Topoi (2010) 29:29–39
DOI 10.1007/s11245-009-9064-5

Intuition and Visualization in Mathematical Problem Solving

Valeria Giardino

Published online: 9 February 2010

Abstract In this article, I will discuss the relationship between mathematical intuition and mathematical visualization. I will argue that in order to investigate this relationship, it is necessary to consider mathematical activity as a complex phenomenon, which involves many different cognitive resources. I will focus on two kinds of danger in recurring to visualization and I will show that they are not a good reason to conclude that visualization is not reliable, if we consider its use in mathematical practice. Then, I will give an example of mathematical reasoning with a figure, and show that both visualization and intuition are involved. I claim that mathematical intuition depends on background knowledge and expertise, and that it allows to see the generality of the conclusions obtained by means of visualization.

Keywords Mathematical intuition · Mathematical visualization · Diagrammatic reasoning · Problem-solving

> Once more, we are forced to retrace our steps and make ourselves aware of phenomena that we have been taking for granted.
>
> - Gian Carlo Rota

1 Introduction: Mathematical Intuition and Visualization

The term ‘mathematical intuition’ designates a particular kind of cognitive relationship between mathematicians and their activity of doing mathematics. Nonetheless, in the literature there does not seem to be an agreement about what kind of knowledge this type of cognitive relationship produces, nor about which aspects of mathematical activity it refers to. Indeed, there are several ways of defining mathematical intuition.

We can think of intuition as the immediate cognition of mathematical *objects*.[1] But mathematical intuition has also been related to the discovery of mathematical *proofs*: intuition would involve an unconscious preparation similar to a gestation, and afterwards an illumination by means of which we get to a new conclusion.[2] In some cases, mathematical intuition has also been discussed as fallible and defeasible, in analogy with other cognitive relationships such as perception: if physical laws are known by

V. Giardino (✉)
Institut Jean Nicod, CNRS, Paris, France
e-mail: valeria.giardino@gmail.com

[1] For example, at the dawn of set theory, Cantor discusses how to access transfinite numbers. According to him, we get to Cantorian sets operating a double act of abstraction from sets of concrete things. The first act of abstraction brings us to the ‘ordinal number’ or *enumeration*; the second act of abstraction brings us to the cardinal number or *power* of the same set. The cardinal number of M, then, is the general concept that arises from the aggregate M by means of our active faculty of thought. It is thanks to this faculty that we can abstract, and, because of that, provide definitions. By abstracting, we obtain a whole *Einheit* (“Unity”) of undifferentiated *Einsen* (“Ones”): according to Cantor, these are ‘objects of our intuition’. See Cantor (1915).

[2] According to Hadamard, both the preparation and the illumination are mostly subconscious. Nevertheless, he does not deny that conscious thinking is necessary. In fact, once this unconscious illumination has occurred, it must be verified by means of conscious thinking. Intuition allows the mathematician to see the conclusion; then, it is only afterwards that this conclusion will be proved by traditional means. See Hadamard (1945).

perception, then *axioms* are known by intuition.[3] To sum up, intuition is defined as intuition of mathematical objects, mathematical proofs, or logical axioms; nonetheless, what seems to be a common feature of all these definitions is the idea that when we reason by intuition, these objects, proofs, or axioms '*force themselves upon us as being true*', to use Gödel's expression.[4] Mathematical intuition is thus eventually described as a form of rationality that puts mathematicians in the position of accepting some mathematical facts, without feeling the urgency of justifying their beliefs: recurring to intuition, mathematicians find a way of 'speeding up' their reasoning, and as a consequence arrive at some mathematical conclusion.

Another particular kind of cognitive relationship between mathematicians and their mathematical activity is 'mathematical visualization'. It can be claimed that mathematical visualization and mathematical intuition are linked. In fact, an alternative way of describing mathematical intuition would be to define it as the capacity of perpetuating the function of vision, but by means other than the eyes: when mathematicians cease being able to visualize a proof, it is then that they turn to intuition. If this is true, then there would be a clear discontinuity between mathematical visualization and mathematical intuition. Nevertheless, there is at least a sense in which the two are similar. First of all, both mathematical visualization and mathematical intuition have a distinct non linguistic character, and it is for this very reason that they have been traditionally opposed to other aspects of mathematical activity that typically possess a propositional and linguistic character, i.e. axioms or deductive proofs. Intuition and visualization are intrinsically not linguistic: therefore, they are supposed not to be reliable and are not legitimate elements of the mathematical discourse.

