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ORIGINAL PAPER

A Macro-scale Approximation for the Running-in Period

I. I. Argatov · Yu. A. Fadin

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Abstract The article presents asymptotic modeling of the running-in wear process with fixed contact zone under a prescribed constant normal load or an imposed contact displacement. The wear contact problem is formulated within the framework of the two-dimensional theory of elasticity in conjunction with Archard's law of wear. The running-in process is considered at the macro-scale level, while the micro-processes associated with roughness changes, tribomaterial evolution, and microstructural alteration in the subsurface layers as a first approximation are neglected. The setting of the steady-state regime for the macro-contact pressure evolution is chosen as the criterion to characterize the completion of running-in. Simple closed-form approximations are derived for the running-in period and running-in sliding distance. The obtained results can be used for estimating the running-in period in wear processes where the evolution of the macro-shape deviations at the contact interface plays a dominant role.

Keywords Dynamic modeling · Contact mechanics · Wear mechanisms

Nomenclature

a	Half-width of the contact zone
C	Asymptotic constant depending on the ratio H/a
c_{2r}, c_{2r+1}	Integration constants
c_m, C_m	Integration constants
d_0	Asymptotic constant
E	Young's elastic modulus
H	Thickness of the elastic layer
k	Dimensional wear coefficient in Archard's wear law
L_{in}	Running-in sliding distance
P	Line normal load in 2D contact problem
$p(x, t)$	Contact pressure
$q(x, t)$	Residual contact pressure
t	Time variable
T_{c}	Characteristic time of the tribological system
T_{in}	Running-in time period
v	Sliding speed of the punch
x	Transverse coordinate in 2D contact problem
x'	Dimensionless transverse coordinate
w	Linear wear
α_{2r}	Eigenvalues of integral equation (4)
β	Auxiliary parameter, $\beta = kv/\vartheta$
$\delta_0(t)$	Variable vertical contact displacement of the punch
δ_0	Constant vertical contact displacement of the punch
$\Delta(x)$	Macro-shape function of the punch
ϑ	Elastic constant, $\vartheta = 2(1 - \nu^2)/(\pi E)$
$\kappa = 3 - 4\nu$	Kolosov's constant for plain strain
ν	Poisson's ratio
λ_m	Eigenvalues of integral equation (12)
ξ	Coordinate integration variable
ξ'	Dimensionless coordinate integration variable
τ	Time integration variable

I. I. Argatov (✉)
Institute of Mathematics and Physics, Aberystwyth University, Ceredigion SY23 3BZ, Wales, UK
e-mail: ivan.argatov@gmail.com

Yu. A. Fadin
Laboratory of Friction and Wear, Institute for Problems in Mechanical Engineering, V.O., Bolshoy pr., 61, 199178 St. Petersburg, Russia

$\varphi_{2r}(x')$ Eigenfunctions of integral equation (4)
$\varphi_m(x')$ Eigenfunctions of integral equation (12)

1 Introduction

It is known that wear testing is a relatively time-consuming process, since the tribotests should be repeated several times for different normal loads. Moreover, in order to obtain correct values for the wear coefficient entering Archard's law of wear, the steady-state wear conditions in tribotests must be achieved, thus ignoring the initial period, which is called the running-in period. At the same time, it may be difficult to judge correctly whether a steady-state wear regime has actually been attained [1, 2]. Owing to the variety of physical, mechanical, and chemical processes that evolve simultaneously in the subsurface layers of contacting solids in friction as well as the complexity to describe them analytically, the underlying mechanisms of the running-in process remain obscure so far and aspects of predicting the running-in period are still under research [3, 4]. This implies that the problem of predicting the running-in period represents a challenging scientific and practical concern.

Running-in is one of the well-known examples of time-dependent behavior in dry or lubricated sliding tribosystems [5, 6]. Depending on loading conditions, operational conditions and material properties of contacting solids, various tribological processes (including acoustic emission accompanying the friction process [7] and wear debris generation [8]) can exhibit different time-dependent behavior and, in particular, steady long-time periodic oscillations of one or another tribological parameter. The long-period oscillations of relaxation type appearing in the wear process of metals under heavy duty sliding conditions were recently mathematically modeled in [9] with the aim of application to structural health monitoring of tribosystems.

