

# On the Determination of Biaxial Residual Stress Fields from Global Indentation Quantities

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**Abstract** In the present study, quantitative relations for the determination of surface residual stresses, using sharp indentation testing, are presented. The relations are based on previous results for equi-biaxial residual fields but further developed to apply also for a general situation. The present analysis relies on theoretical methods, but the results are validated using previous experimental and numerical findings. Cone indentation of classical Mises elastoplastic material behavior is assumed throughout the investigation for clarity but not out of necessity. Further development for a complete characterization of a general residual stress field is discussed in some detail.

**Keywords** Residual stresses · General stress fields · Contact area · Sharp indentation · Mises plasticity

## 1 Introduction

Residual stresses and strains in a material can be determined using various experimental measuring techniques. These techniques include for example indentation crack techniques [1], fracture-surface analysis, neutron and X-ray tilt techniques [2], beam bending, hole drilling [3], and layer removal [4]. These methods can, however, be both complicated and expensive, and therefore, in the present study, a method based on sharp indentation testing is proposed. This is, indeed, of substantial practical importance as the effects of residual stress and strain fields in materials

can be considerable with respect to, for example, fatigue, fracture, corrosion, wear, and friction.

Until recently, the influence of residual stresses and residual strains on the results given by a sharp indentation test, in comparison with the corresponding results for a material without residual stresses or residual strains present, i.e., a virgin material, has been studied only occasionally, and then mainly experimentally. This is in contrast to sharp indentation or hardness testing of virgin materials which is a well-known experimental method used for determination of the constitutive properties of conventional materials such as metals and alloys. In recent years, the method has gained renewed interest due to the development of new experimental devices like the nano-indenter [5], enabling an experimentalist to determine the material properties from extremely small samples of the material. Indentation testing is, for example, a very convenient tool for determining the material properties of thin films in ready-to-use engineering devices. However, returning to the case when residual fields are present, it should be mentioned that already in 1932, Kokubo [6] studied several materials subjected to applied tensile and compressive uniaxial stress. The Vickers hardness was measured, and some very small influence from sign and size of the applied stress was found. However, the observed effect of stress on the hardness value was so small that no decisive conclusions could be drawn from these investigations. These results were confirmed somewhat later by Sines and Carlsson [7].

More recently, several interesting experimental investigations dealing with this issue have been presented, cf. e. g., [8–10]. The basic features of the problem were not fully understood; however, until Tsui et al. [11] and Bolshakov et al. [12] investigated, by using nanoindentation as well as numerical methods, the influence of applied stress

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on hardness, contact area, and apparent elastic modulus at indentation of aluminum alloy 8009, an almost elastic-ideally plastic material. Qualitative results of interest were presented as it was shown that the hardness was not significantly affected by applied (residual) stresses, while the amount of piling-up of material at the contact contour proved to be sensitive to stress (piling-up increased when the applied stresses were compressive and decreased at tensile stresses). Based on these results, further studies have been presented, cf. e. g., [13–20], more directed toward the mechanical behavior of the problem. Perhaps being the first to address this issue, Suresh and Giannakopoulos [13] derived, by making certain assumptions on the local stress and deformation fields in the contact region, a relation between the contact area at indentation of a material with elastic residual stresses (and plastic residual strains) and the corresponding contact area at indentation of a material with no stresses present. The analysis in [13] was restricted to equi-biaxial residual stress and strain fields, but it should be mentioned that for the forthcoming discussion, that these authors clearly distinguished between tensile and compressive residual stresses. The relation was, however, approximated with close to linear functions.

The physical understanding of the problem was further developed by Carlsson and Larsson [14, 15], in a combined theoretical, numerical, and experimental investigation. A somewhat detailed discussion of the results achieved in [14, 15] will be presented in forthcoming sections below, but in short, these authors showed that good correlation between predictions and numerical/experimental results could be achieved if the material yield stress, in relevant indentation parameters, was appropriately replaced by a combination of yield stress and residual stress. Most of the results presented by Carlsson and Larsson [14, 15] were related to equi-biaxial residual stress states, but in [16], the derived relations were extended to apply also for more general residual stress fields. In the latter case though, high accuracy results could not be achieved.

