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Multivalent Semantics for Vagueness and Presupposition

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Abstract Both the phenomenon of presupposition and that of vagueness have motivated the use of one form or another of trivalent logic, in which a declarative sentence can not only receive the standard values *true* (1) and *false* (0), but also a third, non-standard truth-value which is usually understood as 'undefined' (#). The goal of this paper is to propose a multivalent framework which can deal simultaneously with presupposition and vagueness, and, more specifically, capture their projection properties as well as their different roles in language. Now, there is a *prima facie* simple way of doing this, which simply consists in assimilating the two phenomena, and using an appropriate type of trivalent logic. On this view, we just need a compositional system that deals with the 'undefined' truth-value, and does not care about whether the source of undefinedness is 'presuppositional' or related to vagueness. I will argue that such a simple solution cannot succeed, and point out a number of desiderata that any successful approach must meet. I will then present and discuss two seven-valued semantics, inspired, respectively, by the Strong Kleene semantics and by supervaluationism, which meet these desiderata.

Keywords Presupposition · Vagueness · Trivalent logics · Multivalent logics · Projection · Felicity · Supervaluations · Strong kleene

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1 Introduction

Both the phenomenon of presupposition and that of vagueness have motivated the use of one form or another of trivalent logic, in which a declarative sentence can not only receive the standard values *true* (1) and *false* (0), but also a third, non-standard truth-value which is usually understood as 'undefined' (#). In the case of presuppositional sentences, the third truth-value is taken to reflect 'presupposition failure'. To illustrate, the sentence *The king of France is bald* is assigned the value # if there is no king of France. In the case of sentences involving vague terms, the third truth-value is assigned when the sentence is neither clearly true not clearly false. For instance, if there is a king of France but the king of France is a borderline-case of baldness, the sentence *The king of France is bald* ends up undefined. In other terms, trivalent logics can be used to model the *felicity conditions* as well as the *clarity conditions* of sentences. Now, as we have just illustrated with the case of *The king of France is bald*, a single sentence can exhibit both phenomena. Furthermore, there are sentences in which the 'intuitive' presupposition is itself vague, i.e. it is possible for a sentence to have *vague* felicity conditions (rather than simply vague truth-conditions)—a point discussed by Jérémy Zehr in recent work (Zehr 2013, 2014). Consider for instance *Mary knows that Peter is bald*. Because *know* is factive, it triggers the presupposition that its complement clause (in this case *Peter is bald*) is true. But note that this complement clause is itself a vague sentence. If Peter is a borderline case of vagueness, then it is not clear whether the factive presupposition of the sentence is met or not.

It is clear that one needs to capture both presuppositional and vagueness phenomena in natural languages in a single, unified framework. First, this is so because we want to provide a semantics for a natural language as a whole. If a natural language exhibits both phenomena, then this semantics should itself be able to deal with both. Second, as we have just seen, presuppositions and vagueness interact in single sentences, and we need to have a predictive theory of what the felicity and clarity conditions of such sentences are.

The goal of this paper is to propose a framework which can deal simultaneously with presupposition and vagueness, and, more specifically, capture their *projection* properties. Now, there is a *prima facie* simple way of doing this, which simply consists in assimilating the two phenomena, and using an appropriate type of trivalent logic. On this view, we just need a compositional system that will deal with the 'undefined' truth-value, and does not care about whether the source of undefinedness is presuppositional or related to vagueness—indeed, on such a view, it might be that this distinction has no theoretical relevance for semantics proper. In the first section, I will argue that such a simple solution, though appealing, cannot succeed. In the second section, I will present and discuss a multivalent framework, inspired by the Strong Kleene semantics, that meets the *desiderata* mentioned in the first section. In the third section I will discuss a variation on this proposal, which is inspired by supervaluationism.

I should note that Jerémy Zehr's dissertation (Zehr 2014) is probably the first work to have addressed this challenge. I will not discuss here Zehr's approach, which is significantly different from the one I suggest in this paper, but I would like to acknowledge that some of the conversations I had with Jérémy Zehr played an important role in the development of my own ideas. It might be useful in further work to investigate the formal relationships between Zehr's proposals and the proposal I make in this paper.

Importantly, I will only deal with the behavior of sentential connectives (and will focus mostly on conjunction, though the system I will present is predictive for all Boolean operators). Accordingly, the multivalent semantics I will introduce and discuss will be defined for a propositional language. Despite this important limitation, the underlying ideas of my proposal can in principle be extended to a more complex language, such as one with predicates and generalized quantifiers.

2 Trivalent Logics for Presupposition Projection and Clarity Projection

2.1 Background: Middle Kleene as a Solution to the Projection Problem for Presupposition

The projection problem for presupposition is that of finding an algorithm which assigns a presupposition to any complex sentence on the basis of the presuppositions of its part. One possible approach is to adopt a trivalent semantics where sentences can receive a third truth-value, noted #, when their presuppositions are not satisfied. In such a system, the projection problem is not conceived differently from what we could call the problem of truth-value projection. The problem of truth-value projection is simply the problem of deriving the truth-value of a complex sentence on the basis of the truth-values of its atomic parts. In the case of propositional logic, the recursive definition of truth by means of truth-tables is the standard solution. Once we have a third truth-value, the very same device can be used for presuppostion projection. Given a recipe for assigning one of the three truth-values to any sentence, we can use this recipe to determine, for any sentence, the conditions under which it does not receive the third truth-value #. These conditions are then the presupposition of the sentence. Let me illustrate by means of a concrete proposal, namely the trivalent semantics for propositional logic which is sometimes called Middle Kleene (Peters 1979; Beaver and Krahmer 2001; George 2014), which is summarized by the following truth-tables:

ϕ ϕ	$\neg\phi$	A	B	A ∧ B	A	B	A ∨ B	A	B	A → B
1	0	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	0	1	1	0	0
#	#	1	#	#	1	#	1	1	#	#
		0	1	0	0	1	1	0	1	1
		0	0	0	0	0	0	0	0	1
		0	#	0	0	#	#	0	#	1
		#	1	#	#	1	#	#	1	#
		#	0	#	#	0	#	#	0	#
		#	#	#	#	#	#	#	#	#

To see how this can be used to solve the presupposition projection problem, let us enrich our language so that every atomic sentence can be associated, if need be, with a presupposition. We will write q_ψ to denote an atomic sentence which is lexically specified as presupposing some proposition ψ. What this means is that whenever ψ is false, p receives the value #. How can we then compute the presupposition of a sentence of the form, say, $p \wedge q_\psi$? We assume here that p is not presuppositional, hence cannot receive the value #. We end up with the following truth-table, which exhausts the logical space:

ψ	p	q_ψ	$p \wedge q_\psi$
1	1	1	1
1	0	1	0
1	1	0	0
1	0	0	0
0	1	#	#
0	0	#	0

Now, the sentence q_ψ receives the truth-value # when ψ is false (cf. the last two lines), and otherwise receives the value 1 or 0. By assumption, p is bivalent, i.e. never receives the value #. This is why we end up with only 6 lines in the above truth-table. Then, ψ does not play any further role in determining the truth-value of the sentence: we simply apply the truth-table for conjunction to the columns corresponding to p and q_ψ. Now, the sentence's presupposition consists in the set of situations where it receives the truth-value 0 or 1. These situations are exactly those that exclude the fifth line. And this fifth line can be described by the formula $p \wedge \neg\psi$. So the presupposition of the sentence is simply the negation of this statement, which is equivalent to the material conditional $p \rightarrow \psi$. This is the presupposition assigned to sentences of this form by most current theories of presupposition projection (cf, e.g., Heim 1983)

Now, to turn Middle-Kleene into a theory of presupposition, one needs a further ingredient, namely, following Stalnaker (1978), an *assertability condition*. Middle Kleene in itself does not tell us what the third truth-value stands for, how it relates to actual language use. What we need is a bridge from the computation of truth-values to a notion of *felicity* involving in one way or another *common knowledge*. A sentence's presupposition is usually viewed as defining its *felicity conditions*, and more specifically, as constraining what the common ground between speaker and adressee must be for the sentence to be felicitous. The assertability condition that we need is the following:

(1) A sentence S is felicitous in a given context only if in every world compatible with what is common knowledge in this context, S receives either the value 0 or the value 1.

The effect of (1) is that a sentence S is felicitous only if the *context set* (defined as the set of worlds that are compatible with common knowledge) entails the presupposition of S.

Importantly, the Middle Kleene truth-tables are asymmetric. Specifically, while $p \wedge q_\psi$ is predicted to presuppose $p \rightarrow \psi$, $q_\psi \wedge p$ is predicted to presuppose ψ. This can be seen easily by reversing the order of columns in the above truth-table and applying again the Middle Kleene truth-table for conjunction:

ψ	q_ψ	p	$q_\psi \wedge p$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	#	1	#
0	#	0	#

The crucial difference with the previous case can be seen in the last line. In Middle Kleene, when the first conjunct receives the value #, the whole conjunction receives the value #, irrespective of the value of the second sentence. Now, as a result, the sentence $q_\psi \wedge p$ receives the value # exactly when ψ is false, i.e. presupposes ψ. To see why the asymmetry in Middle Kleene might be desirable, let us focus on the two following sentences, in which p is instantiated by *John plays the violin*, q_ψ by *John keeps his musical instrument well hidden*, where ψ is intended to be *John has a musical instrument.*

(2) a. John plays the violin and he keeps his musical instrument well-hidden.
b. John keeps his musical instrument well hidden and he plays the violin.

What Middle-Kleene predicts is that (2a) presupposes the material conditional *John plays the violin* $\rightarrow$*John has a musical instrument*, while (2b) presupposes *John has a musical instrument*. Given that the material conditional *John plays the violin* $\rightarrow$*John has a musical instrument* is a near tautology, the prediction is that (2a) is felicitous even if common knowledge does not entail that John has an instrument, in contrast with (2b). In terms of truth-conditional intuitions, one should view (2a) as false if John doesn't play the violin and has no musical instrument, but in the same situation (2b) should be a presupposition failure. In terms of felicity conditions, the prediction is that by using (2b), one assumes that it is common knowledge that John owns a musical instrument, and that this is not so with (2a). While this is a subtle difference, it seems to be in line with intuitions.[1] The contrast between the two cases might become

[1] There is a recent debate as to whether the predicted asymmetry is real. Schlenker (2008) and Chemla and Schlenker (2012) argue for a kind of ambiguity, whereby the predictions made by the Middle Kleene truth-tables correspond to one of two possible readings (though these proposals do not adopt a trivalent approach to presupposition projection), while the other reading corresponds to what would be predicted by the *symmetric* Strong Kleene truth-tables (see below). Rothschild (2011), on the other hand, claims that there is no real asymmetry in presupposition projection.

clearer when they are turned into polar questions (which allows us to clearly tease apart presuppositions from entailments, assuming that polar questions inherit the presuppositions of their declarative counterparts), as in the following:

(3) a. Is it true that John plays the violin and keeps his musical instrument well hidden?
b. Is it true John keeps his musical instrument well hidden and plays the violin?