In this article, I will go against this line of thought. Mathematical intuition and mathematical visualization may well be two different kinds of mathematical cognitive processes, that is to say, mathematicians may well 'grasp' by intuition precisely what they cannot 'see'. Nonetheless, if we focus on visualization and on its role in the practice of mathematics, we find that at least in some cases of mathematical reasoning, intuitive thinking and visualization are intertwined. To argue in favor of this claim, I will reject the two following misleading assumptions, according to which:

1. in mathematical reasoning, there is a *sharp opposition* between visual processes and linguistic processes;
2. mathematical intuition as well as mathematical visualization represent a kind of *direct access* to mathematical facts.

In sect. 2, I will explain why assumption (1) should be given up in order to explore the complex phenomenon of mathematical activity, which involves lots of different cognitive resources. It is only once this conception of mathematics is assumed as a background, that we can consider the role of intuition and visualization in it. I will present two strategies to reach this goal. In sect. 3, I will focus on two kinds of danger in making use of visualization. In sect. 4 I will show how these dangers are not a good reason to conclude that visualization is not reliable, if we consider its use in mathematical practice. I will give an example of mathematical reasoning with a figure, and show how both visualization and intuition are involved in it, if assumption (2) is rejected. Finally, in sect. 5, I will put forward some conclusions.

2 A (Necessary) Digression on Explanation and Understanding in Mathematics

To discuss the role of intuition and visualization in mathematics means to discuss their role in relationship to other aspects that are central in mathematical activity, but that have been surprisingly neglected by most of the literature on mathematics of the most recent years. Without their acknowledgement, it is difficult to find a way to consider the contribution of informal reasoning in general to the process of mathematical research. According to Barwise and Etchemendy, this neglect was caused by the primacy of a particular notion of mathematical proof, which they define as the 'dogma' of logocentric reasoning. This dogma is very well summarized in the words of Tennant, who claims that

> [The diagram] is only an heuristic to prompt certain trains of inference; […] it is dispensable as a proof-theoretic device; indeed, […] it has no proper place in the proof as such. For the proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array.[5]

[3] According to Gödel, in physics as well as in logic, we are able to describe, and in fact we do describe, the ultimate reality of things. This happens because we access this nature by means of some immediate capacity: by perception in the case of physics, and by intuition in the case of mathematics. It is mathematical intuition that provides mathematical content. The analogy between perception and intuition can be pushed further. Like perception, intuition is fallible: we can fail in our attempts to get to know the abstract world we are facing. This may mean that further and new intuitions are needed. Therefore, axioms are analogous to physical laws, since it is by means of them that we gain knowledge of the relationships among 'things', and we expect experiences to occur in accordance with what these laws prescribe. See Gödel (1986).

[4] Gödel (1986, p. 268).

[5] Barwise and Etchemendy (1996, p. 3).

According to the dogma, a proof:

(a) is a syntactic object
(b) consists only of sentences arranged in a finite and inspectable array

I will not elaborate further the reasons that may have brought to this definition of proof. Rather, I will argue that there are two possible strategies to go against it. My aim is to recover a conception of mathematics as a complex phenomenon, which involves different cognitive resources. I will argue that the second strategy is more successful.

2.1 Barwise and Etchemendy's Program

The first strategy is represented by Barwise and Etchemendy's program.[6] Their project was to reject claim (b) and to reformulate claim (a), arguing that in a mathematical proof semantics does play a role as well as syntax.

According to Shin's reconstruction,[7] the reasons to undertake this project were of three kinds. First, Barwise and Etchemendy relied on *the observation of students' performances*: in teaching logic, semantic concepts are of help to carry out formal proofs in a deductive system. In fact, they assumed that *reasoning is an heterogenous activity*: in the process of reasoning human beings obtain information through many different kinds of media, including diagrams, maps, smells, sounds, as well as written or spoken statements. Finally, they wanted to restore *the unity of teaching and research*: there is no reason to think that we have to choose between the merit of modern logic, i.e. its formalization and rigor, and the merit of multi-modal reasoning, i.e. its practical power, and not to have both of them. Their objective was then to evaluate the logical dimension of diagrams and to create a system which allowed for their rigorous formalization. Formal logic develops out of our daily valid reasoning: it is possible to widen logic in such a way that it finally includes visual representations as well as linguistic propositions. Figures and diagrams are not only good heuristic tools, but proper elements of mathematical proofs. The territory of logic is eventually expanded by freeing it from a single mode of representation, and ordinary reasoning appears not to be in conflict with what has been done in logic and mathematics.

One of the outcomes of the project was Shin's work on Venn diagrams, aimed at giving explicit rules for their rigorous use.[8] A second outcome was the creation of a software to teach logic, *Hyperproof*. As their previous software, *Tarski's World*, this new software used graphics to teach the syntax and semantics of first-order logic. Nonetheless, *Hyperproof* was also teaching the inference making reference to graphics: in fact, it incorporated heterogenous reasoning rules which moved information back and forth between graphical representations of blocks-worlds in a windowpane and sentences of first-order logic below it.

Despite these interesting results, Barwise and Etchemendy strategy seems to work only up to a certain extent, for the following reasons.

