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DLEAC: A Dialetheic Logic with Exclusive Assumptions and Conclusions

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Abstract This paper proposes a new dialetheic logic, a *Dialetheic Logic with Exclusive Assumptions and Conclusions* (DLEAC), including classical logic as a particular case. In DLEAC, *exclusivity* is expressed *via* the speech acts of assuming and concluding. In the paper we adopt the semantics of the *logic of paradox* extended with a generalized notion of *model* and we modify its *proof theory* by refining the notions of *assumption* and *conclusion*. The paper starts with an explanation of the adopted philosophical perspective, then we propose our DLEAC logic. Finally, we show how DLEAC supports the dialetheic solution of the liar paradox.

Keywords Assumptions and conclusions · Exclusivity · Dialetheic logic · Logic of paradox (LP) · Liar paradox

1 Introduction

Let *exclusive negation* be a propositional connective $\sim$ such that, in virtue of its very meaning, A and $\sim A$ are incompatible.

In other words A and $\sim A$ cannot be both true, i.e. it is excluded that A and $\sim A$ are both true. Of course this explanation is circular, because exclusive negation is embedded in the word "cannot", as well as it is involved in the notion of *exclusion* and *incompatibility*. But this circularity is unavoidable in any explanation of a primitive notion. Observe, however, that exclusive negation must be grasped by any competent speaker of the natural language, since it is essential for human verbal communication. Such a kind of negation is standardly called *Boolean negation*; Priest's dialetheism argues for its inexistence (for example, see Priest 1999, 2006a, b).[1]

A standard criticism against those arguing for the inexistence of *Boolean negation* concerns the difficulty of expressing the notion of exclusivity—*exclusively* (or *only*) *true* and *exclusively* (or *only*) *false*—dialetheically. Take, for example, *only true*: It is usually understood as 'true and not false', where the 'not' is the exclusive one, which is not available for a dialetheist.

However, a dialetheist *needs* the possibility of expressing such a notion. For instance, having held that a sentence may be true, false or both, the dialetheist should be able to *reason by cases*, distinguishing three possible cases: *true only*, *false only*, and *true and false*. In order to make up for the lack of *exclusive negation*, Priest introduced the notion of *rejection* of a sentence A, to be clearly distinguished from the *acceptance* of the negation of A.

The goal of this paper is to formulate a Dialetheic Logic with Exclusive Assumptions and Conclusions (DLEAC), including classical logic as a particular case, where *exclusivity* is expressed *via* certain speech acts. We adopt the semantics of the *logic of paradox* (LP)[2] extended with a

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[1] *Dialetheism* holds that there are *dialetheias*, i.e. propositions that are both true and false. Priest uses the terms 'dialetheias' and 'true contradictions' to indicate 'gluts', propositions both true and false, a term coined by Fine (1975). For an introduction to dialetheism, see e.g. Berto (2007). Among the dialetheists, Priest (for example, in Priest 1979, 2002a, b, 2006a, b, 2010) claims that dialetheism supplies the best solution to all self-reference paradoxes. On the same topic see Beall (2009), Colyvan (2009), Weber (2010).

[2] For a general background on LP, see Asenjo (1966), Asenjo and Tamburino (1975), Priest (1979), Routley (1979), Beall (2009).

notion of *model* suitable for DLEAC and we modify its *proof theory* by refining the notions of *assumption* and *conclusion*—understood as speech acts.

First of all, we will explain the adopted philosophical perspective that we think is in agreement with the dialetheic notion of *rejection*. Then, we will propose our DLEAC logic. Finally, we will show how our logic supports the dialetheic solution of the liar paradox.

2 Rejection, Assumptions and Conclusions as Speech Acts

Priest (2006a) tries to recover the exclusivity of negation by introducing the notion of *rejection* or *denial*[3], understood as a *speech act*. He claims that, while it is possible to *accept* both a sentence and its negation[4], one *cannot* accept and reject the same sentence. However, if the 'not' inglobed in the word 'cannot' failed to be exclusive, the claim would be compatible with the possibility of accepting and denying the *same* sentence. In a note, (Priest 2006a, 107) he replies to this objection as follows:

> [(R)] When I said that one cannot accept and reject something, I was denying the claim that one can do this.

However, the mere denial of the claim 'One can accept and deny the same sentence' by no means expresses the thesis that it is impossible to accept and reject the same proposition. It does not say anything that prevents the possibility of denying and accepting that claim. It seems that one cannot express the thesis at issue without using an *exclusive negation*. Similarly, consider the dialetheic thesis:

(1) Negation is not exclusive.

According to the non-exclusivity of 'not', (1) is compatible with the following proposition:

(2) Negation is exclusive.

However, when Priest—or a dialetheist—asserts his thesis (1), he obviously wants to mean that proposition (1) is *only true*. So we think that, in order to accept Priest's reply (R) to the above objection, one has to maintain that the acts of accepting and rejecting the same sentence are *incompatible*, where *incompatibility* is taken as a *non-logical primitive notion*, not definible, in particular, in terms of *logical negation*.[5]

Priest tries to formulate the principles of *asserting* and *rejecting* as follows:

> **Assert (T)**: You may assert A if there is good evidence for A's *truth*.
> **Reject (U)**: One ought to reject A if there is good evidence for its *untruth*.