Middle-Kleene predicts that (3b), but not (3a), should license the inference that John has a musical instrument. My own impression (in line with most of the literature) is that there is a clear contrast in the predicted direction.

2.2 Two Problems for Unifying Clarity Projection and Presupposition Projection

While Middle Kleene seems to provide an adequate account of presupposition projection, this is not so for *clarity projection*. In a trivalent approach to vagueness, vague sentences receive the value # when they are not clearly true or clearly false. For instance, if John is a borderline-case of tallness, the sentence *John is tall* receives the value #. In the same way as a trivalent semantics for presuppositions provides a solution to the projection problem for felicity conditions, it can also provide a solution to the projection problem for clarity conditions. That is, we want to know under which conditions a complex sentence that contains *John is bald* as a subpart receives the truth value 1 or 0, rather than #. A fully explicit trivalent system is able to assign one of the three truth-values to any complex sentence on the basis of the truth-values assigned to atomic sentences, and therefore embodies a solution to the clarity projection problem.

Suppose, then, that we use a specific trivalent semantics to solve the projection problem for presupposition, on the one hand, and a potentially different trivalent semantics to solve the projection problem for vagueness, on the other hand. Then the question arises how we can put these two proposals together—given that natural languages exhibit both phenomena. In the next section, I point out that even if the very same truth-tables were adequate for both presupposition projection and vagueness, we would still face a unification problem. In the subsequent section, I argue that in any case we cannot use the same truth-tables for both phenomena: while Middle Kleene is a good candidate as a model for presupposition projection, it is not for clarity projection.

2.2.1 Clarity Conditions are Not Felicity Conditions

Assume, for the sake of the discussion, that, say, Middle Kleene is adequate for both presupposition projection and clarity projection. One might then think that there is no need to distinguish between the two sources of undefinedness and simply treat all cases of undefinedness in the same way. This, however, would be problematic, because the assertability condition in (1) would then apply indiscrimnately to presuppositional expressions and to vague expressions. Consider for instance the two following questions:

(4) Is John's musical instrument well hidden?
(5) Is John tall?

While (4) intuitively presupposes that John has a musical instrument, (5) does not intuitively presuppose that John is either clearly tall or clearly not tall. To explain the judgment regarding (4), we need to say that a polar question is felicitous only if common knowledge entails that its declarative counterpart is either true or false. However, this would automatically entail that (5) is not felicitous if common knowledge is compatible with the possibility that John is a borderline-case of tallness. This seems to be an undesirable result. We can show this by using the so-called *Wait a minute!*-test, viewed as a test for presupposition (von Fintel 2004). In particular, one can object to (4) by denying that the common knowledge assumption holds, by saying *Wait a minute! I didn't know that John has a musical instrument*. In contrast with this, it would be somewhat odd to reply to (5) by saying *Wait a minute! I didn't know that John is either clearly tall or clearly not tall.*

Somehow, we need the assertibility condition in (1) to be able to distinguish between two sources of potential undefinedness, i.e. presuppositional undefinedness and vagueness-related undefinedness. But if the underlying semantics does not itself make this distinction, it is not trivial to achieve such a result.

2.2.2 Clarity Conditions are Symmetric

Besides the problem I have just noticed, there is another difficulty. While Middle Kleene seems to provide an adequate account of presupposition projection, this is not so for clarity projection. One of the most standard trivalent approaches to vagueness is supervaluationism, which is, crucially, not fully compositional and thus cannot be expressed in terms of truth-tables. The closest compositional trivalent semantics for propositional logic is, clearly, the Strong Kleene (SK) semantics, which is based on a very similar intuition.[2] Let me remind the reader of the SK truth-table for conjunction.

[2] The *Strict-Tolerant* framework of Cobreros et al. (2012), whose original motivation is to provide a theory of vagueness, can be viewed as based on an underlying Strong Kleene semantics, as explained in Cobreros et al. (2015).

A	B	A ∧ B
1	1	1
1	0	0
1	#	#
0	1	0
0	0	0
0	#	0
#	1	#
#	0	0
#	#	#

SK-conjunction is symmetric ($A \wedge B$ and $B \wedge A$ are equivalent). The underlying intuition is the following (Kleene 1952): the truth-value # is treated as representing *uncertainty* about whether the sentence is true or false. Now suppose B has the value #, which we can interpret as characterizing a situation where it is not known whether B is true or false. If A is false, the conjunction $A \wedge B$ is known to be false no matter what, in spite of the uncertainty regarding the actual truth-value of B. In fact, the SK truth-tables can be viewed as derived from bivalent truth-tables by means of the following recipe, which spells out Kleene's original motivation (see George 2014):[3]

(6) Let f be a binary standard Boolean function. Then f', the SK-version of f, is defined as follows.

a. For any pair (x, y) in $\{0, 1, \#\} \times \{0, 1, \#\}$, a *repair* of (x, y) is any pair (x', y') in $\{0, 1\} \times \{0, 1\}$ such that if $x \neq \#$, $x' = x$ and if $y \neq \#$, $y' = y$.

b. $$f'(x,y) = \begin{cases} 0 & \text{if for every repair } (x',y') \text{ of } (x,y),\ f(x',y') = 0 \\ 1 & \text{if for every repair } (x',y') \text{ of } (x,y),\ f(x',y') = 1 \\ \# & \text{otherwise} \end{cases}$$

Let us apply this to the following sentences:

(7) a. Mary is French and she is tall.

b. Mary is tall and she is French.