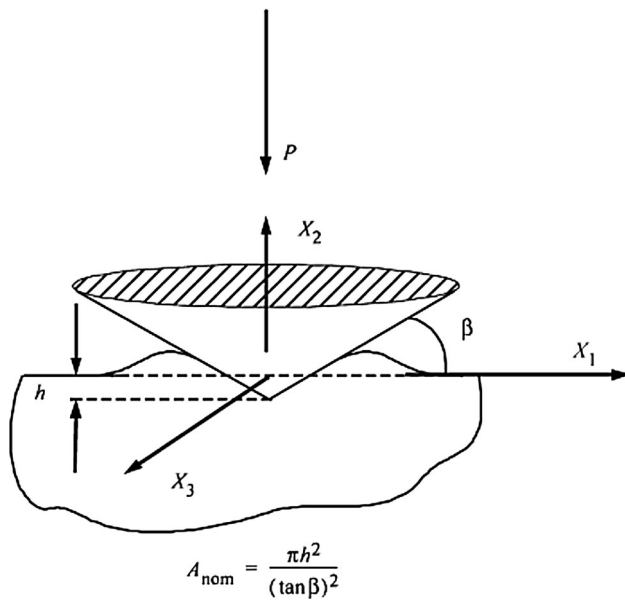
In summary, the studies [13–20] confirmed the invariance of hardness with respect to residual stress fields, and it is now fairly well understood, also from a quantitative point of view, how an equi-biaxial stress field will influence the size of the contact area. However, even though progress has been made also in this case, a quantitative formulation linking a general residual stress to the size (and shape) of the contact has not been achieved.

Further progress regarding the understanding of the problem was achieved by Huber and Heerens [21] and Heerens et al. [22] as these authors analyzed the corresponding problem of residual stress determination using spherical indentation testing. This is a more involved problem (as compared to sharp indentation testing) due to the existence of a characteristic length. Indeed, when

elastic and plastic effects are of similar importance, self-similarity of the problem is lost and a correlation between the indentation contact pressure and the residual stress state as attempted by Huber and Heerens [21] and Heerens et al. [22] becomes very much involved. Despite of this though, also other investigators, cf. e. g., Swadener et al. [23], have suggested that spherical indentation is an attractive approach for residual stress determination. The main reason behind this is that indentation variables are more sensitive to residual stresses in this case (as compared to sharp indentation testing). Presently though, sharp indentation is adhered to basically due to the fact that hardness and relative contact area are independent of indentation depth (due to the fact that the problem is mathematically self-similar with no characteristic length), and this is a particular advantage at interpretation of the results.

It is hopefully clear from the discussion above that much knowledge has been gained regarding the mechanics of indentation of residually stressed materials and structures. It is also clear, however, that a full understanding has not yet been achieved as quantitative relations accurately correlating general residual stresses with relevant indentation quantities are still lacking. In this context though, it should be mentioned that Carlsson and Larsson [15] presented explicit relations for this purpose based on an extension of results presented Carlsson and Larsson [14] for the equi-biaxial case. However, the outcome of the analysis in [15] was somewhat disappointing (as shown by Larsson [24]) due to the fact that compressive residual stresses were not accurately described by Carlsson and Larsson [14]. This problem was recently addressed by Rydin and Larsson [20], and very accurate relations linking both compressive and tensile residual stresses to the size of the contact area were presented. The results in [20] are, however, restricted to the case of an equi-biaxial stress state, and it is the aim presently to extend these findings to a general case relying on the suggestions by Carlsson and Larsson [15]. It should be immediately emphasized though that it is assumed in the analysis that the general stress state is either predominantly tensile or compressive. However, residual stress states ranging from uniaxial to equi-biaxial can be analyzed with the proposed method.

The present analysis will be performed using a theoretical approach where the results are compared with previous experimental and numerical results. In the latter case, such results are based on the finite element method as the resulting boundary value problem is very involved mathematically. General biaxial stresses are accounted for in the analysis with the restriction, as mentioned above, that the stress field should be either predominantly tensile or compressive. The analysis is for clarity confined to cone indentation, see Fig. 1, of elastic-ideally plastic materials relying upon classical Mises plasticity. However, an



**Fig. 1** Schematic of the geometry of the cone indentation test

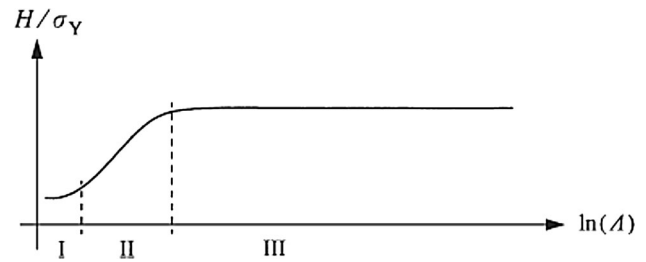
extension to strain-hardening materials can be performed in a straightforward manner according to the theory laid down by Carlsson and Larsson [14].

