is good news in the domain of L2L2. This is also good news in the domain of mathematical problem-solving. The outcomes of the course *Learning to solve mathematical problems together with technology,* in which students learned both mathematical problem-solving and group-learning skills, are positive. The students' overall mathematical problem-solving competency improved after the course in both individual and group levels: students produced more mathematical ideas, correct or not. This is rather reassuring since the domain of mathematical problem-solving is notoriously known to be challenging for students (Lester 1994; Kirschner et al. 2006; Schoenfeld 2007). Students spent more time cooperating in the post-test, which probably explains mathematical problem-solving enhancement. Groups that learned to balance collaboration and cooperation produced more claims and valid arguments, and if they remained coordination they benefit from group-work.

It may be also argued that the progress in mathematical problem-solving performance is related to development. And indeed, Veenman et al. (2004) showed that mathematical problem-solving skills develop over time. Thus, the observed changes in mathematical arguments may then not only originate from the participation to the course. However, the present study focused on patterns of group work, which change over time. The developmental factor cannot be ruled out, but the alternation between collaboration and cooperation in group work found in the post-tests, and the better group problem-solving performance suggests that the observed changes in group-work are at least partly responsible for this progression. We hope that further studies will confirm our findings.

We made a design decision to have students arrange themselves as a group when they solved mathematical problems. One may argue that some students may have learned more by solving problems on their own, without collaborating or cooperating with their peers. While we acknowledge that some people are more effective at working alone, we conjecture that this may be the case for most people. As shown by Schoenfeld (1985), mathematics problems are difficult to learn by individuals and many teachers fail in teaching problem-solving (Schoenfeld 2007). We decided to impose group work and allowed students to choose when they would like to apply which mode of group work. At the same time, we left some freedom to students, so that they could choose when to collaborate and when to cooperate.

We are aware that our methodology for checking the advancement of mathematical problem-solving—tracing the elaboration of claims and arguments, was narrow. A more suitable approach would have been to trace whether and how strategies and heuristics develop during the course in group work. Many methodological problems arise in tracing strategies and heuristics in a course on mathematical problem-solving, though. Among them, the design of the course cannot yield a series of tasks in which learned heuristics are directly applied in the following tasks since the emergence of heuristics cannot be designed deterministically. Tracing arguments and claims may not grasp the development of strategies in mathematical problem-solving; it is limited but constitutes a practical means for checking progress in problem-solving.

The limited number of groups involved in the study could not enable us to consider categories of group settings that are finer than collaborative and cooperative settings. In future research, we aim at observing settings in which all students cooperate, some collaborate while others work on their own, etc. Moreover, in this paper, we focused on the mere act of attending, or not, to the same "thing", with less focus on the nature of cooperation. In future research, we aim at discerning between the objectives, when choosing a setting for group work (e.g., division of labor, individual sense-making). Uncovering these objectives may be worthwhile in designing and understanding further instructional practices that would leverage on the interplay between collaboration and cooperation.

We brought theoretical explanations to the results obtained about (learning to) balance between collaborative and cooperative work. To what extent the results obtained in the context of mathematical problem-solving can be generalized to other domains? Although such a generalization should be the topic of further research, we may raise a reasonable conjecture: When strategies and meta-strategies need to be determined in order to achieve a task, group work with some room for individual investigations is necessary. Mathematical problem-solving is such a domain, but in other challenging domains such as scientific inquiry, it is reasonable that the same conclusions about the necessity of a balance between collaboration and cooperation would hold. Of course, the right balance between collaboration and cooperation depends on the nature of the task and who the students are, but being aware of the necessity to reach this balance is vital for educational design.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Informed consent This study received an authorization of the Israeli Ministry of Education. The parents/legal guardians of the students signed consent forms so the students were able participate in the activities and be filmed for this study.

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