(7a) is predicted to be clearly true if Mary is French and is clearly tall. It is predicted to be clearly false if either Mary is not French (this is sufficient to make the sentence false even if Mary is neither clearly tall nor clearly not-tall) or if Mary is clearly not tall. This is in line with intuitions: if we know that Mary is not French, this is sufficient to claim that (7a) is false, even if we also consider Mary to be a borderline-case of tallness. And things do not seem to be different when the conjuncts are reversed, as in (7b). Again, in a situation where Mary is not French and is a borderline-case of tallness, we clearly judge (7b) to be false rather than undefined. These intuitions provide motivation for the SK truth-table for conjunction.

We are thus faced with a non-trivial unification problem. Namely, the projection properties of presuppositional expressions and of vague expressions appear to be different. And as we saw in the previous section, presuppositions and vagueness are also different from the point of view of their role in language use—presuppositions contribute to felicity conditions in a way that vagueness does not. But we need to have a way to assign both presuppositions and clarity conditions to sentences that display both phenomena. This concerns not only cases like 'Peter is tall and his brother is tall too' but also cases where a single phrase is at the same time presuppositional and vague. For instance, as J. Zehr (p.c.) pointed out to me, a sentence such as 'Mary stopped smoking' presupposes that Mary used to smoke - a proposition that is itself vague (how frequently does one need to have smoked in order to count as someone who used to smoke?).

In the next sections I will make two distinct but related proposals which will provide a unified model of presupposition and vagueness that meet three desiderata: they make it possible to state an assertability condition that relates presuppositions to felicity conditions without undesirable consequences for vague sentences, they account for the distinct projection patterns observed for presupposition and vagueness, and they can deal with sentences in which both phenomena interact.

3 An SK-Inspired Seven-Valued Logic for Presupposition and Vagueness

The solution I will propose is based on an intuition and a specific implementation of this intuition. The intuition can be couched as follows: presuppositions are part of the semantic content of sentences, whereas vagueness should be treated in terms of ambiguity, or, more properly, semantic underdeterminacy. In this sense, the assignment of clarity conditions to sentences is viewed as a post-semantic process that applies *after* the assignment of presuppositions. I will first implement this idea by

[3] There is of course a close relationship between Strong Kleene and Middle Kleene. Middle Kleene can be viewed as an asymmetric variant of Strong Kleene. In Middle Kleene, for every binary connective, whenever the first argument received a standard truth-value (0 or 1), the corresponding line is identical to its counterpart in the Strong Kleene truth-table. When the first argument receives the value #, then the sentence as a whole receives the value #. George 2008 offers a way of deriving in a systematic way Middle Kleene truth-tables on the basis of classical truth-tables, simply by using a different definition of a 'repair' from the one used below in (6). George furthermore extends this account to predicate logic and beyond.

generalizing the procedure that is behing the Strong Kleene approach, following an idea that has been proposed (with a completely different goal in mind) by Priest (1984) and more recenty by Ripley (2013). I will then (in the next section) offer a supervaluationist implementation of the same idea.

Given bivalent connectives, (6) gives us a way to define trivalent connectives in a systematic way. But nothing crucial hinges on the fact the underlying logic is bivalent. We could also start from a trivalent logic, and then define on its basis a multivalent logic where the additional truth-values are assigned to cases where the relevant sentences do not receive clear 'basic' truth-value.

As explained in Sect. 2.2.2, the standard SK-semantics can be viewed as a trivalent system that is derived from a bivalent underlying system, and where the third truth-value represents uncertainty about the actual truth value that a certain atomic proposition has. Suppose now that the underlying system is trivalent. On the basis of the three basic truth-values $\{0, \#, 1\}$, one can define derived truth-values (the 2nd-level truth-values) which will represent the fact that there is uncertainty as to whether a certain proposition is, say, false or undefined, or true or false, etc. These 2nd-level truth-values will be all the non-empty sets of basic truth-values (Ripley 2013). That is, starting from three truth-values 0, 1 and $\#$, the set $\{0, \#\}$ will be the 2nd-level truth-value assigned to a proposition whose basic truth-value is not fully determined but which can only be either 0 or #. At this new level, falsity is then represented as $\{0\}$ and truth as $\{1\}$. The resulting 7 truth-values are the following: $\{0\}$, $\{\#\}$, $\{1\}$, $\{0, \#\}$, $\{0, 1\}$, $\{\#, 1\}$, $\{0, \#, 1\}$.

Now, any binary connective in a trivalent semantics can be represented by a function from $\{0, \#, 1\}$ to $\{0, \#, 1\}$. Call such a function the 1st-level interpretation of the connective. We want to define the 2nd-level interpretation in a systematic way, similarly to the way the SK-semantics can be defined on the basis of a classical semantics, as in (6). Let us take an example. Consider again a sentence of the form $A \wedge B$, where $\wedge$ is interpreted according to the Middle Kleene truth-table for conjunction. Suppose the 2nd-level truth-value of A is $\{1\}$ and that of B is $\{0, \#\}$. This is interpreted as meaning that the 1st-level truth-value of A is 1 and that the 1st-level truth-value of B could be either 0 or $\#$. If the 1st-level truth-value of B were 0, the conjunction as whole would receive the value 0 (given the Middle Kleene truth-table). If it were $\#$, the conjunction as a whole would receive the value $\#$. So the set of 1st-level truth-values that $A \wedge B$ can have (given our uncertainty) is $\{0, \#\}$. As a result, we want this set to be the 2nd-level truth-value of $A \wedge B$.