## 2 Theoretical Background

The basic foundation of the analysis by Carlsson and Larsson [14, 15], as confirmed by finite element calculations, is that a residual stress field will alter the magnitude but not the principal shape of the field variables involved. This immediately suggests that classical indentation analysis still applies but have to be corrected based on the residual stress. In short, it was shown by Carlsson and Larsson [14, 15] that it is possible to correlate the magnitude of the residual stress field with the well-known Johnson [25, 26] parameter

$$A = E \tan \beta / (\sigma_y (1 - \nu^2)) \tag{1}$$

In Eq. (1),  $E$  is the Young’s modulus,  $\nu$  the Poisson’s ratio,  $\sigma_y$  the flow stress, and  $\beta$  is the angle between the sharp indenter and the undeformed surface of the material, see Fig. 1. Johnson [25, 26] suggested that the outcome of a sharp indentation test on an elastic-ideally plastic material falls into one out of three levels, see Fig. 2, characterized by the parameter  $A$  in Eq. (1). In Fig. 2,  $H$  is the material hardness here and in the sequel defined as the average contact pressure. The three levels are schematically shown in Fig. 2 where in level I,  $A \leq 3$ , very little plastic deformation occurs during the indentation test, and an elastic analysis of the problem will be sufficient. In level



**Fig. 2** Normalized hardness,  $H/\sigma_y$  as a function of  $\ln A$ ,  $A$  defined according to Eq. (1). Schematic of the correlation of sharp indentation testing of elastic-ideally plastic materials as suggested by Johnson [25, 26]. The three levels of indentation responses, I, II, and III, are also indicated

II,  $3 < A < 30$ , plastic deformation spreads over the contact area. Finally, in level III,  $A > 30$ , pertinent to most engineering metals and alloys, rigid plastic conditions dominate as plastic deformation is present over the entire contact area and elasticity no longer has any effect on the hardness.

From theoretical, numerical, and experimental results [11, 12, 14, 15] it is, as mentioned above, a well-established fact that the material hardness is not noticeably influenced by stresses at sharp indentation testing. The relative contact area, however, here defined as

$$c^2 = A/A_{nom}, \tag{2}$$

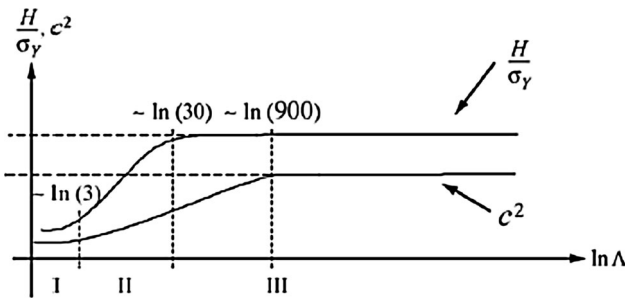
$A$  being the projected true contact area and  $A_{nom}$  the nominal contact area as defined in Fig. 1 for cone indentation can be directly related to the material state (it should be noted in passing that if  $c^2 < 1$  (sinking-in), the resulting contact area is smaller than what could be expected from purely geometrical considerations and the other way around if  $c^2 > 1$  (piling-up)). This finding is of fundamental importance when indentation testing is used to determine residual fields, and subsequently, it was shown by Carlsson and Larsson [14, 15] that when the residual (or applied) stress field is equi-biaxial, the relation

$$c^2 = c^2(\epsilon_{res}, \sigma_{res} = 0) - 0.32 \ln(1 + (\sigma_{res}/\sigma_{res}(\epsilon_{res}))) \tag{3}$$

can be expected to give results of high accuracy at tensile stresses but worse at compressive stresses [24]. In Eq. (3),  $c^2$  is the relative contact area for a material with a (equi-biaxial) residual stress field  $\sigma_{res}$  present (and possibly a residual strain field  $\epsilon_{res}$ ),  $c^2(\epsilon_{res}, \sigma_{res} = 0)$  is the corresponding relative contact area for a material with no residual stress, and  $\sigma(\epsilon_{res})$  is the material flow stress when the effective plastic strain equal  $\epsilon_{res}$ . In case of ideally plastic behavior, as of interest presently, Eq. (3) reduces to

$$c^2 = c^2(\sigma_{res} = 0) - 0.32 \ln(1 + (\sigma_{res}/\sigma_y)) \tag{4}$$

as then the yield stress of the material is independent of the residual strain field.