The first problem is summarized by Mancosu, who claims that "the work carried out by Barwise and Etchemendy on visual arguments in logic and mathematics is motivated in great part by the proof-theoretic foundational tradition".[9] To wit, Barwise and Etchemendy chose to simply reformulate claim (a) without feeling the urgency of re-discussing the very notion of proof as it has been received by the proof-theoretic tradition. They certainly succeeded in widening the class of proof-theoretical devices, in such a way that it includes diagrams as well, but their intention was to do for visual reasoning what Frege and his followers have done for the language-based one. This shows how they were still largely motivated by the proof-theoretic framework: they focused on visual reasoning, but still they had nothing to say about the phenomenology of visualization. In fact, it is the very setting up of the question about mathematical visualization within the traditional framework that is problematic: there is very little clarity on what criteria one can appeal to in order to distinguish linguistic systems from visual systems, beside the fact that only the first ones convey rigorous proofs. The real challenge is to study how in general in a representational system, its linguistic, geometrical and topological features affect its expressivity.

Let us recapitulate what I have been presenting. To discuss informal aspects of mathematical activity such as diagrammatic reasoning, Barwise and Etchemendy's program was aimed at showing that it is possible to add to the traditional model of linguistic rigor rigorous forms of inference with diagrammatic elements in them. I argue that this is not enough, since it only provides a 'diagrammatic' extension of standard 20th century logic, without giving us any insight into what working with diagrams genuinely involves. A diagrammatic logic such as Shin's treatment of Venn circles surely provides a formal system which includes diagrams as well as sentences. Remember that the aim is to reflect what happens in logical reasoning, and how students seem to go back and forth from syntactic to more semantic information, and *Hyperproof* is an attempt to exploit this capacity. Nevertheless, what I claim is that this aim is not completely met. In fact, giving explicit rules to apply in order to visualize as in Shin's system case, or giving explicit

[6] *Ibid.*
[7] Shin (2004).
[8] Shin (1994).
[9] Mancosu (2005, p. 23).

interpretations for each first-order logic sentence as in *Hyperproof*, seems to deprive visualization of its effectiveness and straightforwardness, which are on the contrary its more interesting aspects from a cognitive point of view. Diagrams are given a new formal and rigorous life, in line with the proof-theoretic tradition, but at the same time they seem to lose their character of offering themselves as possible 'reasons' for the truth of some mathematical fact. No argument is given to explain how visualization seems to be more intuitive. Moreover, providing simple models for first-order logic is only a first step in pursuing this strategy, if we want it to be applied to all mathematics. In fact, it is not clear how this model could work in relationship to forms of visualization in mathematical theories that are richer than first-order logic. If we want to widen logic so that it includes more complex and multi-modal reasoning, then we should account also for the most common mathematical cases. I propose that to reach this aim, it is necessary to reject assumption (1) in sect. 1: there is no opposition between visual and linguistic mathematics.

2.2 Mathematical Practice as Searching for Reasons

Let us explore a second strategy, which rejects both claims (a) and (b) and the very opposition between visual processes and linguistic processes in mathematics in assumption (1). As long as this opposition is maintained, visual proofs will always be defined on the basis of what the received notion of proof prescribes: nevertheless, this opposition obscures phenomena that populate the history of mathematics. Instead of focusing on the proof-theoretic notion of mathematical proof, therefore, let us consider mathematical proofs in the light of mathematical practice. I will first discuss which features of mathematical activity emerge once we reject assumption (1), and then I will show how these relate to intuitive thinking and visualization.

Let us look for example at the notion of mathematical explanation of mathematical facts. Mancosu points out that

> it could be reasonably argued that a full proof might not serve as an explanation in the classroom. Indeed, often pictures or informal arguments will play an ideal 'explanatory' role, whereas a full proof will be no explanation at all in that context.[10]

Actually, often mathematical proofs that are in line with claims (a) and (b) do not coincide with an effective mathematical explanation. This seems to be a peculiarity of mathematics, or better of mathematics as it is seen through the lens of proof-theoretic tradition. From the perspective of the received view, there seems to be a gap between the written version of a result, which is the syntactic and sentential proof, and what is needed in order to understand the same result, which can also rely on other kinds of elements such as pictorial or informal arguments. The activity of 'proving' in this narrow sense seems to have obscured the activity of 'searching for reasons'. As Rota explains,

> A realistic look at the development of mathematics shows that the reasons for a theorem are found only after digging deep and focusing upon the possibilities of the theorem. The discovery of such hidden reasons is the work of the mathematician. Once such reasons are found, the choice of particular formal sentences to express them is secondary. Different but exchangeable formal versions of the same reason can and will be given depending on circumstances.[11]

Mancosu tries to challenge the proof-theoretic idea of visualizations as nothing more than useful heuristic tools, and assumes that the investigation of different case studies taken from the history of mathematics can assess the plausibility of the role of diagrams in mathematical explanations: the connections that they reveal maybe do not count as necessary and truth preserving, but still they can count as reasons. As Rota suggests, verification is proof, but verification might not be reason; mathematicians are not satisfied with proving conjectures, since what they want is reasons.