This reasoning illustrates the following general rule for defining 2nd-level functions from 1st-level functions corresponding to binary connectives (again, along the lines of Priest 1984 and Ripley 2013).[4]

(8) Let f be a binary function from $\{0, \#, 1\}^2$ to $\{0, \#, 1\}$. Then f', the 2nd-level extension of f, is the function from
$\{\{0\}, \{\#\}, \{1\}, \{0, \#\}, \{0, 1\}, \{\#, 1\}, \{0, \#, 1\}\}^2$
to
$\{\{0\}, \{\#\}, \{1\}, \{0, \#\}, \{0, 1\}, \{\#, 1\}, \{0, \#, 1\}\}$
such that:

For any (x, y) in $\{\{0\}, \{\#\}, \{1\}, \{0, \#\}, \{0, 1\}, \{\#, 1\}, \{0, \#, 1\}\}^2$,
$f'(x, y) = \{t \in \{0, \#, 1\} : \exists x' \exists y' ((x' \in x) \wedge (y' \in y) \wedge f(x', y') = t)\}$

From now on, I will omit brackets and simply write, e.g., 0# for $\{0, \#\}$. Here is the resulting 2nd-level truth-table for conjunction (in 'Cartesian form'), assuming that at the 1st-level conjunction is interpreted according to Middle Kleene. This 2nd-level truth-table of course inherits the asymmetry of the 1st-level truth-table (note that it displays the truth-value of $A \wedge B$, not of $B \wedge A$, in terms of the truth-values of A and of B).

B / **A**	0	#	1	0#	01	#1	0#1
0	0	0	0	0	0	0	0
#	#	#	#	#	#	#	#
1	0	#	1	0#	01	#1	0#1
0#	0#	0#	0#	0#	0#	0#	0#
01	0	0#	01	0#	01	0#1	0#1
#1	#0	#	#1	0#	0#1	#1	0#1
0#1	0#	0#	0#1	0#	0#1	0#1	0#1

Now, the idea is that # stands for presupposition failure, and 01 for cases where the truth-value is unclear. But there are now additional possibilities. For instance the truth-value #1 would correspond to a case where it is unclear whether the sentence's presupposition is satisfied or not, but where if we decide to interpret the relevant vague

[4] Priest (1984), followed by Ripley (2013), offers a hierarchy of levels, where the semantics at level $n+1$ is derived from the one at level n by means of a generalization of the rule given in (8). The 1st-level semantics in Ripley (2013) is just classical, bivalent semantics. The 2nd-level truth-values consist of the non-empty subsets of $\{0, 1\}$, i.e. $\{0\}$, $\{1\}$, and $\{0, 1\}$, and the associated semantics is then Strong Kleene, with $\{0, 1\}$ playing the role of the third truth-value. The 3rd level in Ripley's system thus corresponds to my 2nd level, and its semantics is derived exactly as stated in (8), but not on the basis of Middle Kleene—so the resulting semantics is different from the one presented here.

material so that the presupposition is satisfied, then the sentence is true. To see what this might mean, consider the sentence *John stopped smoking*, in a situation where we know that in the past John occasionally smoked and that he never smokes now (I owe this example to Jérémy Zehr). The question whether the presupposition that John used to smoke is satisfied is unclear. It depends on whether the kind of occasional smoking that John engaged in counts as smoking in the habitual sense. So we are not sure whether the sentence is defined or not. But, if we have a liberal interpretation of what 'used to smoke' means, the sentence then has a defined truth-value and is in fact true, since now John never smokes. So the set of the 1st-level truth-values that the sentence could receive in this situation is $\{\#, 1\}$, which is therefore the 2nd-level truth-value of the sentence.

The resulting system is in principle able to address the two challenges pointed out in Sect. 2.2. One of the challenges (cf. Subsect. 2.2.2) is that we want to capture the fact that presupposition projection is asymmetric, i.e. is sensitive to linear ordering, whereas clarity projection is symmetric. Now, in the seven-valued semantics I have just characterized, whenever there is no presuppositional material in a sentence, one can safely ignore all the lines and columns where $\#$ appears, i.e. only look at the lines and columns headed by 0, 1 or 01. What remains is then identical to the standard SK-truth table for conjunction, where 01 would stand for the third truth-value. And whenever there is no vague predicate, we can ignore the last four columns and last four lines, and we end up with Middle Kleene. So this system satisfies the desideratum pointed out in Sect. 2.2.2, i.e. is able to capture the fact that presupposition projection is asymmetric and clarity projection is symmetric.

Now, does this system solve the desideratum outlined in Sect. 2.2.1? We want to be able to assign felicity conditions to every sentence as something distinct from clarity conditions. This is now easy to do, because in this system we can distinguish between presupposition failure ($\#$) and unclarity (all non-singleton values). We can for instance state that a sentence is felicitous only if its presuppositions are *clearly* satisfied in all the worlds compatible with common knowledge. This idea takes the form of the following assertability condition:

(9) A sentence S is felicitous in a certain context only if in every world of the context set S receives a 2nd-order truth-value that does not include $\#$.

We will then say that a sentence S is *presuppositional* if there exists an assignment of 2nd-level truth-values to atomic sentences such that S receives a 2nd-order truth-value that includes $\#$ relative to this assignment.