**Fig. 3** Normalized hardness,  $H/\sigma_y$ , and area ratio,  $c^2$ , as functions of  $\ln A$ ,  $A$  defined according to Eq. (1). Schematic of the correlation of sharp indentation testing of elastic-ideally plastic materials. The three levels of indentation responses, *I*, *II*, and *III*, are also indicated

Equations (3) and (4) were derived by Carlsson and Larsson [14, 15] based on the fact that the stress state in the contact region closely resembles the stresses arising at indentation of a virgin material with an initial material yield stress  $\sigma_y + \sigma_{res}$ . This was shown by careful and comprehensive numerical investigations of the behavior of the indentation-induced stress fields close to the contact boundary for materials with and without residual stresses. It is then possible to correlate the experimentally determined  $c^2$  value with the residual stress state based on the universal curve schematically shown in Fig. 3 by introducing an apparent yield stress

$$\sigma_{y,apparent} = \sigma_y + \sigma_{res} \tag{5}$$

in  $A$  in Eq. (1) according to

$$A = E \tan \beta / (\sigma_{y,apparent} (1 - \nu^2)). \tag{6}$$

The usefulness of this feature rests on the fact that elastic effects are more pronounced for  $c^2$ , than for the material hardness, as also shown in Fig. 3, and as a result, level II is the dominating region for this parameter.

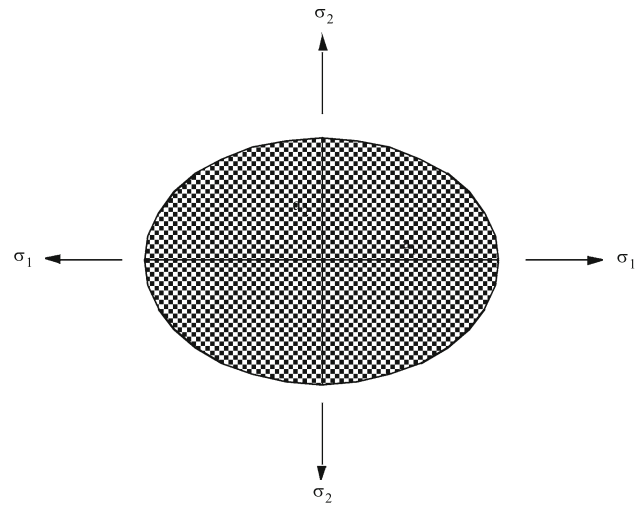
As mentioned above, Eq. (4) is accurate when tensile residual stresses is at issue but not so at compressive fields. The reason for this is that a compressive residual stress state will, cf. Eq. (5), reduce the apparent yield stress  $\sigma_{y,apparent}$  leading to a stronger influence from level III indentation effects. This problem was accounted for by Rydin and Larsson [20], and in this work, it was found, from studying the yield surface at particular points around the contact boundary, that replacing Eq. (5) with the expression

$$\sigma_{y,apparent} = \sigma_y + F\sigma_{res}, \tag{7}$$

where

$$\begin{aligned} F &= 0.52, \sigma_{res} < 0 \\ F &= 0.77, \sigma_{res} > 0, \end{aligned} \tag{8}$$

gave results of very high accuracy both in tension and compression.



**Fig. 4** Schematic of the contact area (shaded) at indentation. The principal residual stresses and the corresponding semi-axes of the elliptical contact area are also indicated

With the above discussed as a background, it was the aim of the present study to take advantage of the results by Rydin and Larsson [20] in order to find quantitative correlation between size of the relative contact area and a general residual stress state. It should be immediately stated that a complete characterization of such stress state requires additional information, for example, the shape of the contact area, and this will also be discussed below in some detail.

### 3 Solution Approach

There has been a few studies presented aiming at the determination of general residual stress states by sharp indentation testing. Many of them, however, are based on an approach where inverse modeling is applied, cf. e.g., Bocciarelli and Maier [19], and it is the intention here to analyze the problem from a purely mechanical point of view. For this reason, the analysis suggested by Carlsson and Larsson [15] constitutes the theoretical foundation for the present study.

