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Approaches to the Impure Logic of Ground

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Abstract
This paper is concerned with the semantics for the logics of ground that derive from a slight variant GG of the logic of Fine (2012b) that have already been developed in deRosset and Fine (2023). Our aim is to outline that semantics and to provide a comparison with two related semantics for ground, given in Correia (2017) and Krämer (2018a). This comparison highlights the strengths and difficulties of these different approaches.

Keywords Impure logic of ground · Truthmaker semantics · Logic of Ground · Ground

This paper concerns the semantics for the logics of ground deriving from a slight variant GG of the logic of Fine (2012b) that have already been developed in deRosset and Fine (2023). Our aim is to outline that semantics and to provide a comparison with two related semantics for ground, given in Correia (2017) and Krämer (2018a). This will serve to highlight the strengths and difficulties of these different approaches. In particular, it will show how deRosset and Fine's approach has a greater degree of flexibility in its ability to acccommodate different extensions of a basic minimal system of ground. We shall assume that the reader is already acquainted with some of the basic work on ground and on the framework of truthmaker semantics. Some background material may be found in Fine (2012b, 2017a, 2017b).

1 The Selection Space Semantics

We shall, first of all, find it helpful to characterize the selection space semantics of deRosset and Fine (2023) for the logic of ground by detailing how it differs from the more standard state space semantics. The standard semantics appeals to three key ideas, two metaphysical and one semantic. In the background metaphysics, it is presupposed that we are given a collection of *states* and that one state may be *part* of another, where the parthood relation is usually taken to be bounded complete in the sense that every set of states S which is bounded by the parthood relation has a *least* such bound $\sup S$. The key semantic idea is the idea of *exact verification* of a sentence by a state. Intuitively, an exact verifier for A makes A true (or would, if the verifier were to obtain); and the exactness of an exact verifier consists in the fact that the entirety of the state has to participate in the verification. It must be possible to see each part of the state as playing a role in making the sentence true and it is also allowed that the verifiers and falsifiers of a sentence may be impossible states, states that cannot possibly obtain. Given the exact verifiers of atomic sentences and given their exact falsifiers (i.e. the verifiers of their negations), we can recursively specify the exact verifiers and falsifiers of more complex sentences. Identifying the truth-condition of a sentence with its set of exact verifiers, its falsity-condition with its set of exact falsifiers and its content with the pairing of its truth- and falsity-conditions, we thereby obtain a recursive specification of the truth- and falsity-conditions and content of each sentence.

The *selection space semantics* for the impure logic of ground shares the general form of the state space semantics, but differs from it in a number of key respects. Whereas the truth- and falsity-conditions assigned to sentences under the standard semantics are defined by appeal to the notion of exact verification, they are assigned *directly* under the selection space semantics, not via their truth- and falsity-makers. Thus truth-conditions in the present semantics are the counterpart of sets of truth-makers in the standard semantics but should not, of course, themselves be taken to be the counterpart of truth-makers. Likewise, the semantic operations

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of conjunction and disjunction - which deRosset and Fine (2023) dub *combination* and *choice* - are also taken to be primitive. It is through this more direct approach that the selection space semantics achieves the desired flexibility in the assignment of semantic values.

A second semantic difference concerns the treatment of negation. In standard state space semantics, $\neg A$'s verifiers are just the falsifiers of A, and $\neg A$'s falsifiers are the verifiers of A. As a result, $\neg\neg A$ gets the same truth- and falsity-conditions as A. But the kind of logic of ground pertinent for selection space semantics requires that $\neg\neg A$ never receive the same semantic value as A, since it is axiomatic in GG that A strictly grounds $\neg\neg A$, and that nothing strictly grounds itself.

Selection space semantics accommodates the distinction between the semantic values of A and $\neg\neg A$ by requiring that the semantic value for the latter be "raised." In particular, though the combination of a single semantic value is identified with the choice of that same semantic value, the singleton combination of A's semantic value always yields something new. This marks a structural difference from state space semantics, in which the application of the semantic analogues of conjunction or disjunction to the single truth-condition a for A is just a itself.[1]

This brings us to the final difference. As with the standard state space semantics, the grounds for A in selection space semantics are given, intuitively, by what it takes for A to be true. Thus, the grounds for A are a function of the truth-condition for A, and so DeMorgan equivalents, which have the same truth-condition, will have the same grounds. Since the falsity-condition for A plays no role in determining what grounds A, we might say that what grounds A has a "positive bias." But the pertinent logics of ground allow for distinctions among DeMorgan equivalents in what they ground. For instance, though $\neg(A \vee B)$ and $(\neg A \wedge \neg B)$ have the same truth-condition, the system GG requires the former, but not the latter, to ground $\neg\neg\neg(A \vee B)$. Thus, what A grounds may be sensitive to A's falsity-condition, and so lacks a positive bias. The difference in falsity-conditions between DeMorgan equivalents must somehow figure into the specification of the truth-condition for a complex sentence like $\neg\neg\neg(A \vee B)$.