We could also view things slightly differently and define *clarity conditions* for felicity itself. That is, we could say the following:

(10) a. A sentence S is *clearly felicitous* in a given context if in every world of the context set, S receives a 2nd-order truth-value that does not include $\#$

b. A sentence S is *borderline-felicitous* in a given context if in every world of the context set, S does not receive the truth-value $\{\#\}$ as its 2nd-order truth-value.

So this system is able to satisfy the two desiderata discussed in Sect. 2.2.

Let me now illustrate in more details what the seven truth-values correspond to intuitively, for a sentence such as *John stopped smoking*.

Situation type	Truth-value
John used to be a heavy smoker and still is	0
John has never smoked	#
John used to be a heavy smoker and never smokes now	1
John was an occasional smoker in the past and is a heavy smoker now	0#
John used to be a heavy smoker and is an occasional smoker now	01
John used to be an occasional smoker and never smokes now	#1
? (To be discussed below)	0#1

It is unclear whether we want 0#1 to be a possible value for *John stopped smoking*. This truth-value should be assigned if it is unclear both whether John used to smoke and whether he smokes now, that is, if he used to be an occasional smoker and still is an occasional smoker. However, one might reason as follows. Assume that John was an occasional smoker in the past and still is. The uncertainty as to whether the presupposition is satisfied has to do with what it means to be a real smoker (as opposed to an occasional smoker). Any decision as to whether the presupposition should be considered to be satisfied is also *ipso facto* a decision about whether John is a smoker now. Suppose that the kind of occasional smoking that John engages in counts as smoking. Then the presupposition is satisfied but then it is false that John stopped smoking. If, however, we decide that John was not really a smoker in the past, then the whole sentence receives the value #. So, depending on how 'smoke' in the habitual sense is interpreted, the sentence will receive either the truth-value 0 or #, but never the truth-value 1. Maybe one should consider a case where John used to be an occasional smoker, and still is now, but smokes even less than before. In such a situation, there is a way of precisifying the meaning of 'being a regular smoker' such that John used to be an habitual smoker in the past but no longer is, and under this precisification, the sentence is then true. But for other precisifications, the sentence could be false or undefined, and so it is reasonable to assign the value 0#1.

Turning now to projection, let us consider the following sentence:

(11) John has tall children, and his children all have blue eyes

Let us see which 2nd-level truth-values are assigned in various situations.

Situation type	1st conjunct	2nd conjunct	(11)
John has no children	0	#	0
John has short children with blue eyes	0	1	0
John has short children who don't have blue eyes	0	0	0
John has borderline-tall children with blue eyes	01	1	01
John has borderline-tall children who don't have blue eyes	01	0	0
John has clearly tall children with blue eyes	1	1	1
John has clearly tall children who don't have blue eyes	1	0	0

The situation types in this table exhaust the logical space. Importantly, we get as a result that the sentence never receives a truth-value that includes #. That is, despite the presence of presuppositional material in the second conjunct, the sentence ends up being non-presuppositional, which is in line with intuitions (ignoring the fact that *John* is itself presuppositional, of course). This is so because all the situations where the second conjunct receives the value # (first line) happen to be situations where the first conjunct is false. In the Middle Kleene semantics, a conjunctive sentence where the first conjunct entails the presupposition of the second one is necessarily non-presuppositional. In this more complex framework, what we derive is this. Let us say that a sentence A *robustly entails* a sentence B if whenever A is assigned either 1 or 01, then B receives the value 1. If A is a non-presuppositional sentence which robustly entails the presupposition of B, then $A \wedge B$ ends up non-presuppositional, in the sense that it is never assigned a 2nd-level truth-value which includes #.

Things are different if the first conjunct entails the presupposition of the second one only in the standard sense (i.e. whenever the first conjunct is true the presupposition of the second conjunct is true). Let us for instance consider the following:

(12) John has tall sons, and his tall children all have blue eyes.

Suppose that all of John's children are boys who are borderline-tall and have blue eyes. Because John's children are borderline-tall, the first conjunct receives the value 01, and the second conjunct's value is #1 (# if John's sons don't count as tall, since in this case the presupposition of the second conjunct is not met, 1 if they count as tall). The resulting truth-value for (12) is thus 0#1. So the sentence can be assigned a truth-value that includes #, and is in this sense presuppositional.

It is not clear however that this is a good result. Let us have a closer look at how this result comes about. The 2nd-level truth-value of a conjunction $A \wedge B$ is the set of 1st-level truth-values that result from all the possible ways of picking a 1st-level truth-value in the 2nd-level truth-value of A and in the 2nd-level truth-value of B. In the case of (12), in the described situation, the reason why # ends up being in the 2nd-level truth-value of the sentence is because we can pick 1 for the first conjunct and then # for the second conjunct. Picking the value 1 for the first conjunct amounts to deciding to interpret *tall* in such a way that John's sons count at tall. But picking the value # for the second conjunct amounts to choosing to interpret *tall* in such a way that the presupposition of *his tall children* are not met, which entails that John's sons do not count as tall. So the choice of 1 for the first conjunct and of # for the second conjunct amounts to picking a different interpretation of *tall* in both conjuncts.[5] Our system is not sensitive to the fact that a single expression occurs several times in a sentence. In this system, the 2nd-level truth-value of a sentence is the set of all the 1st-level truth-values the sentence can have given all the possible combinations of 1st-level truth-values that each *occurrence* of an atomic proposition can have. Such a combination of 1st-level truth-values can be viewed as a specific choice of interpretation for all the vague expressions that occur in the sentence. Seen in this light, the system allows for a non-

[5] As Paul Egré pointed out to me (p.c.), this might actually be reasonable, because the threshold for tallness might be different for boys and girls, or for young children and adult children. I am assuming here that the standard for tallness is uniform for all of John's children. One way of making this natural is to consider a case where John has no daughter, and only adult sons, who happen to be borderline-tall and to have blue eyes.

uniform interpretation of a given atomic sentence across its different occurrences in a single sentence.