The model by Carlsson and Larsson [15] is based on the fact that the indentation-induced in-plane stresses at the contact boundary are compressive and approximately equibiaxial also when general residual stress states are considered (as shown by extensive finite element calculations). Following the discussion above about the equibiaxial case, a direct extension would be to determine the apparent yield stress when an indentation-induced compressive and equibiaxial stress field  $\sigma_{ind}$  is superposed over the surface residual stress field in the material.

The von Mises yield criterion then becomes

$$\sigma_y^2 = (1/2) \left( (\sigma_1 - \sigma_{yind})^2 + (\sigma_2 - \sigma_{yind})^2 + (\sigma_1 - \sigma_2)^2 \right) \quad (9)$$

where  $\sigma_{yind}$  is the apparent yield stress at indentation,  $\sigma_{yind} > 0$ , while  $\sigma_1$  and  $\sigma_2$  are the principal stresses representing the surface residual stress field in the material. The principal stresses are indicated in Fig. 4 where also the resulting elliptic contact area is shown as defined by the semi-axes  $a_1$  and  $a_2$ .

In the equi-biaxial case, the quantity  $\sigma_{res}$  in Eq. (5) represents the change of the apparent yield stress at indentation. Consequently, it was suggested by Carlsson and Larsson [15] that this quantity could represent also a general residual stress field when determined from the expression

$$\sigma_{res} = \sigma_{yind} - \sigma_y. \quad (10)$$

In Eq. (10),  $\sigma_{yind}$  is determined from Eq. (9), and it goes almost without saying that ideally plastic material behavior is assumed.

As already mentioned above, and as also pointed out by Carlsson and Larsson [15], the predictive capability of Eqs. (5), and thereby also Eqs. (9, 10), deteriorates substantially at compressive residual stresses and this is of course a serious drawback as regards the usefulness of the above-described approach to determine residual fields. In the equi-biaxial case, this was corrected by the results derived by Rydin and Larsson [20], and it is the aim presently to apply such results also for the general case. It should then be stated that the outcome of the analysis in [20] can be summarized, based on Eqs. (6) and (7), in the expression

$$c^2 = c^2(\sigma_{res} = 0) - 0.35 \ln(1 + (F\sigma_{res}/\sigma_y)) \quad (11)$$

with  $F$  according to Eq. (8) in order to investigate generality and remembering that the slope of the Johnson [25, 26] curve in Fig. 3 (at level II indentation) is the same regardless of initial residual stress state. In this context, it should be reiterated that the nature of the stress state, determined by the ratio  $\sigma_1/\sigma_2$ , enters the analysis through  $\sigma_{res}$ .

In conclusion then, the solution approach suggested here for comparison with previous experimental and numerical results includes the following steps: (1) identify the principal stresses  $\sigma_1$  and  $\sigma_2$  describing the residual stress state and determine  $\sigma_{yind}$  from Eq. (9), (2) determine  $\sigma_{res}$  from Eq. (10), and (3) determine the area ratio  $c^2$  from Eq. (11) and compare with relevant experimental and numerical findings. Such a comparison will in the present study be performed below in a quantitative manner in order to determine the accuracy of the suggested approach.

It should be immediately stated that in a practical situation, when the determination of unknown residual stresses

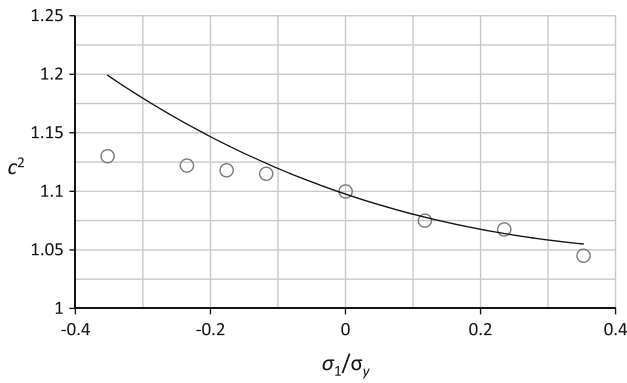
is at issue, the order of the different steps (1)–(3) above would of course be reversed. Accordingly, in such a case, the area ratio  $c^2$  is determined experimentally and  $\sigma_{res}$  calculated from Eq. (11). Following that, the quantity  $\sigma_{yind}$  is given by Eq. (10), and finally then Eq. (9) determines the residual stresses  $\sigma_1$  and  $\sigma_2$ . In order to carry through the last step, additional information about the residual stress state is of course needed and this can be found, for example, from the shape of the contact area. This feature will be discussed further below.