The selection space semantics solves this problem by assigning contents to sentences that comprise both truth- and falsity-conditions, and then supposing that combination and choice are operations, not on conditions, but on contents. Thus the truth-condition for $(A \wedge B)$ will be the combination of the respective *contents* (not truth-*conditions*) of A and B, the truth-condition for $(A \vee B)$ will be the choice of the respective contents of A and B, the falsity-condition for $\neg A$ will be the unit combination of the content of A, and similarly for the other cases. There is thus an interplay between conditions and contents, with contents formed through the pairing of conditions and conditions formed through the combination and choice of contents. So, $\neg(A \vee B)$ may ground $\neg\neg\neg(A \vee B)$ even though its DeMorgan equivalent $(\neg A \vee \neg B)$ does not, since the truth-condition for $\neg\neg\neg(A \vee B)$ will be the singleton combination of the *content* of the former, which may differ from the content of the latter. The difference in content between DeMorgan equivalents induces a difference in truth-conditions one level up. By contrast, the semantic analogues of conjunction and disjunction in standard statespace semantics are more uniform in their application: they operate on conditions to yield further conditions.

2 The Interpretation of Ground

It remains to interpret the notion of ground. As is standard in treatments of the logic of ground, we deploy two orthogonal distinctions among grounding connections: they may be either *strict* or *weak*, and either *partial* or *full*. Further explanation can be found in Fine (2012a, 2012b).

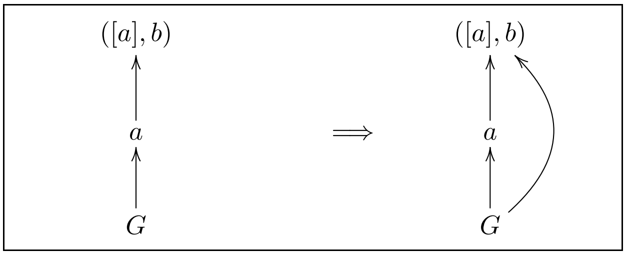
deRosset and Fine (2023) interpret these various claims of ground by appeal to a semantic analogue they call *selection*. They start with a basic notion of an *immediate selection* from a choice or combination. Letting '+' indicate choice and '.' combination, a content a is an immediate selection from any choice of contents $[b + c + \cdots + a + \cdots]$, and $a, b, c, \ldots$ together are an immediate selection from their combination $[a.b.c. \cdots]$. (This reflects the way in which choice is disjunctive and combination conjunctive.) This notion of immediate selection is then used to define further selections, which are the semantic analogues of the different connections of ground. In keeping with positive bias, any immediate selection from the truth-condition of A is ipso facto a selection from the content of A. So, if a is the content of A, then A is a strict ground of each of $\neg\neg A$ (whose truth-condition is $[a]$) and $(A \vee B)$ (truth-condition: $[a + b]$). Similarly, A, B gets to be a strict ground of $A \wedge B$ since (if b is, in addition, the content of B) a, b is an immediate selection from the conjunction's truth-condition $[a.b]$. This gives us a semantic analogue of the notion of unmediated ground ((deRosset 2017), (Fine 2012b, pp. 50-1), (Fritz 2022; Litland 2015)).[2]

[1] The semantic analogue of conjunction for truth-conditions in standard statespace semantics is *component-wise fusion* of truthmakers, and the analogue of disjunction is set-theoretic *union*; see Fine (2017a).

[2] Note, however, that combinations need not be uniquely decomposable into immediate selections, since there is no constraint forbidding the identification of the combination of some contents with the choice or combination of some others. This blocks the inconsistency result of Fritz (2022). In fact, such identifications are crucial to the proof of completeness in deRosset and Fine (2023).

There are two ways to obtain further selections from these basic selections. The first is by ascending the hierarchy of singleton combinations:

Ascent Whenever G is a selection from a content a, it is also a selection from any content of the form $([a], b)$:



Also, whenever G is a selection from a content of the form $([a], b)$, deRosset and Fine (2023) say that it is a *weak selection* from a. This gives us a semantic analogue of full, weak ground. So, ascending the hierarchy amounts to inferring from Δ's being a strict ground of something that it is also a weak ground.

The second way of obtaining further selections is by applying a CUT rule. Say that the contents G are collectively a strict (or weak) selection from the contents $H = \{h_1, h_2, \dots\}$ iff G can be split up into subsets $G_1, G_2, \dots$ such that G_1 is a strict (weak) ground of h_1, G_2 is a strict (weak) ground of $h_2, \cdots$. Thus the contents G are, collectively, a distributive selection from H. The CUT principle then states:

Cut if G is a weak selection from H and H a strict selection from a then G is a strict selection from a, and if G is a strict selection from H and H a weak selection from a then G is a strict selection from a:

$$G \leq H < a \quad \vee \quad G < H \leq a \qquad \Rightarrow \qquad G < a.$$

A conception of ground that appeals only to immediate selections, ASCENT, and CUT is not very informative. We have merely required that strict and weak ground be transitive (or, more generally, subject to CUT), and that weak ground be reflexive and entailed by strict ground. But it is consistent with all this that strict ground and weak ground should be identical. So, deRosset and Fine (2023) need to impose constraints to get an interpretation of grounding claims corresponding to GG. They do have the makings, however, of a definition of weak ground in terms of strict ground: a weak selection from a is just a strict selection from some content of the form $([a], b)$, and it is easy to show that strict selection from any content of that form entails strict selection from all contents of this form. Since the content of $\neg\neg A$ has the relevant form, strictly grounding $\neg\neg A$ is necessary and sufficient for weakly grounding A. Thus, deRosset and Fine's assumptions warrant the following definition of weak ground in terms of strict:

(W/S) Δ weakly grounds A iff Δ strictly grounds $\neg\neg A$ (deRosset and Fine 2023, p. 423).