Reject (U) would be appropriate if untruth were understood as *incompatible* with truth. But for a dialetheist, this is not the case. Consider, for example, the dialetheic solution of the strengthened liar paradox:

(SL) 'This sentence is untrue' is both *true* and *untrue*.

According to this solution, truth and untruth are taken as *compatible*. On this point, Priest observes that:

> The principle concerning rejection might be something like: one ought to reject something if there is good evidence for its untruth, unless there is also good evidence for its truth. Thus **Reject (U)** may be understood as an acceptable *default rule*, but not as an indefeasible one [We inserted bold and italics] (Priest 2006a, 110).

In this vein, Priest accepts that there are *rational dilemmas*. According to him, it might happen that there is a ground for accepting a sentence and also a ground for rejecting it. In this case, as he puts it:

> [By] these two principles [**Assert (T)** and **Reject (U)**], one ought to accept A and one ought to reject A. We have a dilemma, since we cannot do both... (Priest 2006a, 110–111).

Rationality demands that we do something that cannot be done, but Priest says 'arguably, the existence of dilemmas is a fact of life' (Priest 2006a, 111). He contends that a dialetheist cannot rule out the coexistence of a ground for accepting A and a ground for rejecting A. However, this admission undermines the attempt of recovering *exclusivity* through the notion of *rejection*. A *default principle for rejection* cannot help to recover exclusivity without using an exclusive negation. Indeed, if a dialetheist is ready to accept default principles, he can simply accept the principle of the exclusivity of negation as a *default principle*. Insofar as it is default, such a principle would be compatible with its failure in the case of the (strengthened) liar paradox.

We want to carefully distinguish the possibility of *performing* an act from that of *justifying* it. One can perform an act of rejection of any sentence at will. Such an act

[3] For a general background on denial in non-classical theories, see (Ripley 2011, Section 3).

[4] On the thesis see, also, Parsons (1984). For a recent discussion of the topic see also Murzi and Carrara (2015).

[5] This option is also evaluated in Berto (2014). For a different, interesting, point of view on the topic see Beall (2013).

expresses, but by no means *ensures*, that a sentence A is *only false*. We believe that there is a certain confusion in Priest's use of *expressing* and *ensuring*:

> A dialetheist [glut theorist] can express the claim that something, α, is not true in those very words, $\neg\mathsf{T}(\alpha)$. What she cannot do is to ensure that the words she utters behave consistently: even if $\neg\mathsf{T}(\alpha)$ holds, $\alpha \wedge \neg\mathsf{T}(\alpha)$ may yet hold. But in fact, a classical logician [or any explosive logic theorist] can do no better. He can endorse $\neg\mathsf{T}(\alpha)$, but this does not prevent his endorsing α as well.... [C]lassical logic, as such, is no guard against this... (Priest 2006b, 291).

However, the difference between the classicist and the dialetheist concerns only *expressivity*, not *ensurance*. Indeed, for the classicist $\neg\mathsf{T}(\alpha)$ *expresses*—and by no means ensures—that α is only false. In other words, the intended meaning of the Boolean negation $\sim$, likewise the meaning of the other connectives, has nothing to do with any sort of *ensurance*. The act of expressing is performable independently of any justification. What we can express is independent of what we know. In this sense any sentence is capable of being rejected.

As to the justification of an act of rejection, we agree—of course—that in daily life *rational dilemmas* are 'facts of life'. However, at the same time, we think that—from a logical point of view—one is rightfully allowed, *in principle*, to banish them—and so we do. A *rational* human being cannot justify the act of simultaneously accepting and rejecting the same sentence.

For the reasons sketched above, we take as primitive the *absolute* (not default) impossibility of *accepting* and *rejecting* the same sentence.

In this way, we conceive of the rejection of a sentence A as a speech act that, in virtue of its very meaning, expresses the fact that A is *only* false. Similarly, the act of rejecting $\neg A$ expresses the fact that A is *only* true. This use of rejection suggests the idea of a theory of natural deduction, where the acts of assuming and concluding may be understood in an *ordinary* or in an *exclusive mode*.

To assume a sentence in an *ordinary mode* amounts to supposing that it is *at least* true; to assume it in an *exclusive mode* amounts to supposing that it is *only true*. Any sentence can be rightfully assumed in an ordinary or exclusive mode at will. Similarly, to prove a sentence in an ordinary mode amounts to proving that it is (at least) true (under certain assumptions); to prove it in an exclusive mode amounts to proving that it is *only true*. So the acts of proving A and $\neg A$ in an *exclusive mode* are incompatible in the sense that, in principle, they together lead indefeasibly to the rejection of some assumptions they depend on. In particular, the act of concluding A and $\neg A$ in an *exclusive mode*—independently of any hypothesis—cannot in principle be performed by any rational human being.

In this way, we realize the dialetheic aim of taking exclusivity as extraneous to the meaning of logical negation and embedded in the speech acts of *assuming* and *concluding*. Such speech acts will be formalized within a modified natural deduction, where they will be governed by *indefeasible rules.*

3 Semantics of DLEAC

Let L be a language of first-order logic with identity ($\mathsf{FOL}=$) with individual constants and predicates of any *ariety*. For the sake of simplicity we omit function symbols in L. We adopt here the semantics for LP extended with a new, generalized notion of model.