There is also another respect why assumption (1) should be rejected. It could be argued that the distinctive nature of visual proofs would be their being non-linguistic: because of that, visual proofs prove the particular statement 'all together', i.e. in a single display. Most of the time, this is expressed by locutions such as 'reading off' information from a diagram, or extracting information 'at zero cost'. These claims may have a kind of *prima facie* appeal, since they oppose these intuitive features of visual representations to the seriality and sequentiality of language. Nevertheless, these particular advantages can be questioned.

First of all, visual proofs are not really '*proofs without words*'.[12] No one would say that using a diagram is only a matter of possessing some kind of perceptual capacity: we have to know the mathematical statement in question to be able to find it 'in' or 'represented by' the diagram. If we do not, then reasoning with a diagram would be equivalent to finding the solution of a riddle such as "from this figure, find the statement in it" or "which statement is represented by the figure?". In other words, we always need some linguistic explanation or justification for our use of that particular diagram. Moreover, a mathematical problem is not a riddle because there is a historical context it refers to. Secondly, visual proofs are not really conveying different

[10] Mancosu (2001).

[11] Rota (1997, p. 191).

[12] 'Proofs without Words' is the title of Nelsen (1997, 2001).

pieces of information simultaneously, where proofs as defined by claims (a) and (b) would do that in discrete steps. In fact, a visual proof consists as well in step by step instructions on how to organize space. As I will show, in a diagram the process of discovery appears to be intertwined with the process of justification: the diagram convinces us without putting us in the position of explaining this feeling of confidence we experience.

This second strategy therefore rejects assumption (1) and gives an account of mathematics as the activity of searching for *reasons*. In this context, proofs are not the objects prescribed by claims (a) and (b), but they involve different kinds of multi-modal reasoning. If this is the case, then it is a matter of finding good historical case studies which show how intuitive thinking or visualizations are proper elements in the process of finding a solution to a problem or in feeling justified in our beliefs.

Wa can now rephrase the limitations of Barwise and Etchemendy strategy. According to them, a mathematical proof proves the truth of a single proposition, and it does that because there are some truth-preserving rules of inference that have been defined and that have been shown to be reliable. Shin's system makes Venn circle rigorous, and *Hyperproof* goes back and forth from sentences to the concrete-world model: they both give explicit criteria for having a correct and reliable visualization. Nevertheless, this strategy does not seem to reflect what actual mathematical proofs are: in providing a proof, we happen to intuitively recur to visualizations, and we do that not because we apply some explicit rules that have been given, but simply because the recourse to visualizations may in some cases have a role in our searching for reasons.

In the rest of this article, I will discuss to what extent intuition and visualization have a proper role in this searching for mathematical reasons. According to Feferman, looking at mathematical practice, it is easy

> to recognize the ubiquity of intuition in the common experience of teaching and learning mathematics, and the reasons for that […]. In sum, no less than the absorption of the techniques of systematic, rigorous, logically developed mathematics, *intuition is necessary for the understanding of mathematics*.[13]

I will try to give evidence to suggestions such as this, and I will argue that to do that it is necessary to reject also assumption (2). Intuition as well as visualization are not a kind of direct access to mathematical facts, but rather they are mediated by background knowledge and expertise. In the next section, I will present two dangers in relying on visualizations and intuition. In sect. 4, I will give my proposal for a general framework that accounts for them.

[13] Feferman (2000).

3 Two Problems

3.1 Dangerous Discoveries

I will present here the first danger in visualizing: *figures can induce false conclusions*. This danger would call into question the possibility that giving a visualization is helpful in mathematical discovery and creativity, and is an important issue for mathematical *research*.

In 1908, Klein presented the case of a diagram which is apparently impeccable, but which in fact induces us to draw a false conclusion. His aim was to show that figures are not reliable. His example is the unsound proof that brings to the conclusion, false, that all triangles are isosceles.

Consider an arbitrary triangle ABC and draw the bisector line from the angle A and the perpendicular to side BC which goes to its middle point D. If these two lines were parallel, the bisector would also be the altitude of the triangle and the triangle would obviously be isosceles. Assume instead that these two lines meet. Two cases are possible: the meeting point O may be *inside* the triangle or *outside* the triangle. In both cases, draw the segments OE and OF that are perpendicular to AC and AB, respectively. Finally, join O to B and to C.