(W/S) is, in effect, a kind of maximality principle. We know that the content a of A is a strict selection from the content of $\neg\neg A$. (W/S) says, in effect, that a is the maximal such content, in the sense that any strict selection from the content of $\neg\neg A$ is a weak selection from a. One can do no better than A, so to speak, in grounding $\neg\neg A$.

There is another assumption that may plausibly be taken to relate weak and strict selection. Say that a is a *weak partial* selection from b if it is one of the items in a weak selection from b and that a is a *strict partial* selection from b iff a is a weak partial selection from b but b is not a weak partial selection from a; and say that the weak selection G from b is *irreversible* iff b is not a weak partial selection from any item of G. The assumption then states:

Irreversibility Any irreversible weak selection is a strict selection (where the corresponding ground-theoretic principle is that any irreversible weak ground is a strict ground) deRosset and Fine (2023, p. 424).

We might take the converse:

Any strict selection from an item is an irreversible weak selection

as an additional assumption. Alternatively, it might be derived from some further assumptions. For suppose the contents G are a strict selection from b. By the above principle of ASCENT, G is a weak selection from b. Now suppose, for *reductio*, that b is a weak partial selection from some item w in G. By CUT, b is a strict selection, on its own or with other items, from b. But this, given:

Non-Circularity No item is part of a strict selection of itself

is a contradiction.

These assumptions justify for deRosset and Fine (2023) the following definition of strict ground in terms of weak ground:

(S/W) Δ strictly grounds A iff Δ irreversibly weakly grounds A (deRosset and Fine 2023, p. 424).

Thus, given these various assumptions weak and strict ground will be inter-definable.

There are two other assumptions deRosset and Fine (2023) need to make, connecting weak and strict selection to combination and choice:

Maximality Any items which constitute a strict selection from $[a.b.\cdots]$ will constitute a weak selection from $a, b, \ldots$;

> Any items which constitute a strict selection from $[a + b + \cdots]$ will constitute a weak selection from some subset of $a, b, \ldots$. (deRosset and Fine 2023, p. 424)

These assumptions generalize the semantic analogue of the previous maximality principle for $\neg\neg A$.

In summary, deRosset and Fine's (2023) semantics for the impure logic of ground appeals to a selection space of contents and conditions, with operations of choice and combination taking sequences of contents to conditions. Choice and combination are constrained so that the singleton combination $[a]$ is identical to the singleton choice of a. Strict selection is defined by appeal to immediate selections from choices and combinations, ASCENT, and CUT. Choice and combination are constrained to obey IRREVERSIBILITY and MAXIMALITY. A model interprets a language suitable for expressing grounding claims by mapping sentences of the language to contents, and interpreting grounding claims as selection relations among the contents. The result is a logic in which weak and strict ground are interdefinable in the ways required by GG. In fact, GG is sound and complete for this semantics. Details of the semantics are given in an appendix and further developed in deRosset and Fine (2023).

We turn now to a comparison of this approach with those of Krämer (2018a; b) and Correia (2017). What these three approaches most significantly have in common is their conformity to the basic structural rules of the pure logic of ground in Fine (2012a) and the basic introduction and elimination rules for the truth-functional connectives of the impure logic of ground in Fine (2012b). Beyond that, there are some further points of contact and several points of contrast, largely relating to (i) the underlying conception of propositional content (where for us a propositional content is the pairing of a truth-condition with a falsity-condition), (ii) the semantical treatment of the truth-functional connectives, (iii) the account of strict ground and its relation to weak ground, and (iv) the resulting logic of ground.

3 Correia's Approach

Correia (2017) works with a very fine-grained conception of propositions; they essentially have the same structure as formulas but for the fact that conjunction and disjunction are taken to be commutative (519). He assumes, in particular, that the classes of disjunctive, conjunctive and negative propositions are pairwise disjoint. Such a fine-grained view is compatible with deRosset and Fine's (2023) approach but is not required, since, as we have already noted, the selection space semantics allows a range of further propositional identities to hold. It can allow, for example, for $(A \vee B)$, $(A \wedge A)$ and $\neg\neg A$ to be ground-theoretically equivalent when B weakly grounds A.

For Correia, the semantics for the truth-functional connectives is given by primitive algebraic operations on propositional contents that correspond to the various connectives whilst, for deRosset and Fine, these operations are explained in terms of the underlying operations of combination and choice.

When it comes to strict ground, as with the connectives, Correia posits a semantic primitive. But it is a simple notion of ground that merely connects simple propositions (atomic propositions or their negations); and, given the simple notion, he then shows how it can be used to define a general notion of ground, that is applicable to all propositions whatever (520). deRosset and Fine's approach is quite different. The notion of ground is not given externally, so to speak, but is defined, via the mechanism of selection, on the basis of the internal structure of the propositions.