Moreover, though Priests maintains that a dialetheist should adopt dialetheic logic even in the metalanguage[6], we think that the exclusive negation is essential in the metalanguage, as well as in the natural language, and that Priest himself uses it over and over. So we will use the exclusive negation in the metalanguage in order to treat the non-exclusive negation in the object language.

Let us summarize the semantics for LP.[7]

A dialetheic interpretation of the propositional logic consists of an evaluation v that assigns to each atomic formula a member of the set $\{\{1\}, \{0\}, \{0, 1\}\}$. v is extended to the complex formulas by the following clauses:

$$
\begin{aligned}
(\vee)\, & v(A \vee B) = \{1\} \text{ if either } 0 \notin v(A) \text{ or } 0 \notin v(B);\\
& v(A \vee B) = \{0\} \text{ if } 1 \notin v(A) \text{ and } 1 \notin v(B);\\
& v(A \vee B) = \{0,1\} \text{ otherwise.}\\
(\wedge)\, & v(A \wedge B) = \{1\} \text{ if } 0 \notin v(A) \text{ and } 0 \notin v(B);\\
& v(A \wedge B) = \{0\} \text{ if either } 1 \notin v(A) \text{ or } 1 \notin v(B);\\
& v(A \vee B) = \{0,1\} \text{ otherwise.}\\
(\neg)\, & v(\neg A) = \{1\} \text{ if } v(A) = \{0\};\\
& v(\neg A) = \{0\} \text{ if } v(A) = \{1\};\\
& v(\neg A) = \{0,1\} \text{ otherwise.}
\end{aligned}
$$

A sentence A *is true* if $1 \in v(A)$, *is false* if $0 \in v(A)$; A *is exclusively true* if $0 \notin v(A)$, *is exclusively false* if $1 \notin v(A)$.

[6] See, for example the following quotation from *On Contradiction*:

> The distinction between a theory (...) and its metatheory makes perfectly good sense to a dialetheist. But there is no reason to insist that the metatheory must be stronger than, and therefore different from, the theory. The same logic must be used in both 'object language' and 'metatheory' (Priest 2006b, 98).

[7] For details see (Priest 2006b, sez. 5.2, 5.3).

This semantics is extended in a similar way to first order logic with identity. We make the simplifying assumption that there is a name in the language L for every object of the domain D of quantification.

An evaluation v assigns to every individual constant a member of the domain D, to every unary predicate P two subsets of D: the extension P^+ and the counter-extension P^-, possibly overlapping, with the only constraint that $P^+ \cup P^- = \mathsf{D}$. Then:

$$v(Pa) = \{1\} \quad \text{if } a \in P^+ - P^-$$
$$v(Pa) = \{0\} \quad \text{if } a \in P^- - P^+$$
$$v(Pa) = \{0,1\} \quad \text{if } a \in P^+ \cap P^-$$

Similarly for predicates of degree >1.

The constraints for the identity sign (=) are the following:

> $(=)^+ = \{(a,a) : a \in \mathsf{D}\}$, while $(=)^-$ is arbitrary with the only constraint that $(=)^+ \cup (=)^- = \mathsf{D}$.

So, according to these constraints, while a dialetheic interpretation cannot identify two distinct objects,[8] nevertheless it may regard some single object as *both identical and not identical to itself*.

The clauses for the universal and the existential quantifiers are analogous to those of conjunction and disjunction respectively.

We extend the semantics of LP by introducing a notion of model suitable for DLEAC.

Let S be any set of sentences of a first order language L, some of which may be starred (i.e. marked by a star *). Observe that stars * do not belong to the object language L.

> A model M of S is an LP-interpretation in which all sentences of S are true and the starred ones are exclusively true.

> A sentence A (a starred sentence A*) is a *semantical consequence* of a set S of possibly starred sentences, in symbols $S \models A(*)$, if it is true (exclusively true) in every model of S.

4 DLEAC: Deductive Rules

Let A, B, C... be formulas of a first order language L and Γ a finite set of possibly starred formulas.

A sequent is an expression of the form:

$\Gamma : C(^*)$

to be read: *From the assumptions in* Γ, *one can infer the conclusion C* (*in an ordinary or exclusive mode*).

The non starred formulas in Γ are understood to be assumed in an *ordinary mode*, and the starred ones in an *exclusive mode*. Similarly, the conclusion C is to be understood in an *ordinary* or in an *exclusive mode*.

4.1 Basic Deductive Rules for DLEAC

We list the primitive inference rules. When some stars occur in parentheses ( ) the rule holds in the double form:

- with all stars in parentheses at work,
- with all stars in parentheses deleted.

Reflexivity

$A(^*) : A(^*)$

$A^* : A$

The informal reading of the first rule is the following: From the assumption that A is only true (at least true), it follows that A is only true (at least true). The informal reading of the second rule is: from the assumption that A is only true it follows that A is (at least) true.