In Fig. 1, O is *inside* the triangle. The right triangles AOE and AOF are congruent: they have the hypothenuse AO in common; the angles in A are equal; also the two right angles are equal. Therefore, $AF = AE$. Analogously, the two right triangles OCD and OBD are congruent: they have OD in common, $DB = DC$, and the right angles are equal. Therefore, $OB = OC$. Now, because of the first congruence, $OE = OF$; then, it is possible to derive the congruence of triangles OEC and OFB. Hence, $FB = EC$. If we add equals to equals, we get to the conclusion that $AB = AC$. Therefore, the triangle ABC would be isosceles.

In Fig. 2, O is *outside* the triangle. In the same way, it can be inferred that the pairs of triangles OFA and OEA, OBD and OCD, OBF and OCE are all congruent. Therefore, $AF = AE$, $FB = EC$. If we subtract equals to equals, we get to the conclusion $AB = AC$. The triangle ABC would be once more isosceles.

The proof took into account the two possible cases, and in both cases it showed that the triangle ABC is isosceles: something must have gone wrong. According to Klein, "the only thing in this proof that is false is the figure", because "the argument is always based upon inaccurate figures, with perverted order of points and lines".[14] In the first attempt, the error which yields the false conclusion is the assumption that the point O is inside the triangle, because such a situation will never occur. In the second

[14] Klein (2004, p. 202).

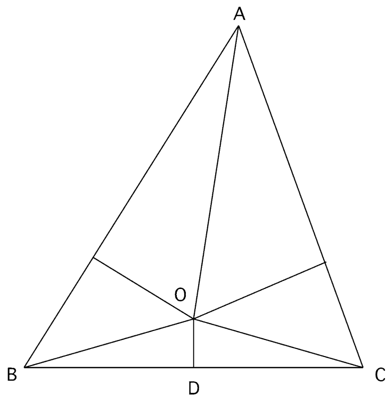

Fig. 1 *O* is *inside* the triangle

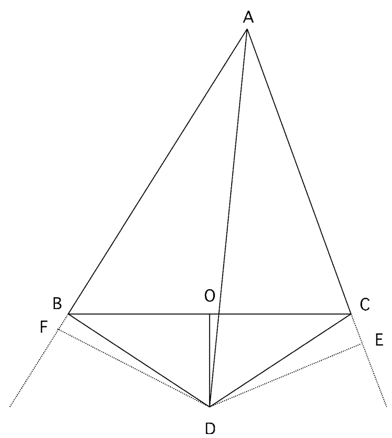

Fig. 2 *O* is *outside* the triangle

attempt, the error which yields the false conclusion is the claim that OF and OE can be drawn like in Fig. 2. The right figure is Fig. 3: of the two feet E and F, one must lie inside, the other outside the side on which it lies, as shown. If this is the case, then $AB = AF - BF$, and $AC = AE + CE = AF + BF$.

Klein's worry is clear: if there exist such inaccurate figures, then we should not give credit to them and to what they show. Intuition and visualization would not be reliable in the process of mathematical discovery. In the next paragraph I will elaborate more on this kind of worry.

3.2 Dangerous Familiarities

The second danger in visualizing is that *figures can mislead our reasoning*. This can happen when the reasoning is performed on the particular image that represents the mathematical statement without considering the consequences implied by it. This would call into question the idea that giving a visualization is helpful because it provides a more 'familiar' format on which to reason, and is an important issue for mathematical *education*.

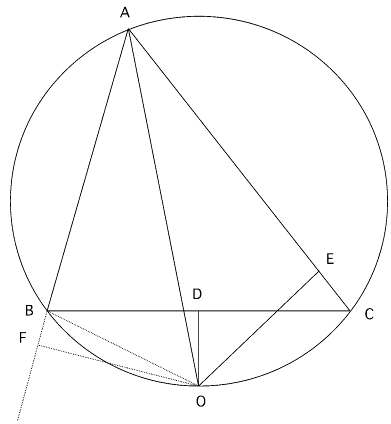

Fig. 3 The right figure

In this section, I will first present the results of a recent study conducted by Bråting and Pejlare on Klein's ideas on the limits of our spatial intuition, and secondly a simple example given by Fischbein. Before discussing the examples, I will introduce some ideas on the importance and the limits of intuition in mathematical visualization.

In 1873, Klein distinguished between *naive* and *refined* intuition. He characterized naive intuition as inexact and fallible. For example, if we imagine a line, it is impossible for us to imagine it not having a certain thickness, because it is only with some approximation that we imagine a sign that corresponds to what the mathematical definition prescribes. But it is only this first kind of intuition that is subject to these limits: refined intuition is instead the logical deduction starting from the axioms which are considered as exact truths. Nevertheless, these axioms are neither arbitrary nor a priori truths, but they constitute an idealization of the inexact data that are obtained by applying naive intuition. Informal reasoning plays therefore a role in the practice of mathematics and especially in mathematical discovery: mathematics do not coincide with logic intended in the narrow sense.