Correia adopts the following characterization of weak ground in terms of strict[3]:

> Some propositions weakly ground a given proposition iff either (i) they are all ground-theoretically equivalent to the proposition or (ii) they all strictly ground the proposition or (iii) some are ground-theoretically equivalent to the proposition and the rest strictly ground the proposition (2017, p. 516).

We can see this definition as arising from the following line of thought: all that weak ground essentially adds to strict ground is the fact that a proposition is to ground itself; add this fact to the strict grounding facts, close under chaining and ground-theoretic equivalence, and we get the weak notion.

However, it is not clear that this is an acceptable line of thought, since we would like to be able to say that, for

[3] Although Correia's logic embodies this definition (516), it should be noted that he is in a position to accept the previous definition (W/S) of weak ground and also the previous definition (S/W) of strict ground.

distinct bodies x, y, and z, x being of the same mass as y and y the same mass as z weakly grounds x being of the same mass as z. But neither x being of the same mass as y nor y being of the same mass as z is ground-theoretically equivalent to x being of the same mass as z, nor do they strictly ground x being of the same mass as z. Or, we would like to be able to say its being chilly, its being windy, and its being chilly, windy, and sunny weakly grounds its being chilly, windy, and sunny, and yet its being chilly and its being windy do not strictly ground its being chilly, windy and sunny. Thus there may be more to what weak grounding adds to strict ground than just identity or equivalence.

Of course, Correia could just stipulate that this is what he *means* by weak ground. But then the elimination rules for negation (and also the other connectives) would, from the present point of view, no longer be valid. For plausibly, x being of the same mass as y and y being of the same mass as z will strictly ground that $\neg\neg$(x is the same mass as z) even though, for Correia, they do not weakly ground that x is of the same mass as z. We see from such examples that Correia's definition of weak ground is not without its consequences and that it will lead, in conjunction with the elimination rules, to a very severe restriction on the notion of strict ground. deRosset and Fine's semantics, by contrast, is built around the idea that neither the weak nor the strict notions are to be restricted in this way.

There are a number of relatively superficial differences between Correia's logic of ground and GG. He adopts strict ground, weak ground, ground-theoretic equivalence and their negations as primitives in his formal language, whilst deRosset and Fine adopt strict and weak full ground and strict and weak partial ground as primitives and do not allow these notions to be negated (which makes it somewhat harder for them to establish completeness). He adopts, moreover, a view of sentence-letters under which they stand for atomic propositions, those which are not negations or conjunctions or disjunctions, whilst deRosset and Fine adopt a view under which they stand for arbitrary propositions. Thus, what should be taken to correspond to the derivable inferences of GG are the derivable inferential-schemes of his system, so that the derivability of $S \vdash T$ in GG would correspond to the derivability of each substitution-instance of $S \vdash T$ for him. Even under this correspondence, however, there will be a mismatch, as we shall see, between the two systems.

For Correia's approach requires the purely structural principle that if Δ, A fully weakly grounds A, for any non-empty Δ, then Δ alone fully weakly grounds A. His approach also requires the principles connecting weak and strict ground implied by the line of thought discussed above, on which weak ground adds only identity and ground-theoretic equivalence to strict ground. Also, his approach requires that $(A \wedge B)$ is never ground-theoretically equivalent to $(C \vee D)$, whilst deRosset and Fine's approach requires no such principles. One might perhaps attribute the difference on this latter point to a difference in aim. Fine (2012b, 67) notes a lacuna in his system in regard to questions of propositional identity (or ground-theoretical equivalence). But whereas Correia's target is a maximal system in which all such questions are settled in favor of a highly fine-grained conception of propositional identity or equivalence, deRosset and Fine's target is a minimal system, such as GG, in which all such questions are as far as possible left open.

Of course, Correia could move in the direction of deRosset and Fine's approach and drop the strict conditions that he imposes in obtaining a maximally fine-grained conception of the identity of propositions. But his definition of general ground in terms of simple ground could no longer be guaranteed to work, since his metalogical results depend upon his propositions having a well-founded logical structure; and so it looks as if he would then be forced to adopt the general notion of ground as a semantical primitive. Since he would then need to impose conditions on the general notion corresponding to the rules of inference of his favored system, the semantics would end up being a mere rewrite of the proof theory in quasi-algebraic terms.

4 Krämer's Approach

We turn to the "mode-ified" semantics of Krämer (2018a) (and also of Krämer (2018b)).[4]

Like deRosset and Fine (2023), he adopts a bilateral conception of propositions under which they may be regarded as ordered pairs of unilateral contents – a truth-condition, or positive content, on the one side and a falsity-condition, or negative content, on the other side. However, his conception of the truth- and falsity-conditions is rather different from deRosset and Fine's. A truth-condition for him is a *set* of modes of verification and a falsity-condition a *set* of modes of falsification, where, intuitively, a mode of verification is not simply given by a verifier, or some verifiers, but also by the manner in which they verify (and similarly for modes of falsification). A disjunction $(A \vee A)$, for example, can be verified either via the left disjunct or via the right disjunct. But for deRosset and Fine a truth- or falsity-condition is either a combination, a choice, or a more basic "urelement" from which contents, combinations, and choices are composed.