Weakening:

$$\frac{\Gamma : A(^*)}{\Gamma\ \Delta : A(^*)}$$

Cut:

$$\frac{\Gamma : A(^*), \Delta\ A(^*) : B}{\Gamma\ \Delta : B}$$

$$\frac{\Gamma : A(^*), \Delta A(^*) : B^*}{\Gamma\Delta : B^*}$$

Conjunction:

$$\mathsf{I}\wedge \frac{\Gamma : A(^*), \Delta : B(^*)}{\Gamma\Delta : A \wedge B(^*)}$$

$$\mathsf{E}\wedge \frac{\Gamma : A \wedge B(^*)}{\Gamma : A(^*)}$$

$$\mathsf{E}\wedge \frac{\Gamma : A \wedge B(^*)}{\Gamma : B(^*)}$$

Disjunction:

$$\mathsf{I}\vee \frac{\Gamma : A(^*)}{\Gamma : A \vee B(^*)}$$

$$\mathsf{E}\vee \frac{\Gamma A : C(^*), \Delta B : C(^*), \Lambda : A \vee B}{\Gamma\Delta\Lambda : C(^*)}$$

$$\mathsf{E}\vee \frac{\Gamma A^* : C(^*), \Delta B^* : C(^*), \Lambda : A \vee B^*}{\Gamma\Delta\Lambda : C(^*)}$$

[8] Observe that *distinction* is a metalinguistic notion.

Double Negation:

$A(^*) : \neg\neg A(^*)$

$\neg\neg A(^*) : A(^*)$

Introduction of absurd (IA):

$$\frac{\Gamma : A^*, \Delta : \neg A}{\Gamma\Delta : A \wedge \neg A^*}$$

The informal justification of IA is the following: From A and $\neg A$ follows $A \wedge \neg A$. Furthermore, since A is only true, it cannot be a *dialetheia*; hence also $\neg A$ cannot be a *dialetheia*. As a result, neither of the conjuncts of $A \wedge \neg A$ can be also false and, therefore, $A \wedge \neg A$ is *only* true.

Since $\neg(A \wedge \neg A)$ is a *dialetheic logical law*, the conclusion $A \wedge \neg A^*$ is an *authentic absurd*, i.e. a conclusion unacceptable even by a dialetheist. Since, dialethically, $A \wedge \neg A$ might be true, it does not count as an absurd. For this reason, by an *absurd*, we mean a formula $A \wedge \neg A$ that is *only true*.

Reductio ad absurdum (RAA):

$$\frac{\Gamma A^* : B \wedge \neg B^*}{\Gamma : \neg A}$$
$$\frac{\Gamma A : B \wedge \neg B^*}{\Gamma : \neg A^*}$$

Informally, RAA works in this way: If the assumption that A is true (only true) leads to the *authentic absurd*, it cannot be true (only true), hence it is *only false* (at least false).

The rules for the quantifiers are analogous to those of conjunction ($\wedge$) and disjunction ($\vee$). The rules for identity are the following.

Introduction of identity (I =)

$: x = x$

Elimination of identity (E =):

$x = y, Px : Py$

$$\mathsf{E}= \frac{\Gamma : A^* : \neg(t = t)^*}{\Gamma : \neg A}$$
$$\mathsf{E}= \frac{\Gamma : A : \neg(t = t)^*}{\Gamma : \neg A^*}$$

Observe that, according to the semantics of identity (=), a sentence having the form $(t = t)$[9] cannot be *exclusively false*.

[9] where 't' is an individual constant or a variable.

4.2 Derived Deductive Rules for DLEAC

The following ones are derived rules:

Material conditional:

$$\frac{\Gamma A(^*) : B(^*)}{\Gamma : \neg A \vee B}$$
$$\frac{\Gamma A^* : B}{\Gamma : \neg A \vee B}$$
$$\frac{\Gamma A : B(^*)}{\Gamma : \neg A \vee B(^*)}$$

Elimination of absurd (Ex absurdo quodlibet) (EA):

$$\mathsf{EA}\frac{\Gamma : A \wedge \neg A^*}{\Gamma : B^*}$$

Notice that EA is a derived rule.

Modus ponens (MPP):

$$\mathsf{MPP}\frac{\Gamma : A^*, \Delta : \neg A \vee B}{\Gamma\Delta : B}$$
$$\mathsf{MPP1}\frac{\Gamma : A, \Delta : \neg A \vee B^*}{\Gamma\Delta : B^*}$$

For an example of how DLEAC works, here is the derivation of MPP1:

1	1.	A	Assumption
2	2.	$\neg A \vee B^*$	Assumption
3	3.	$\neg A^*$	Assumption
1, 3	4.	$A \wedge \neg A^*$	IA
1, 3	5.	B^*	EA
6	6.	B^*	Assumption
6	7.	B^*	*Reflexivity*
1, 2	8.	B^*	E∨

Other derived rules of DLEAC are the *De Morgan rules:*

$$\frac{\Gamma : \neg(A \wedge B)(^*)}{\Gamma : \neg A \vee \neg B(^*)}$$
$$\frac{\Gamma : \neg A \vee \neg B(^*)}{\Gamma : \neg(A \wedge B)(^*)}$$
$$\frac{\Gamma : \neg(A \vee B)(^*)}{\Gamma : \neg A \wedge \neg B(^*)}$$
$$\frac{\Gamma : \neg A \wedge \neg B(^*)}{\Gamma : \neg(A \vee B)(^*)}$$

The Law of non-contradiction:

$\Gamma : \neg(A \wedge \neg A)$

The Law of the excluded middle:

$\Gamma : (A \vee \neg A)$

5 Tarski's Rules for the Truth Predicate T

Let L be, as above, a first order language. Extend L to L' by means of a new monadic predicate symbol T and names for all sentences of L'. For every sentence A of L' let $\lceil A \rceil$ be the name of the sentence A. T is intended to be the truth predicate of L'.