Phenomenological mathematical models of the running-in process and its transition to the steady-state wear regime were developed in [4, 10]. Such phenomenological models employ empirical relations for the running-in wear rate, running-in period and steady-state wear rate as functions of normal load, and they require determining the best-fit fitting parameters for the model equations using the experimental wear date.

Analytical approach for wear contact problems with fixed contact area was developed by Galin [11], and lately extended in a number of publications [12, 13]. Applying an optimal control approach [14] recently developed to study the asymptotic behavior of elasto-viscoplastic structures, Peigney [15] has solved the problem of determining the asymptotic state reached by an elastic half-plane subjected to wear and submitted to a normal and tangential loading by a cyclically moving rigid indenter with the prescribed vertical displacement.

In this article, we employ asymptotic modeling approach for modeling the running-in wear process governed by Archard's law in the two-dimensional wear contact problem with fixed contact zone under a prescribed constant normal load or an imposed contact displacement. Analytical approximations for the running-in period and running-in sliding distance are obtained in a very simple closed form.

2 Sliding Wear Contact Problem with Prescribed Vertical Displacement

We consider the two-dimensional sliding contact problem for a wearable rigid punch moving along the surface of an elastic layer of relatively large thickness (see Fig. 1). In the case of fixed contact zone, the wear contact problem for determining the variable contact pressure $p(x, t)$ can be reduced to the following integral equation [16]:

$$\frac{2(1-\nu^2)}{\pi E}\int_{-a}^{a} p(\xi,t)\left(\ln\frac{H}{|x-\xi|}-d_0\right)\mathrm{d}\xi + vk\int_0^t p(x,\tau)\mathrm{d}\tau = \delta_0(t)-\Delta(x) \qquad (1)$$

Here, H is the thickness of the elastic layer, a is the half-width of the contact zone, E and ν are Young's modulus and Poisson's ratio of the elastic layer, k is the dimensional wear coefficient in Archard's wear law [17]; $\delta_0(t)$ is the vertical contact displacement of the punch, the function $\Delta(x)$ describes the macro-shape of the punch, v is the sliding speed of the punch, d_0 is an asymptotic constant given by the following formulas (see for e.g., [18]) with $\kappa = 3 - 4\nu$ being Kolosov's constant for plain strain:

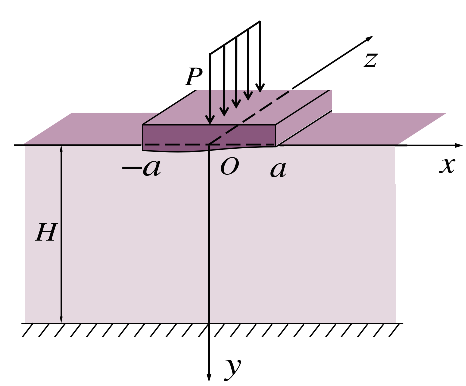

Fig. 1 Schematic representation of two-dimensional contact

$$d_0 = \int_0^\infty \frac{1}{u}(1 - e^{-u} - L(u))du,$$
$$L(u) = \frac{2\kappa \sinh(2u) - 4u}{2\kappa \cosh(2u) + 1 + \kappa^2 + 4u^2}$$

Note that the asymptotic constant d_0 is determined from the singular solution of the two-dimensional elastic problem for an elastic strip loaded by a point force. In the case of the elastic strip firmly attached to a rigid base, the value of d_0 depends on Poisson's ratio ν (see Fig. 2). In particular, for $\nu = 0.3$, we have $d_0 \approx 0.527$. In the case, when the elastic strip is in frictionless contact with the rigid base, $d_0 \approx 0.352$ [16].