## 4 Results and Discussion

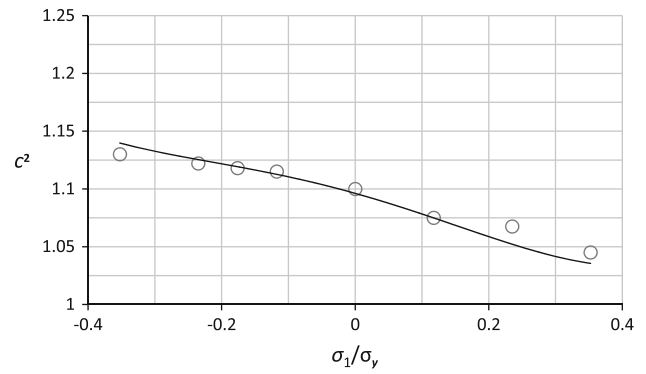
It was thought advisable not to include any new numerical (FEM) calculations in the present study as the amount of material in the literature, to be used for comparison with the present results, is quite sufficient. This includes of course the experimental results by Tsui et al. [11] but also numerical ones as sharp indentation has been studied for quite a long time using finite element methods (cf. e. g., [27–30]) and reached some maturity. In the present case, results pertinent to sharp indentation of materials with general residual stresses is at issue, and in this context, Larsson and Blanchard [31] (followed by a related study, Larsson and Blanchard [32], more directed toward the issue of hardness invariance) recently presented a finite element study with results directly suitable for a comparison with the ones derived here.

As mentioned repeatedly above, and also being the background to this study, the main drawback with the analysis by Carlsson and Larsson [15] concerns the fact that it gives low accuracy results at compressive residual stresses. This is evident from the results shown in Fig. 5 where the experimental results for uniaxial loading by Tsui et al. [11] are compared with the predictions by Carlsson and Larsson [15]. In the latter case, the quantity  $\sigma_{res}$  is determined from Eqs. (9) and (10), assuming a uniaxial stress state, and subsequently, the area ratio  $c^2$  is given by Eq. (4). The material is an aluminum 8009, which is an almost elastic-ideally plastic material. It is very much clear from the results shown in Fig. 5 that the approach by Carlsson and Larsson [15] has to be improved at compression and basically that the fundamental behavior at such stress states is not captured by this approach. It should be noted in the context of Fig. 5 that the experimental results by Tsui et al. [11] are pertinent to Berkovich indentation, while the theoretical ones in [15] are derived from cone indentation results. However, it has been shown by Larsson [33] that the difference between cone and Berkovich indentation results is small for the global properties at issue here (but not so for field variables).

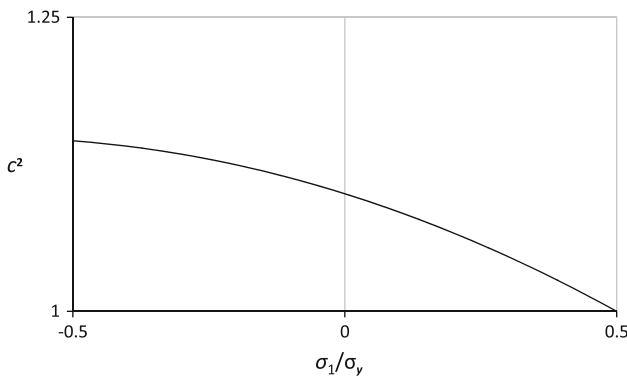




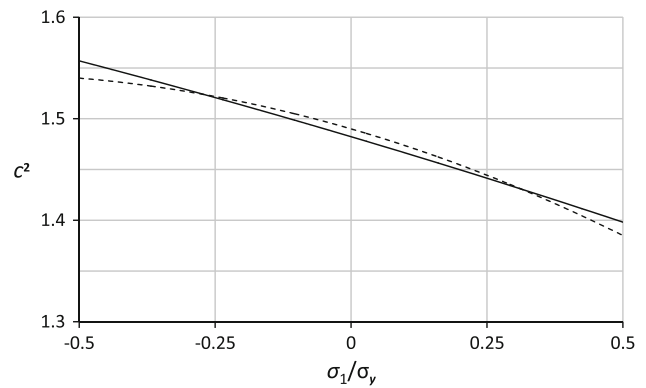
**Fig. 5** Berkovich indentation of an aluminum alloy 8009 ( $E = 82.1$  GPa,  $\nu = 0.31$ ,  $\sigma_y = 425.6$  MPa (this is the peak stress after a small amount of initial work-hardening),  $c^2(\epsilon_{res} = 0, \sigma_{res} = 0) = 1.10$ ). The area ratio  $c^2$  as function of an applied uniaxial stress (ratio)  $\sigma_1/\sigma_y$ . *Open circles* experimental results by Tsui et al. [11]. (*Solid line*), predictions by Carlsson and Larsson [15] based on Eq. (5). Results taken from Carlsson and Larsson [15]