Moreover, he adopts what one might call a *cumulative* conception of truth-conditions, under which they are

[4] (Krämer 2018b, §§4.2-4) contains a comparison between his semantics and that of Correia (2017). He also compares his semantics with the fundamentality-based account of Correia (2018) and the syntactic account of Poggiolesi (2016, 2018)).

composed of the modes of verification which correspond to both the immediate and the mediate grounds for the given proposition. deRosset and Fine's view, by contrast, is one in which the truth-condition for a given proposition corresponds to its immediate grounds. We can, of course, recover the mediate grounds for a proposition through chaining, but it is not in general possible to recover the immediate grounds from the total grounds, since there is nothing in principle to stop total grounds from coinciding when immediate grounds do not - as with $(\neg\neg A \vee \neg\neg A)$ and $(\neg\neg A \vee A)$.

Krämer (2018a, p. 800) adopts semantical clauses for the connectives somewhat similar to those of deRosset and Fine (2023, p. 427). Thus the falsity-condition for $\neg A$ will involve "raising" the truth-condition of A; and the truth-conditions for conjunction and disjunction will involve operations of combination and choice ($\sqcup$ and $+$) that need not be commutative. But there are some significant differences. For deRosset and Fine, the truth-condition for $A \wedge B$, for example, will be the combination of the *bilateral* contents of A and B, whereas for him it will be the combination of the positive *unilateral* contents of A and B; and similarly for the other connectives. Also, he does not adopt a primitive operation of choice but takes the choice of two unilateral contents to be the union of those contents (which, recall, are sets of modes of verification) along with the combinations of those modes.

Krämer – like deRosset and Fine (2023), but in contrast to Correia (2017) – adopts a flexible approach to propositional identity (although he also argues for a particular conception of propositional identity). If, for example, modes of verification are insensitive to order, so that modes corresponding to the sequences of propositions P, Q and Q, P are the same, then it will turn out that the positive and negative contents of $(A \wedge B)$ and $(B \wedge A)$ will be the same; and otherwise not. Similarly, if modes of verification are insensitive to repetition, then it will turn out that the positive and negative contents of $(A \vee A)$, $(A \wedge A)$ and $\neg\neg A$ will be the same (Krämer 2018b, 3,17). How exactly the two approaches compare in regard to which propositional identities they allow is not altogether clear and is worthy of further study.

When it comes to ground, Krämer (2018a; b) adopts essentially the same definition of weak ground in terms of strict as Correia (2017); he takes some unilateral propositions to strictly ground a given unilateral proposition just in case they correspond to a mode of verification for the given proposition; and he takes some bilateral propositions to strictly ground a given bilateral proposition just in case the corresponding relation of strict ground holds among their positive contents (Krämer 2018b, p. 17). Thus his notion of ground is positively biased both to the left and to the right of the grounding relation, whereas deRosset and Fine's is only positively biased to the right hand side of a ground-theoretic statement. Also, given his cumulative conception of propositional content, the grounds for a given proposition can be directly read off from its modes of verification whereas, for deRosset and Fine, they can only be indirectly ascertained via the selections from its positive content.

Krämer does not attempt to axiomatize his semantics (although in Krämer (2018b), he does axiomatize various notions of propositional identity). However, it should be clear that the logic resulting from his semantics will be significantly stronger than GG. For one thing, he adopts the same restrictive account of weak ground as Correia, and so there will be the same addition in the structural principles for weak ground and its relation to strict ground. But there are also differences in the principles governing strict ground. For, harking back to our previous example, $\neg(A \vee B)$ will have the same positive content as $(\neg A \wedge \neg B)$ (Krämer 2018b, p. 25) and, in general, if C and D have the same positive content then so do $\neg\neg C$ and $\neg\neg D$ (Krämer 2018b, p. 26) and so, in particular, $\neg\neg\neg(A \vee B)$ will have the same positive content as $\neg\neg(\neg A \wedge \neg B)$. But $\neg(A \vee B)$ is a strict ground for $\neg\neg\neg(A \vee B)$ and so, since in Krämer's treatment strict ground only depends upon positive content, $\neg(A \vee B)$ will be a strict ground for $\neg\neg(\neg A \wedge \neg B)$. But this is exactly the kind of conclusion avoided in deRosset and Fine's semantics by making combination and choice a function of (bilateral) contents rather than conditions. Thus even though Krämer's semantics distinguishes the bilateral contents of $\neg(A \vee B)$ and $(\neg A \wedge \neg B)$ (2018b, p. 26), it does not distinguish their ground-theoretic roles.