Introducing the notation

$$C = \ln\frac{H}{2a} - d_0, \quad \vartheta = \frac{2(1-\nu^2)}{\pi E}, \quad \beta = \frac{k\nu}{\vartheta}$$

we rewrite Eq. 1 as follows:

$$\int_{-a}^{a} p(\xi, t)\left(\ln\frac{2a}{|x-\xi|} + C\right)\mathrm{d}\xi + \beta\int_0^t p(x,\tau)\mathrm{d}\tau = \frac{1}{\vartheta}(\delta_0(t) - \Delta(x)) \tag{2}$$

Following Aleksandrov et al. [16], we consider the sliding contact problem (2) under the assumption of the prescribed constant vertical displacement, i.e., $\delta_0(t) = \delta_0$, where δ_0 is a constant. The solution of Eq. 2 can be represented in the form

$$p(x,t) = \sum_{r=1}^{\infty}\left[c_{2r}\varphi_{2r}\left(\frac{x}{a}\right)\exp\left(-\frac{\beta}{a}\alpha_{2r}t\right) + c_{2r+1}\varphi_{2r+1}\left(\frac{x}{a}\right)\exp\left(-\frac{\beta}{a}\alpha_{2r+1}t\right)\right] \tag{3}$$

where the eigenvalues α_{2r} and eigenfunctions $\varphi_{2r}(x')$ are determined as solutions of the homogeneous integral equation

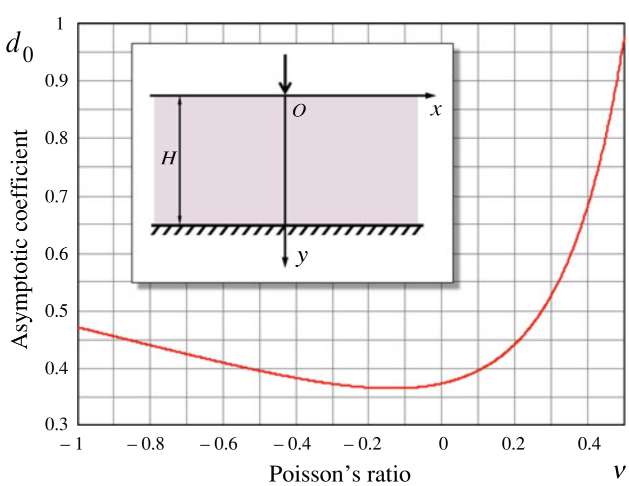

Fig. 2 Asymptotic constant d_0 as function of ν

$$\varphi_{2r}(x') - \alpha_{2r}\int_{-1}^{1}\left(\ln\frac{2}{|x'-\xi'|} + C\right)\varphi_{2r}(\xi')\,\mathrm{d}\xi' = 0 \tag{4}$$

where x' and ξ' are dimensionless variables, $x', \xi' \in (-1, 1)$.

If the function $\Delta(x)$ is even, then restricting ourselves to the first term in the expansion (3), we get

$$p(x,t) \cong c_2\varphi_2\left(\frac{x}{a}\right)\exp\left(-\frac{\beta}{a}\alpha_2 t\right) \tag{5}$$

where the coefficient c_2 is determined from the initial condition at $t = 0$. Formula (5) becomes more accurate as the time t increases. It is clear that formula (5) shows that the pressure distribution tends toward zero.

Remark 1 Let us discuss the tribological meaning of the exponent in Eq. 5. From the viewpoint of the approach used in [9], the negative value of the exponent's argument determines a relaxation process for the contact pressures under the punch's base. According to the notation used in Eq. 2, we have

$$\frac{\beta}{a}\alpha_2 t = \frac{k\nu}{\vartheta a}\alpha_2 t$$

Since both sides of the equality above are dimensionless as well as the eigenvalue α_2, the dimensional quantity $T_{\rfloor} = \vartheta a/(k\nu)$ turns out to represent a characteristic time of the tribological system under consideration. It should be emphasized that the characteristic time T_c depends not only on the material characteristics ϑ and k, but also on a characteristic size of the contact zone, a, and the sliding speed of the punch ν, which is an operational characteristic of the tribological system.