**Fig. 7** Berkovich indentation of an aluminum alloy 8009 ( $E = 82.1$  GPa,  $\nu = 0.31$ ,  $\sigma_y = 425.6$  MPa (this is the peak stress after a small amount of initial work-hardening),  $c^2(\epsilon_{res} = 0, \sigma_{res} = 0) = 1.10$ ). The area ratio  $c^2$  as function of an applied uniaxial stress (ratio)  $\sigma_1/\sigma_y$ . *Open circles* experimental results by Tsui et al. [11]. (*Solid line*), present predictions



**Fig. 6** The area ratio  $c^2$  as function of an applied uniaxial stress (ratio)  $\sigma_1/\sigma_y$ . The qualitative functional form of the schematic curve is given by carrying through the present analysis where the quantity  $\sigma_{res}$  is determined from Eqs. (9) and (10), assuming a uniaxial stress state, and the area ratio  $c^2$  is determined by Eqs. (8) and (11)



**Fig. 8** Cone indentation of an elastic-ideally plastic material,  $c^2(\epsilon_{res} = 0, \sigma_{res} = 0) = 1.49$ ). The area ratio  $c^2$  as function of an applied uniaxial stress (ratio)  $\sigma_1/\sigma_y$ . *Dotted line* finite element results by Larsson and Blanchard [31]. (*Solid line*), present predictions

The main difference between the present approach and the one suggested by Carlsson and Larsson [15] is of course that compressive and tensile residual stresses are treated in a different manner according to Eqs (8) and (11). It could then be expected that in a uniaxial case, a curve with a qualitative functional form as the one schematically shown in Fig. 6 would be the outcome when the quantity  $\sigma_{res}$  is determined from Eqs. (9) and (10), assuming a uniaxial stress state, and the area ratio  $c^2$  is given by Eqs. (8) and (11). Indeed, this would qualitatively be in agreement with the experimental results by Tsui et al. [11], but it remains to see whether or not quantitative agreement can be found.

The issue of quantitative agreement is of course investigated further, and the validation of the present theoretical

approach is undertaken below. In doing so, the results derived here are compared with (1) experimental results [11] pertinent to a uniaxial stress field, (2) FEM results [31] pertinent to a uniaxial stress field, and (3) FEM results [31] for stress ratios  $\sigma_1/\sigma_2$  taking on other values than 0 and 1.

First of all then, in Fig. 7, the present results are compared with the ones by Tsui et al. [11]. It is obvious that there is excellent agreement between the two sets of results giving some true confidence in the present approach. Once again, it should be stated that in Fig. 7, the values from this study are derived in such a way that the quantity  $\sigma_{res}$  is determined from Eqs. (9) and (10), assuming a uniaxial stress state, and the area ratio  $c^2$  is given by Eqs. (8) and (11). It should be noted once again that the experimental results by Tsui et al. [11] are pertinent to Berkovich

indentation, while the theoretical ones in [15] are derived from cone indentation results, as discussed above.

Further comparisons in order to test the relevance of the presently suggested approach are made with finite element results presented by Larsson and Blanchard [31]. In [31], among other things, the area ratio  $c^2$  was determined at cone indentation of materials with general residual stresses. In Fig. 8, the results from [31], again pertinent to a uniaxial residual stress state, are compared with the present ones in the same manner as in Fig. 7, and obviously, the agreement is good.