A further peculiarity of Krämer's semantics might be noted. For its ability to distinguish the positive content of $(A \vee B)$ and $(B \vee A)$ depends upon adopting an inclusive interpretation of disjunction under which the modes of verification for the conjunction are among those for the disjunction, since otherwise the positive content of $(A \vee B)$ and $(B \vee A)$ alike would simply be the union of the positive contents of A and B. Thus under a non-inclusive interpretation of disjunction, the positive contents of $(A \vee B)$ and $(B \vee A)$ would be the same and hence would play the very same ground-theoretic role. So, for example, given that $(A \vee B)$ is a strict ground for $\neg\neg(A \vee B)$, $(B \vee A)$ will also be a strict ground for $\neg\neg(A \vee B)$, which is a commitment deRosset and Fine avoid. We see, in this way, the distinctive role that the operation of choice can play in providing an alternative semantics for disjunction.

In sum, we may say that the main differences between deRosset and Fine's (2023) semantics and those of Correia (2017) and Krämer (2018a; a) arise from deRosset and Fine adopting a more liberal conception of how strict and weak ground might be related and a more flexible approach to the question of ground-theoretic equivalence, one under which choice and combination are operations on contents rather than conditions and which thereby allows us to have positive bias "on the right" without also having it "on the left". These differences then enable deRosset and Fine to target a minimal system of ground, such as GG, rather than one of the stronger systems favored by Correia and Krämer.

Appendix

A *selection system* is a triple $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$, where Σ and Π are each operations on finite sequences (including the empty sequence) of ordered pairs of members of F, taking each such sequence into a member of F, with $\Sigma(\langle v \rangle) = \Pi(\langle v \rangle)$. We use lower case letters 'a'-'g' (sometimes with numerical superscripts) for members of F, lower case letters 'u'-'z' (sometimes with numerical superscripts) for pairs of members of F, and upper case letters 'G'-'K' (sometimes with numerical subscripts or superscripts) for sets of pairs of members of F. Thus, if $G = F \times F$, then $\Sigma, \Pi : G^{<\omega} \rightarrow F$. For a pair v, we write $v_\oplus$ for v's first element, and $v_\ominus$ for its second element. Intuitively, F is a set of *conditions*, and pairs of such conditions are *contents*.

Abusing notation, we indicate unions of sets of contents by comma-separated lists, and we often omit brackets for singletons of contents in these lists. So, for instance, G, H, v is used for $G \cup H \cup \{v\}$. We will occasionally write expressions of the form (x_i) for the indexed set $\{x_i | i < \alpha\}$, leaving the upper bound of the ordinal indices implicit. For instance, we will sometimes write (ϕ_i) instead of $\phi_1, \phi_2, \ldots$, (E^i_j) instead of $E^1_1, E^1_2, \ldots, E^2_1, E^2_2, \ldots, E^k_1, E^k_2, \ldots$, and $(G_i < v)$ instead of $G_1 < v; G_2 < v, \ldots$.

Write $[v^0 + v^1 + \cdots]$ for $\Sigma(\langle v^0, v^1, \ldots \rangle)$ and $[v^0.v^1. \cdots]$ for $\Pi(\langle v^0, v^1, \ldots \rangle)$. $[v^0 + v^1 + \cdots]$ is the *choice* of $v^0, v^1, \ldots$, and $[v^0.v^1. \cdots]$ the *combination* of $v^0, v^1, \ldots$. Since the *choice* of a single content v is just the same as the *combination* of v, we denote it by $[v]$, which is neutral between the '+' notation for choice and the '.' notation for combination. We use '$\ll_\mathfrak{F}$' to indicate the relation of *immediate selection* between sets (not sequences) of contents and choices and combinations, where $v^i \ll_\mathfrak{F} [v^0 + v^1 + \cdots]$ for each i, and $v, w, \cdots \ll_\mathfrak{F} [v.w. \cdots]$ (and that is all).

Given a selection system $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$, the relation of *strict selection* $<_\mathfrak{F}$ between a finite set of contents G and a content v is defined inductively in terms of immediate selection. In this definition, the *weak selection* relation $G \leq_\mathfrak{F} v$ abbreviates $(\exists d) G <_\mathfrak{F} ([v], d)$:

Definition 1

1. *Basis*: if $G \ll_\mathfrak{F} v_\oplus$, then $G <_\mathfrak{F} v$;
2. *Ascent*: if $G <_\mathfrak{F} w$ and $[w] = v_\oplus$, then $G <_\mathfrak{F} v$;
3. *Lower Cut*: if $(G^i \leq_\mathfrak{F} v^i)$ and $(v^i) <_\mathfrak{F} v$, then $(G^i) <_\mathfrak{F} v$; and
4. *Upper Cut*: if $(G^i <_\mathfrak{F} v^0)$ and $(v^i) \leq_\mathfrak{F} v$, then $(G^i) <_\mathfrak{F} v$.

Relations of partial selection are defined in terms of $<_\mathfrak{F}$:

- $w \preceq_\mathfrak{F} v$ iff there is an H such that $w, H \leq_\mathfrak{F} v$; and
- $w \prec_\mathfrak{F} v$ iff $w \preceq_\mathfrak{F} v$ but $v \npreceq_\mathfrak{F} w$.

Let a *covering* of G be a family of sets $G_0, G_1, \ldots$ such that $G = G_0 \cup G_1 \cup \ldots$.