Remark 2 According to Archard's wear law, the linear wear rate is given by

$$\frac{dw}{dt} = kp\nu$$

where w is the linear wear defined as the ratio of the worn volume to the contact area. In view of Eq. 5, in the case of the wear contact problem with prescribed displacement, the linear wear rate will be exponentially decaying function. Figure 3 shows the behavior of the normalized dimensionless relative wear rate with the increasing relative time for different values of the eigenvalue α_2. Observe that the initial wear rate is determined by the normal contact load P that, in turn, is determined by the value of the vertical displacement δ_0. In particular, for a flat-ended indenter, the contact load P is related to the displacement δ_0 by the following equation [16]:

$$P = \frac{\delta_0}{\vartheta(\ln(2H/a) - d_0)}$$

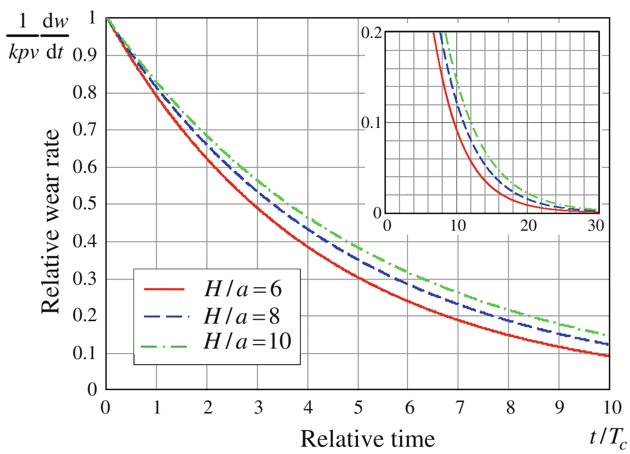

Fig. 3 Wear rate variation with time in the wear contact problem with prescribed displacement

Thus, the experimental data for different values of the relative layer thickness H/a should be compared after the normalization as it is shown in Fig. 3.

3 Sliding Wear Contact Problem with Prescribed Normal Load

Following Komogortsev [13], we consider the sliding contact problem (2) under the assumption of prescribed normal load P. We note the misprints in article [13] due to the assumption $a = 1$ made in the intermediate calculations.

In this case, the following equilibrium equation takes place:

$$\int_{-a}^{a} p(\xi, t)\mathrm{d}\xi = P \tag{6}$$

Setting $t = 0$ in Eq. 2 and subtracting the equation obtained from (2), we will have

$$\int_{-a}^{a} (p(\xi,t) - p(\xi,0)) \ln\frac{H}{|x-\xi|}\mathrm{d}\xi + \beta\int_{0}^{t} p(x,\tau)\,\mathrm{d}\tau = \frac{1}{\vartheta}(\delta_0(t) - \delta_0(0)) \tag{7}$$

Eliminating the unknown right-hand side from (7) by integrating over the contact zone (see [13] for the complete detail), we arrive at the equation

$$\int_{-a}^{a}\left\{\ln\frac{H}{|x-\xi|} - \frac{1}{2a}\int_{-a}^{a}\ln\frac{H}{|x-\xi|}\mathrm{d}x\right\}(p(\xi,t) - p(\xi,0))\,\mathrm{d}\xi + \beta\int_{0}^{t} p(x,\tau)\mathrm{d}\tau = \frac{\beta}{2a}Pt \tag{8}$$

The solution of Eq. 8 can be represented as follows:

$$p(x,t) = \frac{P}{2a} + q(x,t) \tag{9}$$

where it is assumed that

$$\lim_{t\to\infty} q(x,t) = 0,\ x \in (-a,a); \qquad \int_{-a}^{a} q(x,t)\mathrm{d}x = 0, \quad t \geq 0 \tag{10}$$