Also according to Fischbein, there is a perceptual element which appears to be a crucial element of intuition. This element is extremely important and ubiquitous in mathematics (Fischbein 1987). Beliefs, expectations, pictorial prompts, analogies and paradigms are not mere residuals of more primitive forms of reasoning, but proper components of mathematical reasoning—and, in general, of every kind of scientific reasoning. These properly mathematical features are genuinely productive as they are active ingredients of any type of reasoning: without this direct intervention of empirical information, many subjects would not be able to spontaneously rely only on their logical schemas for drawing correct formal conclusions. Nevertheless, as for Klein's naive and refined intuition, there exists a deep tension between the nature of

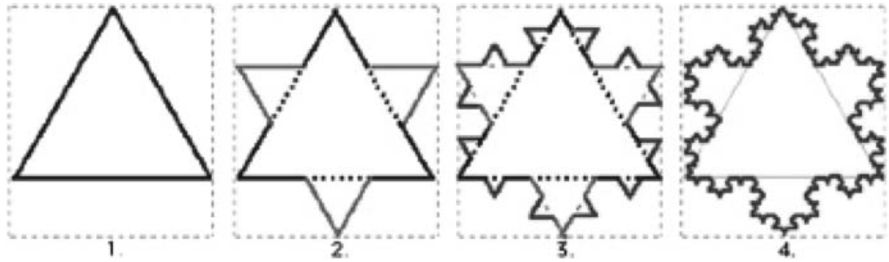

Fig. 4 Bråting and Pejlare (2008, p. 355) (With kind permission of Springer Science and Business Media)

mathematical reasoning and the nature of mathematics. The possible practical connotation of some mathematical notion—for example, the fact that the notion of number is linked to the notion of concrete magnitude—is not a criterion for the acceptance of that mathematical notion as mathematically valid: on the contrary, mathematical validity is based on the possibility of giving definitions and of being consistent with certain given axiomatic structures.

Let us turn to my first example now, and consider a case in which students use figures in order to speculate on a theorem, but they do not realize that they are in fact speculating on the properties of the very image and not on the theorem itself. Bråting and Pejlare's aim was to evaluate this informal and fallible form of intuition and to show what the role of expertise is in reasoning with visualizations.[15] Do students have problems in seeing some mathematical conclusion as correct in a visualization?

To answer to this question, they took into account the so called 'snowflake'. This visualization was elaborated by the Swedish mathematician von Koch to represent an everywhere continuous but nowhere differentiable function. Weierstrass had already given a formalization of this kind of function, but von Koch was not satisfied by this analytic version of it because, according to him, the analytic version did not reflect the intrinsic geometrical nature of the function. By contrast, von Koch believed that in order to understand something, it is necessary to 'see' it. This is what the snowflake does: it makes an everywhere continuous but nowhere differentiable function *visible*. It is precisely thanks to what Klein has labeled naive intuition that we see that it is not possible to draw the tangent in any point of the curve. Bråting and Pejlare disagree: to recognize this impossibility is a matter of possessing the necessary expertise, and not of giving the right visualization.

Thirty-nine university students in mathematics were given the following task.

Consider the construction that follows (represented in Fig. 4):

- Start with an equilateral triangle where each side has length 1.
- On the middle third of each of the three sides, build an equilateral triangle with sides of length 1/3. Erase the base of each of the three new triangles.
- On the middle third of each of the twelve sides, build an equilateral triangle with sides of length 1/9. Erase the base of each of the twelve new triangles.
- Repeat the process with this 48-sided figure.

Please answer the following questions as carefully as you can!

1. For how long can you repeat the process?
2. What figure will you get at the end? Is it continuous? Is it differentiable?

This curve has the property of auto-similarity, that means that in each of its parts there is a set of details that are as rich and as complex as the preceding ones, and this process can go on at infinity. For this reason, the snowflake is continuous but it does not admit any unique tangent in any point: each part of the flake, also the smallest one, possesses the property of auto-similarity, that means it contains in turn a richness of details and of minuscule snowflakes.

The results show that the majority of students do not have problems with the first question, and are perfectly aware that the process can go on at infinity. The difficulties come with the second one, which is answered in very different ways. Sixteen students think that the figure at the limit is uniform everywhere except in a certain finite number of points. Seven among these 16 students believe that it is going to be a circle or a square, and 9 of them a 'flower'. One student in this last group comments that at the limit the triangles will become so small that the figure will become an uniform curve, continuous and differentiable. Fourteen students do not think that the figure at the limit will be everywhere uniform, and therefore show to be familiar with the flake curve. Nine students do not give pertinent answers.