Definition 2 A *frame* is a selection system $\mathfrak{F}$ meeting two constraints (which can be shown to be satisfiable (deRosset and Fine 2023, C3.8,T8.6)):

1. *Irreversibility*: $G <_\mathfrak{F} v$ iff $G \leq_\mathfrak{F} v$ and $(\forall w \in G) v \npreceq_\mathfrak{F} w$; and
2. *Maximality*:
   (a) $G <_\mathfrak{F} ([v^0.v^1. \cdots], d)$ only if there is a covering (G_i) of G such that $(G_i \leq_\mathfrak{F} v^i)$; and
   (b) $G <_\mathfrak{F} ([v^0 + v^1 + \cdots], d)$ only if there is a non-empty subset (w^j) of (v^i) and a covering (G_j) of G such that $(G_j \leq_\mathfrak{F} w^j)$.

Suppose we are given a propositional language $\mathscr{L}$, whose connectives are conjunction, disjunction, and negation. We will identify $\mathscr{L}$ with the set of its sentences. Let $<, \leq, \prec$, and $\preceq$ be *fresh* symbols. (That is, they are pairwise distinct from one another and from every connective and sentence of $\mathscr{L}$.) The *grounding claims of* $\mathscr{L}$ then consist of the following:

$$\Delta < \phi \qquad \Delta \leq \phi \qquad \phi \prec \psi \qquad \phi \preceq \psi$$

for any $\Delta \subseteq \mathscr{L}$ and any sentences ϕ, ψ of $\mathscr{L}$. We will continue to use the lower-case Greek letters ϕ, ψ, δ, and θ (sometimes with superscripts) for sentences of $\mathscr{L}$ and upper-case Greek letters Δ, Γ, Σ, and Θ (sometimes with superscripts) for sets of such sentences. The Greek letters σ and τ (sometimes with subscripts) are used for grounding claims of $\mathscr{L}$, and upper-case letters S and T for sets of grounding claims of $\mathscr{L}$. An *interpretation* for a language $\mathscr{L}$ into a frame $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$ is a function $\bar{\bar{\cdot}}$ mapping each atomic sentence ϕ in $\mathscr{L}$ to a content $\bar{\bar{\phi}}$. We extend interpretations to molecular sentences by means of the following recursive clauses:

1. $\overline{\overline{\neg\phi}} = (\bar{\bar{\phi}}_\ominus, [\bar{\bar{\phi}}])$;
2. $\overline{\overline{(\phi \wedge \psi)}} = ([\bar{\bar{\phi}}.\bar{\bar{\psi}}], [\overline{\overline{\neg\phi}} + \overline{\overline{\neg\psi}}])$; and
3. $\overline{\overline{(\phi \vee \psi)}} = ([\bar{\bar{\phi}} + \bar{\bar{\psi}}], [\overline{\overline{\neg\phi}}.\overline{\overline{\neg\psi}}])$.

We extend the notion of an interpretation to sets of sentences of $\mathscr{L}$ in the standard way: $\bar{\bar{\Delta}} = \{\bar{\bar{\delta}} | \delta \in \Delta\}$.

Definition 3 A *model* $\mathfrak{M}$ for a language $\mathscr{L}$ is a tuple $\langle \Sigma, \Pi, F, \bar{\bar{\cdot}} \rangle$, where $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$ is a frame, and $\bar{\bar{\cdot}}$ is an interpretation for $\mathscr{L}$ into $\mathfrak{F}$.

If $\mathfrak{M} = \langle \Sigma, \Pi, F, \bar{\bar{\cdot}} \rangle$ is a model and $\mathfrak{F}$ is the frame $\langle \Sigma, \Pi, F \rangle$, we write $\leq_{\mathfrak{M}}$ for $\leq_{\mathfrak{F}}$, and, similarly, for the other relations of ground.

Definition 4 Let $\mathfrak{M}$ be a model $\langle \Sigma, \Pi, F, \bar{\bar{\cdot}} \rangle$. *Truth in a model* for grounding claims is defined by the following clauses:

1. $\mathfrak{M} \vDash \Delta \leq \phi$ iff $\bar{\bar{\Delta}} \leq_{\mathfrak{M}} \bar{\bar{\phi}}$;
2. $\mathfrak{M} \vDash \Delta < \phi$ iff $\bar{\bar{\Delta}} <_{\mathfrak{M}} \bar{\bar{\phi}}$;
3. $\mathfrak{M} \vDash \phi \preceq \psi$ iff $\bar{\bar{\phi}} \preceq_{\mathfrak{M}} \bar{\bar{\psi}}$; and
4. $\mathfrak{M} \vDash \phi \prec \psi$ iff $\bar{\bar{\phi}} \prec_{\mathfrak{M}} \bar{\bar{\psi}}$.