Following [13], we represent the function $q(x, t)$ by the series

$$q(x,t) = \sum_{m=1}^{\infty} c_m\varphi_m\left(\frac{x}{a}\right)\exp\left(-\frac{\beta}{a}\lambda_m t\right) \tag{11}$$

where the eigenvalues λ_m and eigenfunctions $\varphi_m(x')$ are determined as solutions of the homogeneous integral equation

$$\varphi_m(x') - \lambda_m\int_{-1}^{1}\left\{\ln\frac{1}{|x'-\xi'|} - \frac{1}{2}\int_{-1}^{1}\ln\frac{1}{|x'-\xi|}\mathrm{d}x'\right\}\varphi_m(\xi')\mathrm{d}\xi' = 0 \tag{12}$$

where x' and ξ' are dimensionless variables, x', $\xi' \in (-1, 1)$. Note that when the function $\varphi_m(x')$ satisfies the integral equation (12), the following condition [see (10)] is automatically satisfied:

$$\int_{-1}^{1} \varphi_m(\xi')\mathrm{d}\xi' = 0 \tag{13}$$

Following form (9), the coefficients c_m of series (11) are found from the initial condition

$$p(x,0) - \frac{P}{2a} = \sum_{m=1}^{\infty} c_m\varphi_m\left(\frac{x}{a}\right) \tag{14}$$

as follows:

$$c_m = \frac{1}{a}\int_{-a}^{a}\left(p(x,0) - \frac{P}{2a}\right)\varphi_m\left(\frac{x}{a}\right)\mathrm{d}x \tag{15}$$

It is assumed that the eigenfunctions $\varphi_m(x')$ satisfy the normalization condition

$$\int_{-1}^{1}\varphi_m^2(x')\mathrm{d}x' = 1(m = 1, 2, \ldots) \tag{16}$$

Substituting the expressions (9) and (11) into the left-hand side of Eq. 7, we derive

$$\delta_0(t) - \delta_0(0) = \frac{kvP}{2a}t - \vartheta \sum_{m=1}^{\infty} C_m \left(1 - \exp\left(-\frac{\beta}{a}\lambda_m t\right)\right) \tag{17}$$

where the following notation was introduced

$$C_m = \frac{c_m}{2a} \int_{-a}^{a} \int_{-a}^{a} \varphi_m\left(\frac{\xi}{a}\right) \ln \frac{R}{|x-\xi|} \mathrm{d}x\, \mathrm{d}\xi \tag{18}$$

Restricting ourselves to the first terms in the expansions (11) and (17), we get

$$p(x,t) \cong \frac{P}{2a} + c_1 \varphi_1\left(\frac{x}{a}\right) \exp\left(-\frac{\beta}{a}\lambda_1 t\right) \tag{19}$$

$$\delta_0(t) - \delta_0(0) \cong \frac{kvP}{2a}t - \vartheta C_1 \left(1 - \exp\left(-\frac{\beta}{a}\lambda_1 t\right)\right) \tag{20}$$

where the coefficients c_1 and C_1 are determined by formulas (15) and (18).

Formulas (19) and (20) become more accurate as the time t increases. We emphasize that the asymptotic formulas (19) and (20) differ from the approximations derived in [13]. In fact, the results obtained by Komogortsev [13] follow from formulas (19) and (20) by substituting the approximation for $\varphi_1(x')$ obtained by Galerkin's method in terms of the Chebyshev polynomials of the first kind. It is interesting that the eigenvalues λ_m do not depend on the elastic layer's thickness. Aleksandrov et al. [16] obtained that $\lambda_1 \approx 1.0996$.