Observe that, when dealing with dialetheias, the material conditional trivializes Tarski's schema:

$\mathsf{T}(\lceil A \rceil) \leftrightarrow A$

If A is a dialetheia, it fails to express truth-preservation from A to $\mathsf{T}(\lceil A \rceil)$ and *vice versa*. That is, if

$\mathsf{T}(\lceil A \rceil) \leftrightarrow A$

is understood as

$(\neg \mathsf{T}(\lceil A \rceil) \vee A) \wedge (\neg A \vee \mathsf{T}(\lceil A \rceil)),$

when A is a dialetheia, then $\mathsf{T}(\lceil A \rceil)$ may have any value. However, we can express Tarski's laws in DLEAC as inferential rules.

Primitive Tarki's rules:

$$\frac{\Gamma : A(*)}{\Gamma : \mathsf{T}(\lceil A \rceil)(^*)}$$

$$\frac{\Gamma : \mathsf{T}(\lceil A \rceil)(^*)}{\Gamma : A(^*)}$$

Observe that from the above rules it follows that, semantically, $\mathsf{T}(\lceil A \rceil)$ and A possess the same truth-values.

Derived Tarki's rules:

$$\frac{\Gamma : \neg \mathsf{T}(\lceil A \rceil)(^*)}{\Gamma : \mathsf{T}(\lceil \neg A \rceil)(^*)}$$

$$\frac{\Gamma : \mathsf{T}(\lceil \neg A \rceil)(^*)}{\Gamma : \neg \mathsf{T}(\lceil A \rceil)(^*)}$$

To exemplify, let us demonstrate the second derived rule, i.e.:

$$\frac{\Gamma : \mathsf{T}(\lceil \neg A \rceil)(^*)}{\Gamma : \neg \mathsf{T}(\lceil A \rceil)(^*)}$$

in the non-starred version or the starred version:

1	1.	$\mathsf{T}(\lceil \neg A \rceil)$	Assumption
2	2.	$\mathsf{T}(\lceil A \rceil)*$	Assumption
2	3.	$A*$	Tarski
1	4.	$\neg A$	Tarski
1, 2	5.	$A \wedge \neg A*$	IA
1	6.	$\neg \mathsf{T}(\lceil A \rceil)$	RAA

1	1.	$\mathsf{T}(\lceil \neg A \rceil)*$	Assumption
2	2.	$\mathsf{T}(\lceil A \rceil)$	Assumption
2	3.	A	Tarski
1	4.	$\neg A*$	Tarski
1, 2	5.	$A \wedge \neg A*$	IA
1	6.	$\neg \mathsf{T}(\lceil A \rceil)*$	RAA

6 Completeness of DLEAC

Let S be any set of sentences of possibly starred sentences.

We say that S is *dialetheically consistent* (d–*consistent*) if no conclusion of form $(A \wedge \neg A)*$ is derivable from S.

Theorem 1 *If S is d–consistent, then it has a model $\mathcal{M}$.*

Proof Let S be d–*consistent*. Extend the language L to a language L' with an infinite sequence of new individual constants $c_1, c_2, \ldots, c_n, \ldots$. Let

$A_1, A_2, \ldots, A_n, \ldots$

be a sequence of all L'-sentences. We inductively define the sequence:

$S_0, S_1, \ldots, S_n, \ldots$

of sets of (possibly starred) L'-sentences as follows:

1. $S_0 = S$;
2. $S_{n+1} = S_n$ if A_{n+1} is derivable from S_n and is not an existential sentence;
3. $S_{n+1} = S_n \cup \{B(c)(*)\}$ if $A_{n+1} = \exists x B(x)$ and $S_n \vdash \exists x B(x)(*)$, where c is the first constant not occurring in S_n nor in A_{n+1};
4. $S_{n+1} = S_n \cup \{\neg A_{n+1}(*)\}$ if A_{n+1} is not derivable from S_n.

Let us define:

$S_\omega = \cup_{n \in \mathsf{N}} S_n$

One can prove by induction that each S_n is d–*consistent*, so that S_ω is d–*consistent*.

Consider, for example, the case 3). Suppose, by reduction, that S_{n+1} is inconsistent. If $S_{n+1} = S_n \cup \{B(c)\}$, then $S_n \vdash \neg B(c)^*$ and hence $S_n \vdash \forall x \neg B(x)^*$, against the d–*consistency* of S_n. If $S_{n+1} = S_n \cup \{B(c)^*\}$, then $A_{n+1} = \exists x B(x)^*$. Then $S_n \vdash \neg B(c)$ and hence $S_n \vdash \forall x \neg B(x)$, against the d–*consistency* of S_n. □

S_ω is *deductively complete*: for any L′-sentence, if not $S_\omega \vdash A$, then $S_\omega \vdash \neg A^*$.

Define an interpretation I of L′ as follows. Take as domain the set D of all individual constants. Define the evaluation v as follows:

$1 \in v(A)$ *iff* $S_\omega \vdash A$, $0 \in v(A)$ *iff* $S_\omega \vdash \neg A$, for every atomic L′-sentence.