We inherit a property of the standard Strong Kleene approach. In the Strong Kleene approach to vagueness, as well as in our seven-valued semantics, a sentence such as *John is tall and John is not tall* is not necessarily false, because if John is a borderline-case of tallness, both conjuncts receive the undefined truth-value (i.e. 01 given our approach), and as a result the whole conjunction itself receives this value. Yet one may think that even if we are uncertain whether John counts at tall, we know that whatever decision we make regarding the meaning of *tall*, *John is tall and John is not tall* comes out false. In this case again, the SK-system does not 'see' that the word *tall* occurs twice. In the same way as this might be viewed as a limitation of Strong Kleene, which we directly inherit, the prediction of our system for (12) can also be viewed as inadequate. It is not at all clear, at an intuitive level, that if John's sons are borderline-tall and have blue eyes, there is any feeling that (12) is possibly infelicitous.

The supervaluationist approach to vagueness (Fine 1975) can be viewed as based on the same intuition as Strong Kleene, but as taking into account the fact that a certain vague expression occurs several times in a sentence is taken into account (at the cost of giving up the possibility of computing truth-values in a compositional way). In Sect. 4, I will define a supervaluationist seven-valued logic which will predict (12) can never receive a truth-value that includes #. The seven-valued logic developed in this section can be seen as the closest compositional counterpart to the supervaluationist seven-valued logic that will be discussed in Sect. 4.

Before turning to this supervaluationist version, let me consider another interesting case, where there is no entailment between the first conjunct and the presupposition of the second conjunct (nor the other way around):

(13) Mary has babies, and her twins have blue eyes.

Ignoring for a moment the fact that *baby* has a vague meaning (at what age exactly does a child stops being a baby?), the presupposition predicted for (13) by standard approaches, be it dynamic semantics (Heim 1983) or trivalent approaches (Peters 1979; Beaver and Krahmer 2001; George 2008; Fox 2009; George 2014), is the material conditional *Mary has babies* $\rightarrow$ *Mary has twins*.

Let us see what is predicted by our logic with 7 truth-values for such a sentence, when we take into account that fac that *baby* is a vague term. In the table below, 'infant' is used to refer to children who are clearly babies, and 'toddlers' to children for which it is unclear whether they count as babies.

Situation type	1st conjunct	2nd conjunct	(13)
Mary has no young children but has twins with blue eyes	0	1	0
Mary has no young children but has twins who don't have blue eyes	0	0	0
Mary has toddlers and twins with blue eyes	01	1	01
Mary has toddlers and twins who don't have blue eyes	01	0	0
Mary has toddlers but no twins	01	#	0#
Mary has infants and twins with blue eyes	1	1	1
Mary has infants and twins who don't have blue eyes	1	0	0
Mary has infants but no twins	1	#	#

We see right away that (13) receives a truth-value that includes # just in case Mary has children who are either clearly babies or borderline-cases of babies and no twins. So the situations where the sentence does not receive a truth-value that includes # are exactly those where the following material conditional is true: *Mary has children who are either clearly babies or borderline-cases of babies* $\rightarrow$ *Mary has twins* is true. Now, note that these are exactly the situations where the **Strong Kleene** material conditional *Mary has babies* $\rightarrow$ *Mary has twins* is true. So, if we define the presupposition of a sentence as the set of worlds where the sentence's truth-value does not include #, the sentence's presupposition appears to be the Strong Kleene material conditional *Mary has babies* $\rightarrow$ *Mary has twins*. Now, given that neither *Mary has babies* nor *Mary has twins* are presuppositional, the SK interpretation of this material conditional is exactly the same as the interpretation assigned by our 7-valued logic (because when we restrict our 7-valued truth-tables to lines and columns where # does not appear, we get exactly the SK-truth-table). So we seem to get the result that a sentence of the form $A \wedge B_p$ (where A has no presupposition but is possibly vague and B is not vague but presupposes p, which is itself not vague) presupposes $A \rightarrow p$, interpreted according to the 7-valued logic we have defined. In a sense, then, we thus reproduce the predictions of standard approaches to presupposition.

4 A Seven-Valued Supervaluationist Semantics

As we saw, the SK truth-tables can be viewed as *derived* from the classical, bivalent truth-tables by means of a general and well-motivated rule (stated in (6)). This is what allowed us to formulate a multivalent system based on the very same intuition as the Strong Kleene semantics, but where the 'primitive' logic, instead of being bivalent, is trivalent. In a completely similar way, supervaluationism defines a trivalent semantics on the basis of a classical, bivalent semantics, by means of a very general principle (Van Fraassen 1966; Fine 1975). And again, nothing prevents us from applying this very same principle to a system that is underlyingly trivalent (instead of being bivalent as in standard supervaluationism), giving rise again to a seven-valued semantics. This is the purpose of this section—though I will not discuss the resulting predictions in any detail.

Again, we define a set of 2nd-level truth-values on the basis of the set of 1st-level truth-values, $\{0, \#, 1\}$, in exactly the same way as before, i.e. as the set of non-empty subsets of $\{0, \#, 1\}$. Let me now give a number of definitions.