4 Approximation for the Running-In Period

Formula (19) shows that the pressure distribution tends toward uniform. Rewriting formula (19) in the form

$$p(x,t) \cong \frac{P}{2a}\left\{1 + \frac{2ac_1}{P}\varphi_1\left(\frac{x}{a}\right) \exp\left(-\frac{\beta}{a}\lambda_1 t\right)\right\} \tag{21}$$

we can estimate the factor $2ac_1/P$ in (21) by applying Schwarz's inequality as follows:

$$\begin{aligned} \frac{2ac_1}{P} &= \frac{1}{a}\int_{-a}^{a}\left(\frac{2a}{P}p(x,0) - 1\right)\varphi_1\left(\frac{x}{a}\right)\mathrm{d}x \\ &\leq \sqrt{\frac{1}{a}\int_{-a}^{a}\left(\frac{2a}{P}p(x,0) - 1\right)^2 \mathrm{d}x}\sqrt{\int_{-1}^{1}\varphi_1^2(x')\mathrm{d}x'} \end{aligned} \tag{22}$$

Substituting, for e.g., the Hertz distribution $p(x,0) = (2P/\pi a)\sqrt{1 - x^2/a^2}$ into the first integral (22) and taking into account the normalization condition (16), we obtain the estimate $2ac_1/P \leq \sqrt{1 + 4/(3\pi^2)} \approx 1.065$.

Let for $t = T_{\text{in}}$, the second term in the braces in (21) is less than 1% in the L_2-norm sense. Then, taking into account the estimate (22), we obtain the approximation

$$T_{\text{in}} = \frac{5}{\lambda_1}\frac{\vartheta a}{kv} \tag{23}$$

where $\lambda_1 \approx 1.0996$.

The main conclusion that can be drawn from formula (23) is that the running-in time, T_{in}, does not depend on the load level.

We can express the running-in time in terms of the running-in sliding distance $L_{\text{in}} = vT_{\text{in}}$. From (23), it follows that

$$L_{\text{in}} = \frac{5}{\lambda_1}\frac{\vartheta a}{k} \tag{24}$$

In the case of an imposed contact displacement, according to (5), we arrive at the following approximations:

$$T_{\text{in}} = \frac{5}{\alpha_2}\frac{\vartheta a}{kv}, \quad L_{\text{in}} = \frac{5}{\alpha_2}\frac{\vartheta a}{k} \tag{25}$$

Here, the eigenvalue α_2 is given by Table 1. It should be noted that α_2 depends on the relative size of the contact region.

In the case of two elastic solids in contact, following the standard procedure in contact mechanics [19], we can generalize formulas (23–25) by setting

$$\vartheta = \frac{2(1-\nu_1^2)}{\pi E_1} + \frac{2(1-\nu_2^2)}{\pi E_2}$$

where E_1, E_2 and ν_1, ν_2 are elastic characteristics of the contacting solids.

It should be emphasized that formulas (23–25) are derived from the asymptotic solution of the sliding wear contact problem with fixed contact zone. In the case of increasing contact zone, formulas (23–25) with a standing for the initial half-width of contact should provide the lower bounds for the running-in time and the number of running-in cycles.

5 Discussion

First of all, it is interesting to observe that in accordance with Eqs. 23 and 25, the running-in period under an imposed contact displacement is about five times greater than that under a prescribed constant normal load. In fact,

Table 1 The first eigenvalue in the wear contact problem with prescribed displacement

H/a	6	7	8	9	10
α_2	0.239	0.223	0.210	0.200	0.192

the ratio λ_1/α_2 ranges from 4.6 (for $H/a = 6$) to 5.7 (for $H/a = 10$). At that, Eq. 25 implies that the running-in period under an imposed contact displacement depends on the relative layer's thickness H/a (increasing with its increase). This circumstance was overlooked in [16], and, to the best of the authors' knowledge, was not subjected to experimental verification.

In the case of a prescribed constant normal load, the experimental setup employing the moving pin with a spiral wear track can be adopted from [2], where a test methodology was proposed for transient wear tests. The case of an imposed contact displacement requires a specific approach, because the displacement δ_0 should be imposed quasistatically. In the latter case, a general solution constructed in [16] for displacements linearly varying with time can be used in developing the corresponding test methodology for verifying the analytical formulas (25). It should be underlined that though two-dimensional wear contact problems are considered in the present study, the obtained results can be applied for the experimental verification in three-dimensional tests. Indeed, as it follows from the dimensional analysis, formulas (23–25) will hold true also in three-dimensional wear contact problems with the only difference that λ_1 and α_2 are eigenvalues of the corresponding eigenvalue problems for the integral operator of three-dimensional contact problems.