$S \vDash T$ iff, for every model $\mathfrak{M}$, if $\mathfrak{M} \vDash \sigma$ for each $\sigma \in S$, then $\mathfrak{M} \vDash \tau$, for some $\tau \in T$. So, sets of grounding claims are treated conjunctively on the left-hand side and disjunctively on the right-hand side of $\vDash$. $\mathfrak{M} \vDash S$ iff $\mathfrak{M} \vDash \sigma$, for some $\sigma \in S$.

deRosset and Fine (2023, pp. 428-9) specify a system GG, which they show to be sound and complete for the semantics just specified (deRosset and Fine 2023, T3.1, T8.6). GG comprises the following rules and axioms, which inductively define a derivabiliy relation $\Vdash$ among *finite sets* of grounding claims:

Structural Rules:

THINNING	If $T \Vdash S$, then $T, T' \Vdash S, S'$
SNIP	If $\sigma, S \Vdash T$ and $S' \Vdash T', \sigma$, then $S, S' \Vdash T, T'$

In the statement of the structural rules, T' and S' are finite sets of grounding claims. Since $\Vdash$ relates sets, contraction and permutation rules are not needed.

The Pure Logic of Ground:

IDENTITY	$\sigma \Vdash \sigma$	
SUBSUMPTION	$(\leq / \preceq): \ \Delta, \phi \leq \psi \Vdash \phi \preceq \psi$	$(< / \leq): \ \Delta < \phi \Vdash \Delta \leq \phi$
	$(< / \prec): \ \Delta, \phi < \psi \Vdash \phi \prec \psi$	$(\prec / \preceq): \ \phi \prec \psi \Vdash \phi \preceq \psi$
TRANSITIVITY	$(\preceq / \preceq): \ \phi \preceq \psi; \ \psi \preceq \theta \Vdash \phi \preceq \theta$	$(\preceq / \prec): \ \phi \preceq \psi; \ \psi \prec \theta \Vdash \phi \prec \theta$
IRREVERSIBILITY	$\phi \preceq \psi \Vdash \phi \prec \psi; \ \psi \preceq \phi$	
REFLEXIVITY	$\Vdash \phi \leq \phi$	
NON-CIRCULARITY	$\phi \prec \phi \Vdash \emptyset$	
CUT	$\Delta \leq \phi; \quad \phi, \psi_0, \psi_1, \ldots, \psi_n \leq \psi \quad \Vdash \quad \Delta, \psi_0, \psi_1, \ldots, \psi_n \leq \psi$	
REVERSE SUBSUMPTION	$\phi_0, \phi_1, \ldots, \phi_n \leq \psi; \ \phi_0 \prec \psi; \ \phi_1 \prec \psi; \ \cdots; \ \phi_n \prec \psi \quad \Vdash \quad \phi_0, \phi_1, \ldots, \phi_n < \psi$	

Let $S_0, S_1, \ldots$ be finite sets of grounding claims. Then $S \Vdash (S_0|S_1| \ldots)$ is defined to hold iff $S \Vdash \sigma_0, \sigma_1, \ldots$ for each set $\sigma_o, \sigma_1, \ldots$ such that $\sigma_i \in S_i$. It is easily shown that a model $\mathfrak{M}$ verifies every such set $\sigma_0, \sigma_1, \ldots$ iff, for some S_i, $\mathfrak{M}$ verifies every grounding claim in S_i.

Introduction Rules:

$$\Vdash \phi < \neg\neg\phi$$

$$\Vdash \phi < (\phi \vee \psi) \qquad \Vdash \psi < (\phi \vee \psi)$$

$$\Vdash \phi, \psi < (\phi \wedge \psi)$$

$$\Vdash \neg\phi < \neg(\phi \wedge \psi) \qquad \Vdash \neg\psi < \neg(\phi \wedge \psi)$$

$$\Vdash \neg\phi, \neg\psi < \neg(\phi \vee \psi)$$

Elimination Rules:

$$\Delta < \neg\neg\phi \Vdash \Delta \leq \phi$$

$$\Delta < (\phi \wedge \psi) \Vdash (\ \Delta^0_\phi \leq \phi; \Delta^0_\psi \leq \psi \ | \ \Delta^1_\phi \leq \phi; \Delta^1_\psi \leq \psi \ | \ \ldots \ )$$

$$\Delta < (\phi \vee \psi) \Vdash \Delta \leq \phi; \quad \Delta \leq \psi; \quad \Delta < (\phi \wedge \psi)$$

$$\Delta < \neg(\phi \vee \psi) \Vdash (\ \Delta^0_\phi \leq \neg\phi; \Delta^0_\psi \leq \neg\psi \ | \ \Delta^1_\phi \leq \neg\phi; \Delta^1_\psi \leq \neg\psi \ | \ \ldots \ )$$

$$\Delta < \neg(\phi \wedge \psi) \Vdash \Delta \leq \neg\phi; \quad \Delta \leq \neg\psi; \quad \Delta < (\neg\phi \wedge \neg\psi)$$

In the statement of the elimination rules for $\wedge$ and $\neg\vee$, $\langle \Delta^0_\phi, \Delta^0_\psi \rangle, \langle \Delta^1_\phi, \Delta^1_\psi \rangle, \ldots$ are taken to be all of the ordered pairs $\langle \Delta^n_\phi, \Delta^n_\psi \rangle$ for which $\Delta = \Delta^n_\phi \cup \Delta^n_\psi$. For any sets S and T of grounding claims, let $S \vdash T$ iff there are $S' \subseteq S$ and $T' \subseteq T$ such that $S' \Vdash T'$.

Data availability Not applicable.

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