One can prove, by induction on the complexity of a sentence A, that $v(A) = \{1\}$ *iff* $S_\omega \vdash {}^*A, v(A) = \{0\}$ *iff* $S_\omega \vdash {}^* \neg A$, $v(A) = \{0, 1\}$ *iff* $S_\omega \vdash A$ and $S_\omega \vdash \neg A$.

It follows that I is a model of S_ω and hence of S.

Completeness. If $S \models A(^*)$ then $S \vdash A(^*)$

Proof Let $S \models A$. Suppose, by reduction, that not $S \vdash A$. Then $S \cup \{\neg A^*\}$ is d–*consistent* and hence has a model where $\neg A$ is only true, against the hypothesis. Similarly if $S \models A^*$. □

7 Extending a Theory with the Truth Predicate

Aim of this section is to show that any dialetheic interpretation of a first order language L can be extended to an interpretation of a language L′ capable of expressing its own truth predicate.

Let L be a first order language with predicates and individual constants (for simplicity we ignore functions). Let I be any interpretation of L and D its domain of quantification. Extend L with a new predicate symbol T and infinitely many individual constants. Extend D to D′ by adding to D all L′-sentences. Let *I’* map the new constants 1-1 onto D′ so that any member of D′ has an L′-name. If A is an L′-sentence, we indicate by $\lceil A \rceil$ its name.

I’ puts all sentences in the counter-extension of the L-predicates and the members of D in the counter-extension of T. We want to show that it is possible to fix the interpretation of T in such a way that it turns out to be the truth predicate of *I’*, i.e. in such a way that, for all L′-sentences A, A and $\mathsf{T}(\lceil A \rceil)$ have the same truth values.

Theorem 2 *There is an extension of I to an interpretation I′ of* L′ *such that, for every* L′*–sentence A, A and* $\mathsf{T}\lceil A \rceil$ *have the same truth values, while the values of the* L*–sentences, relativized to* D*, are unchanged.*

An evaluation v' is a *sub-evaluation* of v, in symbols $v' \subset v$, if v' is obtained from v by suppressing a truth value of some atomic dialetheias.

Lemma *If a sentence has a unique v-value, this is also the unique v′-value, for any* $v' \subset v$.

Proof The proof is obtained by an easy induction on the complexity of the sentence. □

Proof of the theorem 2 Define, by transfinite induction, the following sequence:

$v_0 \supset v_1 \supset \ldots \supset v_\alpha, \ldots$(for all ordinals)

of evaluations of sentences of form $\mathsf{T}(\lceil A \rceil)$ for all L′-sentences A:

$v_0(\mathsf{T}(\lceil A \rceil) = \{0, 1\}$ for all A;

$v_{\alpha+1}(\mathsf{T}\lceil A \rceil)$ is defined by cases:

(i) $v_{\alpha+1}$ $(\mathsf{T}\lceil A \rceil) = v_\alpha(A)$ if $v_\alpha(A)$ is a singleton while v_α $(\mathsf{T}\lceil A \rceil)$ is not;

(ii) $v_{\alpha+1}$ $(\mathsf{T}\lceil A \rceil) = v_\alpha$ $(\mathsf{T}\lceil A \rceil)$ otherwise.

$v_\beta(\mathsf{T}(\lceil A \rceil)) = \cap_{\alpha<\beta} v_\alpha(\mathsf{T}(\lceil A \rceil))$ for β limit.

One can prove, by transfinite induction, that:

For all α, if v_α $(\mathsf{T}\lceil A \rceil)$ is a singleton, then v_α $(\mathsf{T}\lceil A \rceil) = v_\alpha(A)$.

1. $\alpha = 0$. Trivial
2. $\alpha = \beta + 1$. Let v_α $(\mathsf{T}\lceil A \rceil)$ be a singleton. We distinguish the two cases above (i) and (ii):
   (i) v_α $(\mathsf{T}\lceil A \rceil) = v_\beta$ (A) and, since v_β (A) is a singleton and v_α is a sub-evaluation of v_β, by the *lemma* v_α (A) = v_β(A).
   (ii) v_α $(\mathsf{T}\lceil A \rceil) = v_\beta$ $(\mathsf{T}\lceil A \rceil)$. So, v_β $(\mathsf{T}\lceil A \rceil)$ is a singleton. By the induction hypothesis, v_β $(\mathsf{T}\lceil A \rceil) = v_\beta(A)$ and, by the lemma, $v_\alpha(A) = v_\beta(A)$.
3. α limit. If v_α $(\mathsf{T}\lceil A \rceil)$ is a singleton, then v_α $(\mathsf{T}\lceil A \rceil) = v_\beta$ $(\mathsf{T}\lceil A \rceil) = v_\beta(A)$, for some $\beta < \alpha$. By the lemma, $v_\alpha(A) = v_\beta(A)$. □

Observe that, if $v_\beta \neq v_\alpha$, with $\beta < \alpha$, there is some sentence A such that $v_\beta(A) = \{0, 1\}$, while $v_\gamma(A)$ is a singleton for all $\gamma > \beta$. And since only countably many sentences can satisfy–for some ordinal–this condition, it follows that at least from the first uncountable ordinal on, the sequence becomes stationary. If δ is such an ordinal, clearly v_δ is the required evaluation.