(14) a. A 2nd-*level valuation* is an assignment of 2nd-level truth-values to atomic sentences.
b. A 1st-*level valuation* is an assignment of 1st-level truth-values to atomic sentences.

We assume that a 1st-level valuation assigns a truth-value to every sentence (atomic or complex) on the basis of some specified truth-tables, such as, for instance, the Middle Kleene truth-tables. Given a 1st-level valuation w and a sentence S, we note $[\![S]\!]^w$ the truth-value assigned to S by w (if S is atomic, $[\![S]\!]^w = w(S)$).

In order to be able to assign 2nd-level truth-values to complex sentences on the basis of a 2nd-level valuation, we need the following auxiliary notion:

(15) Given a 2nd-level valuation v, a 1st-level valuation w is a *precisification* of v if the following holds: For every atomic sentence p, $w(p) \in v(p)$.

That is, w is a precisification of v just in case w represents a way of 'choosing', for each atomic proposition, a 1st-level truth-value that belongs to the 2nd level truth-value that v assigns to the proposition. Now we can state the rule for assigning 2nd-level truth-values to complex sentences.

(16) Given a 2nd-level valuation v and a sentence S, the 2nd-level value assigned to S by v, noted $[\![S]\!]^v$, is defined as follows:
$[\![S]\!]^v = \{t \in \{0, \#, 1\} :$ there exists a 1st-level valuation w which is a precisification of v such that $[\![S]\!]^w = t\}$

The intuition behind this definition is very similar to the one developed in Sect. 3. The idea is that the 2nd-level truth-value of a proposition relative to a 2nd-level valuation v is the set of 1st-level truth-values that the proposition could have according to every way of turning v into a 1st-level valuation that is consistent with v (i.e. a way of 'precisifying' v).

For most of the cases discussed in the previous section, the end-result is the same. Cases where the supervaluationist version makes a difference are those where some vague material occurs several times in a single sentence. Consider again (12), repeated below as (17):

(17) John has tall sons, and his tall children all have blue eyes.

Recall that in the system constructed in Sect. 3, there were situations in which (17) receives a 2nd-level truth-value that includes #. In the supervaluationist seven-valued semantics I have just defined, we can show that (17) can never receive a truth-value that includes # as its 2nd-level truth-value, hence is predicted to be non-presuppositional (assuming the 1st-level semantics is Middle Kleene).

Proof that (17) can never be assigned a 2nd-level truth-value that includes #:

Since our semantics is defined for a propositional language only, we first have to capture the relevant entailment relationships between expressions (e.g., *having sons* entails *having children*) by means of meaning postulates. Let us thus represent (17) by the sentence (18a), with the associated meaning postulates in (18b):

(18) a. $p' \wedge q_p$
b. q_p presupposes p, p' and p are vague but not presuppositional, and p' asymmetrically entails p.
In other terms:
For every 1st-level valuation v:
1. $v(p) \neq \#$ and $v(p') \neq \#$
2. If $v(p') = 1$, then $v(p) = 1$
3. $v(q) = \#$ if and only if $v(p) = 0$

Let me prove that (18) can never receive a 2nd-level truth-value that includes #. Assume, to the contrary, that for some 2nd-level valuation w, $[\![p' \wedge q_p]\!]^w$ includes #. Then there must be a 1st-level valuation v which is a precisification of w such that $[\![p' \wedge q_p]\!]^v = \#$. Pick such a 1st-level valuation v. Given the meaning postulates and the Middle Kleene truth-tables, one cannot have $[\![p' \wedge q_p]\!]^v = \#$ unless $v(q_p) = \#$, i.e., given the meaning postulates above, $v(p) = 0$. So $v(p) = 0$. But since p' entails p, if $v(p) = 0$, then $v(p') = 0$. Given the Middle Kleene truth-tables, then, $[\![p' \wedge q_p]\!]^v = 0$, which contradicts $[\![p' \wedge q_p]\!]^v = \#$.

It follows that $p' \wedge q_p$ can never receive a 2nd-level truth-value that includes #, and is thus predicted not to be presuppositional.

In general, a sentence can receive a value that includes # in the seven-valued semantics defined here only if it can receive the value # in the underlying trivalent semantics. That is, there exists a 2nd-level valuation v such that $[\![S]\!]^v$ includes # only if there exists a 1st-level valuation w such that $[\![S]\!]^w = \#$. In other terms, a sentence is presuppositional relative to the seven-valued semantics if and only if it is presuppositional relative to Middle Kleene.

5 Conclusion

To conclude, I would like to note that the approach outlined here can be implemented in various ways. For instance, I assumed that the system that deals with presupposition projection is Middle Kleene. But if one has any reason to favor another type of trivalent semantics for presupposition projection, the two constructions proposed in this paper can still be applied. Note in particular that the supervaluanist way of defining a 2nd-level semantics on the basis of a trivalent semantics (Sect. 4) is defined independently of the specific nature of the 1st-level trivalent semantics. In fact, the 1st-level trivalent semantics could itself be a supervaluationist semantics, or some variation on the supervaluationist approach that can account for the observed left-right asymmetry observed in presupposition projection (cf. Fox 2009). Likewise, one might also view what I called here the 1st-level semantics, i.e. Middle Kleene, as not being primitive, but as being derived from a bivalent semantics by a modification of the SK-method which cares about linear ordering (see George 2008, 2014 for an account exactly along these lines). My hope is to have offered a general perspective which can be modified and developed, and extended to more realistic models of natural languages, such as formal languages with predicates, variables and generalized quantifiers.

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