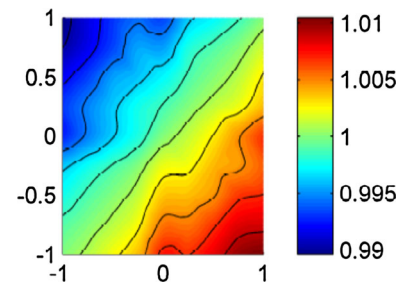
It would of course be interesting for completeness to compare the results from the present approach with results where the stress state is more complicated than the uniaxial case (or the equi-biaxial case). Larsson and Blanchard [31] presented pertinent results for this purpose for a number of different stress ratios. Results ( $c^2$  values) from the present approach were compared with the corresponding finite element results in [31]. In summary, this part of the investigation showed that good agreement (as good as in Figs. 7, 8) between the two sets of results was found when both principal residual stresses were either tensile or compressive. However, when one of the stresses were tensile and one compressive, the present predictions were not in agreement with the numerical results. This is exemplified here by presenting the results for two cases, first of all, a case when tensile stresses dominate,  $\sigma_1 = \sigma_y$  and  $\sigma_2 = \sigma_y/4$ . In this situation, the prediction from the present analysis is excellent as  $c^2$  then takes on the value 1.32, which is in very good agreement with the corresponding FEM result by Larsson and Blanchard [31] predicting  $c^2 = 1.325$ . Secondly, a case with both tensile and compressive stresses,  $\sigma_1 = \sigma_y/2$  and  $\sigma_2 = -\sigma_y/2$ , was investigated and with the present approach, using  $F = 0.52$  [remembering that  $\sigma_{\text{res}}$  is negative as given by Eqs. (9), (10)] in Eq. (8), and  $c^2 = 1.595$  was determined. Larsson and Blanchard [31] predicted the value  $c^2 = 1.49$  so clearly the two sets of results differ significantly when the two principal stresses have different sign.

This issue is of course a major drawback for the present approach. A possible solution would of course be to perform a comprehensive parameter study and determine the constant  $F$  in Eq. (8) in a tentative expression

$$F = F(\text{sign}(\sigma_1), \text{sign}(\sigma_2), \sigma_1/\sigma_2). \quad (12)$$

Such an approach is, however, left for future studies.

It should also be mentioned in the present context, that the use of a conical indenter is not ideal for determining a general residual stress state. As mentioned many times above, additional information is definitely needed as the proposed analysis will give the magnitude of the stress state but not an indication of the ratio between the principal stresses. The geometry of the contact area, as represented



**Fig. 9** Semi-axes ratio  $a_1/a_2$ , see Fig. 4, as function of the stress ratios  $\sigma_1/\sigma_y$  (horizontal axis) and  $\sigma_2/\sigma_y$  (vertical axis). Explicit values on  $a_1/a_2$  are determined by the colors on the right-hand side of the figure. The value on the Johnson [25, 26] parameter is  $\Lambda = 100$ . Results taken from Larsson and Blanchard [31] (Color figure online)

by  $a_1$  and  $a_2$  in Fig. 4, can give some guidance here. However, it was shown by Larsson and Blanchard [31] that in case of cone indentation,  $a_1$  and  $a_2$  are very weakly dependent on the principal stress ratio  $\sigma_1/\sigma_2$ , see Fig. 9. In short, it would not be possible to use the dependence shown in Fig. 9 to experimentally determine the ratio  $\sigma_1/\sigma_2$  with acceptable accuracy, and this indicates that a more complex indenter geometry, such as the Knoop indenter (relying upon the rhombic shape of this indenter), should be used for this purpose.

Finally, it should be reiterated that the present analysis is restricted to cone indentation of elastic-ideally plastic materials with residual stresses present at contact. This restriction was enforced as the main feature of interest in this investigation was to determine in an accurate manner the theoretical foundation for residual stress determination, both in tension and compression, using indentation methods (and it could be suggested that this approach is qualitatively valid also at scratch testing [34–39]). It is the intention to also include material strain-hardening effects and other more complicated indenter geometries, like the Knoop indenter as mentioned above, in future studies.

## 5 Conclusions

Cone indentation of elastic-ideally plastic materials with general residual stresses was investigated theoretically based on previous experimental and numerical findings. The aim of the investigation was to analyze the possibility of determining residual stress states using indentation techniques.

The most important conclusions given by this investigation can be summarized as follows:

- The proposed method accurately predicts the magnitude of the residual stresses when the stresses are predominantly tensile or compressive but ranging from

uniaxial to equi-biaxial. Quantitative relations for this purpose are presented and evaluated producing results of good accuracy.

- The accuracy of the predictions deteriorates when the principal residual stresses have different signs. A possible remedy for this problem is discussed.
- The proposed method predicts the magnitude of but not the ratio between the principal residual stresses. It is suggested that this feature can be determined from the geometry of the contact area. A complex indenter geometry is then advantageable, and possibly, a Knoop indenter should be relied upon for this purpose remembering the rhombic shape of this indenter.

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