Observation 1 The guiding idea of the constructed evaluation is that, when Tarski's rules fail to determine a unique value for a sentence of form $\mathsf{T}(\ulcorner A \urcorner)$, there is no reason to arbitrarily choose one of the two truth values for $\mathsf{T}(\ulcorner A \urcorner)$, which therefore is to be evaluated as a dialetheia.

Observation 2 If in the original language the recursive functions are representable (as in arithmetic), then in the extended language self-reference is at work. In such a case, from a classical point of view—due to the Tarski's theorem—the extended language cannot be semantically closed. Dialetheically, however, the semantical closure is possible because of the presence of dialetheias. If, for some constant k, $k = \ulcorner \neg \mathsf{T}(k) \urcorner$, the initial evaluation $v_0(\mathsf{T}(k) = \{0, 1\})$ persists along all further evaluations. Similarly if $k = \ulcorner \mathsf{T}(k) \urcorner$. So the liar and the truth teller turn out to be dialetheias.

Observation 3 The extended language could be made more expressive by introducing some sorts of restricted quantifiers. Indeed, it would be desirable to quantify over all sentences as well as over all relativized sentences of the original languages. Here, we disregarded such refinement of the language for the sake of simplicity.

Conservativity. The extension of any theory by means of the predicate T with Tarski's rules is *conservative*.

Proof It is derivable from *Completeness* and *Theorem* 1. □

8 The Liar in DLEAC

At the beginning of the paper we observed that, according to Priest, dialetheism supplies the best solution to all self-reference paradoxes.[10]

What about the Liar in DLEAC?

Let A be a sentence of form $\neg\mathsf{T}(\ulcorner A \urcorner)$. You obtain that $\mathsf{T}(\ulcorner A \urcorner)$ is a dialetheia:

1	1.	$\mathsf{T}(\ulcorner A \urcorner)$	Assumption
1	2.	$\neg\mathsf{T}(\ulcorner A \urcorner)$	Tarski
	3.	$\neg\mathsf{T}(\ulcorner A \urcorner) \vee \neg\mathsf{T}(\ulcorner A \urcorner)$	*Material conditional*
	4.	$\neg\mathsf{T}(\ulcorner A \urcorner)$	Assumption
	5.	$\neg\mathsf{T}(\ulcorner A \urcorner)$	*Reflexivity*
	6.	$\neg\mathsf{T}(\ulcorner A \urcorner)$	E∨
	7.	$\mathsf{T}(\ulcorner A \urcorner)$	Tarski
	8.	$\neg\mathsf{T}(\ulcorner A \urcorner) \wedge \mathsf{T}(\ulcorner A \urcorner)$	I∧

Notice that–according to *conservativity*–it is impossible to use the liar paradox to obtain an *absurd*. Indeed, while you obtain a formula saying that the liar is false, there is no formula saying that it is *only false*. Nor can you get an *absurd* by using starred assumptions. Consider, for example the following proof:

1	1.	$\mathsf{T}(\ulcorner A \urcorner)^*$	Assumption
1	2.	$\neg\mathsf{T}(\ulcorner A \urcorner)^*$	Tarski
1	3.	$\neg\mathsf{T}(\ulcorner A \urcorner) \wedge \mathsf{T}(\ulcorner A \urcorner)^*$	IA
	4.	$\neg\mathsf{T}(\ulcorner A \urcorner)$	RAA
	5.	$\mathsf{T}(\ulcorner A \urcorner)$	Tarski
	6.	$\neg\mathsf{T}(\ulcorner A \urcorner) \wedge \mathsf{T}(\ulcorner A \urcorner)$	I∧

9 DLEAC and Other Formal Systems Dealing with *Rejection*

There are some calculi, developed in last years like, for example, the *refutation* or *rejection* calculi whose aim is to formalize the notion of *rejection*.

For a general introduction to these calculi in propositional logic see Skura (2011). Skura's refutation calculi [(developed in Skura (1992, 1996, 2009)] are based on a Łukasiewicz-style refutation calculi for propositional logics (see on this Łukasiewicz 1957). Skura proposed a system of this kind for the modal logic of S4 in Skura (1996). With the same purpose, Wansing (2016), introduced a natural deduction calculus whose central idea is to begin with pairs comprising a set of assertions and a set of rejections, obtaining by inference a similar pair. In a nutshell, Wansing's idea is to dualize the introduction and elimination rules for intuitionistic propositional logic with a primitive notion of dual proof so obtaining a kind of *bi-intuitionistic* propositional logic combining *verification* and its dual, i.e. *falsification*. His project is part of an "inferentialist, procedural specification of the meaning of the logical operations" (Wansing 2016, 426). Wansing on this observes that:

> From a falsificationist perspective it is of interest to consider whether there exists a unary operation that internalizes some notion of verification into the object language of a suitable logic, and from a verificationist perspective it is of interest to consider whether there is a one-place connective that internalizes some notion of falsification into the object language of a suitable logic. If such unary operations exist, it is a separate issue whether they are or are not genuine negations according to certain criteria. In particular, if falsification is seen

[10] See on this Priest (2006a, (2006b, (2010); on the same topic see Beall (2009), Colyvan (2009), Weber (2010).