It could be possible to object that this case is so problematic because it has to do with infinity. Maybe it is the use of a figure brought 'to the limit' that is problematic, and not the figure *per se*. Nevertheless, there are also simpler cases in which the use of a concrete reference for reasoning on a mathematical statement is misleading. For this reason, let us turn to our second example.

[15] Bråting and Pejlare (2008).

As I have already discussed, Fischbein argued that there exists a tension between two different poles typical of mathematical activity. The first pole is represented by the research of an ideal model; the second pole by the use to this aim of tools which possess concrete and psychological constraints. His hypothesis is that in some cases concrete objects are introduced to study abstract ones, as they provide more familiar and meaningful interpretations. Nevertheless, once this is done, we should be careful not to apply to these objects kinds of manipulations that are proper to the concrete objects in question, but that do not correspond to any sort of manipulation of the abstract objects they represent. According to Fischbein, intuition is a particular kind of cognition, that is our natural way of treating mathematical conditions as if they were more like practical and empirical conditions. This intuition is not mediated by language, and the mathematical system would be sterile without it. Nevertheless, visualization can bring students to errors.

Consider two parallel axes *X* and *Y*. Draw two parallel lines *AB* and *CD* perpendicular to *X* and *Y*. Let *a* be the constant distance between *AB* and *CD*. Draw two curves *EF* and *GH* such that the distance between two corresponding points, measured on a line parallel to the *X* and the *Y* axes, remains constant and equal to *a* as in Fig. 5.

Subjects were asked to compare the areas of the figures *ABCD* and *EFGH*. In general, they affirmed that the area *EFGH* is greater than the area *ABCD*. They insisted that it was because *EFGH* was 'longer'. The areas are instead equivalent.

Therefore, also from these two examples, it would seem that intuition and visualization, despite their importance in mathematical reasoning, and not reliable and therefore cannot be proper elements of mathematical valid reasoning. In the following section, I will argue against this conclusion.

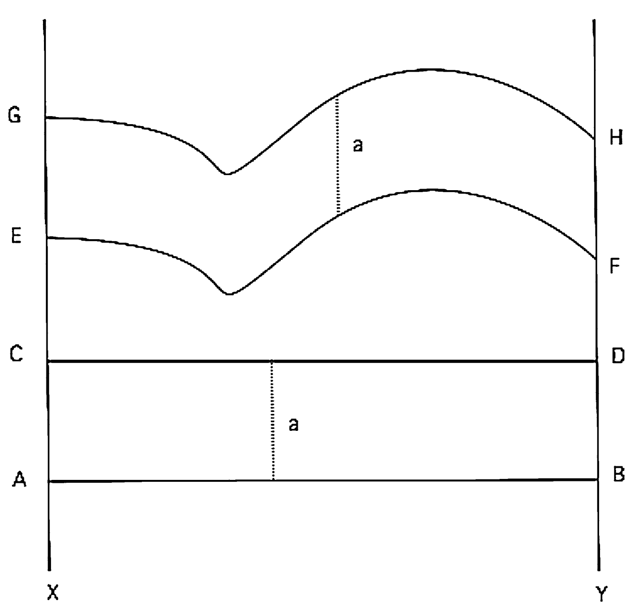

Fig. 5 Fischbein's example (With kind permission of Springer Science and Business Media)

4 Learning (and Teaching) a Practice of Manipulation

4.1 How to Solve it Revised

Polya in 1945 tried to answer to the question about the process that brings to the solution of a mathematical problem.[16] According to him, a problem is solved in the following four phases:

I. Understanding the problem
II. Devising a plan
III. Carrying out the plan
IV. Looking back to check the result

In phase (I), the problem solver is recommended to draw a figure and to introduce a suitable notation. Why? Polya's suggestion is that a detail pictured in our imagination may be forgotten; but the detail traced on paper remains, and, when we come back to it, it reminds us of our previous remarks and it saves us some of the trouble we have in recollecting our previous consideration. But what exactly does this mean? Let us consider once again the nature of mathematical explanation and mathematical understanding along the lines that I have discussed in sect. 2 and revisit the examples of errors and non reliability of visualization that I have given in the previous section.

Concerning the triangle example, we can object to Klein's interpretation.[17] In fact, in this case, it is not the figure that is incorrect and that brings us to the false conclusion according to which all triangles are isosceles. Rather, what is misleading is the reasoning 'behind' the figure. We are wrong in thinking that the right figure can be traced as in Figs. 1 and 2, because there is a series of hypotheses behind these two figures that are incorrect. These incorrect—propositionally expressed—hypotheses activated an inaccurate figure, and this is what brings to a false conclusion. Klein claims that the order of points and lines is perverted in the figure, because point *O* lies outside the triangle and point *E* and *F* are misplaced. Nevertheless, this does not mean that the figure is incorrect *as a figure*, but *as the activation* of some incorrect hypotheses. In other terms, the incorrect figure is like an impeccable identikit obtained from the description given by an unreliable witness. Therefore, the error is in the informal reasoning which is behind the construction of these figures, and not in

16 Polya (1945).

17 See Giardino and Piazza (2008), Ch. III.

the very figures, or in the possibility of putting them to the test.