It should be underlined that the mathematical model based on the integral equation (1) is developed at the macro-scale level. This means that the tribological processes at the micro-scale level associated with roughness changes, tribomaterial evolution, and microstructural alteration in the subsurface layers as a first approximation are neglected in the final results represented by (23)–(25).

Finally, the exponential behavior observed in Eqs. 5, 19, and 21 agrees with the phenomenological mathematical models of wear evolution presented in [2, 4, 20]. The problem of wear evolution in the running-in period is very complicated for mathematical modeling, if the tribological processes at the micro-scale level should be taken into account. The phenomenological mathematical models used in [2, 4] are based on a range of experimental results for specific types of materials, but they provide little information about the mechanisms of wear. On the contrary, the presented mathematical model and analytical approximations are derived from first principles of elasticity theory in the framework of Archard's law.

6 Conclusion

A mathematical model has been developed to estimate the running-in period in the two-dimensional wear contact problem with fixed contact zone in the framework of Archard's law under a prescribed constant normal load or an imposed contact displacement. The exponents in Eqs.5, 19, 21 can be explained from the expected tribological behavior [4, 21]. The inverse proportionality of the characteristics of running-in period (23–25) to the wear coefficient agrees with the experimentally observed behavior [2]. Formula (23) for estimating the running-in period at the macro-scale level constitutes the main result of this study.

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Appendix: Determining Eigenvalues in the Wear Contact Problem with Prescribed Displacement

Following Aleksandrov et al. [16], we employ the biorthogonal expansion

$$\ln\frac{2}{|x'-\xi'|} + C = 2\left[\ln 2 + \sum_{l=1}^{\infty}\frac{T_l(\xi')T_l(x')}{l}\right] + C$$

where $T_l(x') = \cos(l\arccos x)$, $l = 1, 2, \ldots$, are the Chebyshev polynomials.

Using the representation

$$\varphi_{2r}(x') = \sum_{m=0}^{\infty} a_m^{(r)}\tilde{P}_{2m}(x'), \quad \tilde{P}_{2m}(x') = \sqrt{\frac{4m+1}{2}}P_{2m}(x')$$

where $P_{2m}(x')$ are the Legendre polynomials, one can derive the following infinite linear algebraic system for determination of the coefficients $a_m^{(r)}$:

$$a_m^{(r)} = \alpha_{2r}\sum_{l=m}^{\infty}\frac{b_{mn}}{l^*}\sum_{s=0}^{l} b_{sl}a_s^{(r)} \quad (m = 0, 1, \ldots) \tag{26}$$

Here we used the notation

$$l^* = l \geq 1, \quad 0^* = (2\ln 2 + C)^{-1},$$

$$b_{ml} = \int_{-1}^{1}\tilde{P}_{2m}(\xi')T_{2l}(\xi')\,d\xi'$$

Finally, we transform the system (26) into the following one:

$$\alpha_{2r}^{-1}a_0^{(r)} = \left((2\ln 2 + C)b_{00}^2 + \sum_{n=1}^{\infty}\frac{b_{0l}^2}{l}\right)a_0^{(r)} + \sum_{s=1}^{\infty}a_s^{(r)}\sum_{l=s}^{\infty}\frac{b_{0l}}{l}b_{sl} \tag{27}$$

$$\alpha_{2r}^{-1} a_m^{(r)} = \sum_{l=m}^{\infty} \frac{b_{ml}}{l} b_{0l} a_0^{(r)} + \sum_{s=1}^{m} a_s^{(r)} \sum_{l=m}^{\infty} \frac{b_{ml}}{l} b_{sl} + \sum_{s=m+1}^{\infty} a_s^{(r)} \sum_{l=s}^{\infty} \frac{b_{ml}}{n} b_{sl} \quad (m \geq 1) \tag{28}$$

The results of numerical calculations based on the homogeneous system (27), (28) are presented in Table 1. We note the computational misprints in article [16].

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