> as a kind of reasoning that is dual to verification and if intuitionistic negation is viewed as a negation with respect to verification, one may wonder what is the counterpart of intuitionistic negation from the point of view of falsification (Wansing 2016, 426).

He formulates a logic motivated by a dualization of the natural deduction rules for intuitionistic propositional logic. It is outside the scope of this paper to enter into the details of the system and to confront Wansing's proposal with DLEAC. It is sufficient to observe that the chosen approach is a bi-intuitionistic propositional logic, deeply different from our dialetheistic perspective. Just to exemplify a difference: in a constructive approach *tertium non datur* does not work, whereas in our proposal, following dialetheism, *tertium non datur* is perfectly at work.

Finally, from a semantical point of view, it has been observed[11] that DLEAC reflects some intuition behind the notion of *consistency operator* $\circ$ introduced by da Costa Newton (1974) and studied by Carnielli, Coniglio, and Marcos in Carnielli et al. (2007). Carnielli, Coniglio, and Marcos have developed a family of systems known as *Logics of Formal Inconsistency* based on da Costa *consistency operator* $\circ$. In the *Logics of Formal Inconsistency* they expand da Costa's driven idea to introduce a consistency operator whose role is to control the behavior of contradictions, so avoiding explosion. In a nutshell, the idea is that in a three-valued semantics a *normality operator* $\circ$ singles out the classical formula:

$\circ A$ takes value $\{1\}$ iff A does not take value $\{0,1\}$.

In truth-table terms:

	$\neg A$	$\circ A$
1	0	1
i	*i*	0
0	1	1

Why not considering * as a semantic operator and adding it to an extension of LP, mapping classical truth values to $\{1\}$ and the inconsistent truth value to $\{0\}$? In such a perspective * could be considered as a classicalicality operator like $\circ$. Doing so we could obtain formulas with nested operators more expressive than our approach developed in DLEAC.

Some remarks, starting on this last point: we think that the above suggestion goes against to our main purpose of DLEAC: committing to speech acts the job of expressing exclusivity without affecting the dialetheic meaning of logical constants. And speech acts cannot be nested. It is of particular interest the fact that, in spite of this limitation, our acts of assuming and concluding are adequate to our purpose of recovering *exclusivity* in a dialetheic perspective. Observe that the dialetheic flavor of our speech acts is suggested by Priest's treatment of *rejection*.

Besides, we want to emphasize that our starred formulas are by no means to be regarded as classical. A star merely indicates that the marked sentence, when occurring in a proof, is assumed or concluded in an exclusive mode. Semantically, the star * indicates that a model of the marked sentence is, by definition, a dialetheic interpretation at which the sentence has the value $\{1\}$. In both contexts a star has nothing to do with the content of the marked sentence; in particular it does not affect the non-exclusive meaning of negation.

10 Conclusions

Our theory is inspired to Priest's treatment of the notions of negation and rejection. In particular, we have considered the following dialetheic theses:

1. Logical negation is non-exclusive;
2. The acts of accepting and rejecting the same sentence are incompatible.

Theses 1 and 2 suggest that, according to the dialetheic perspective, while one cannot express the fact that a sentence is *only true* or *only false* by means of a logical connective, nevertheless one can assume or conclude that a sentence is only true or only false, endorsing exclusivity in the *force* of the speech act of assuming or concluding.

If such speech acts are, as we think, dialetheically intelligible, DLEAC counts as a dialetheic theory of deduction. Our logic—DLEAC—is dialetheic in the same sense as LP: it rejects the general validity of *modus ponens* and *reductio ad absurdum* just because these rules may fail when dealing with dialetheias. It includes, as a particular case, classical deduction. If we take as starred assumptions all sentences having the form $\neg(A \wedge \neg A)$, it is easy to verify that all classical inference rules are recovered; as a result, we obtain a dialetheic interpretation of classical logic. Classical reasoning is dialetheic reasoning under the assumption that all contradictions are *only false.*

In Priest's perspective, the material conditional is not a genuine conditional because, in general, it does not permit the validity of MPP[12]. In our approach, the validity of MPP is appropriate under a starred assumption. This way, we obtain the following reading of the quasi-validity of

[11] Thanks to the referee for the suggestion.

[12] For an extended discussion of this topic see Carrara and Martino (2014).

MPP for a dialetheist: MPP is appropriate when at least one of the two premises is starred.

DLEAC brings forward the *Introduction of absurd*, an authentic absurdity, whose truth is to be rejected even by a dialetheist. This notion provides a dialetheic version of the *reductio ad absurdum*.

It is worth noticing that DLEAC shows the equivalence of falsity and untruth. Priest's truth theory does not validate the inference from falsity to untruth. The failure of this inference is clearly justified for a *gap-theorist*, because if a sentence is neither true nor false, it is, in particular, not true, i.e. untrue. However, this argument is not available for a dialetheist, who adopts *tertium non datur* as a logical law. At first glance, a dialetheic reason for the failure of the inference at issue may seem to be the consideration that the falsity of a sentence does not exclude its truth. But untruth does not necessarily exclude truth either, nor can we see other dialetheic reasons for regarding untruth as more general than falsity. Consequently, we believe that our theory supplies a dialetheic reason for the equivalence of falsity and untruth.

Finally, DLEAC does justice to the dialetheic claim that the (strengthened) liar sentence is a dialetheia.

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