This kind of error in using figures is *pre-visual,* since it depends on wrong hypotheses that are made *before* the figures are drawn.

Concerning von Koch example, Bråting and Pejlare conclude from their results that, though students know the mathematical concepts of continuity, derivability, and convergence, most of them are not capable of solving the snowflake test. Therefore, knowing the right mathematical definitions is not sufficient to visualize in the right way: what is necessary is also to know *how to use them* in order to visualize. As a consequence, it is not true that to provide a visualization of the Weierstrass function *immediately* translates into seeing that the function is continuous but not differentiable at any point. Moreover, the student who says that the figure at the limit will be uniform, continuous and differentiable, is not able to discern the figure on the paper from the mathematical object, since he thinks that the edges will become so many and so small that the curve will become uniform and singularities will disappear. The same kind of error occurs in the area example. According to Fischbein, this example recalls the Piagetian clay ball problem and the non-conservation reaction of pre-operational children. If we take a ball of clay and roll it into a long thin rod, or even split it into ten little pieces, the pre-operational child, who is under 7 or 8 years old, does not understand that there is still the same amount of clay. The analogy between these two misinterpretations is explained by Fischbein in the following way: in the area problem, as in the clay-ball problem, subjects are focusing only on one dimension – the length – which becomes the dominant one. By contrast, they do not consider enough the width dimension.

These kinds of error in using figures are *post-visual*, since they depend on wrong hypotheses that are made *on* the drawn figure.

The interesting point in discussing these pre-visual and post-visual errors is that they show that visualizations, hypotheses and propositional knowledge are all intertwined to make up a particular proof. To give a visualization is to give a contribution to the organization of the available information: in visualizing, we are referring also to background knowledge with the aim of getting to a global and synoptic representation of the problem. As Fischbein suggests, such visualizations can sometime play a role in anticipating the solution, because they are based on the way they are constructed and can be manipulated, and as a consequence they are of help in transforming the problem into a problem of figurative composition.

What is at stake in giving a mathematical proof is not the knowledge of single propositions and of the reliability of the connections among them. What is at stake in giving a mathematical proof, be it visual or not, is not a set of more or less explicit given rules but a set of *procedures*. In fact, what is a stake is manipulation practices that mathematicians apply in doing mathematics and that students should learn. Syntactic rules are piecemeal while procedures are holistic: the visualization becomes the mathematician's worksite, where operations, plans, and experiments are made in a search for solutions and for reasons for these solutions. The antidote against the logocentric approach to mathematics is to acknowledge that there is a continuous 'dialogue' between language and figures, that is a solid interaction that has as outcome the manipulation of the figure with the aim of obtaining a conclusion.

My proposal then is that a diagram is dynamic and invites to apply some procedure; it is for this very reason that it can bring about an experiment by means of the application of these procedures. Informal inferences take the form of transformations, but in the range of all possible transformations only some of them are legitimate within the theory considered. It is only when one knows which manipulations are legitimate and which are not, that one demonstrates to know how the representational system in question works. The rules of diagrammatic representations are normally *externalized* as procedures. As a consequence, what we learn is not a single manipulation, but rather a set of procedures, not abstract rules, but *instructions* on how to act on the diagram and to read and interpret it correctly. Even though this framework is promising, there is still the problem on how the application of these instructions is safeguarded from the possible pre-visual or post-visual errors. I will analyze an example in order to show which place intuition has in this account.

4.2 Intuition and Generality

In the famous example of the Platonic dialogue *Meno,* the slave has to solve the problem of finding a square which is twice the area of a given square. To do that, he is invited to use a figure like Fig. 6:

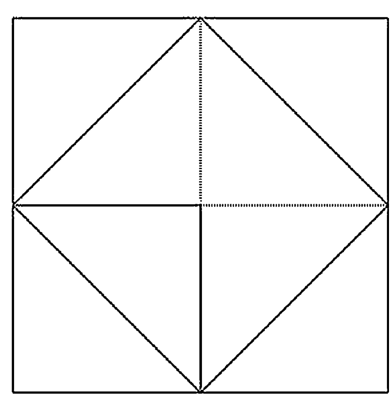

Fig. 6 The figure to solve *Meno's* problem

According to my framework, seeing this figure is not a brute act of perception. In fact, many visual properties of Fig. 6, such as its color or its dimension, are not relevant to solve the problem, though they are clearly *visible*. It is only once the slave has individuated the problem that his visual exploration of the figure omits accidental information and discards non salient conditions. To do that, the slave must already possess the knowledge that ‘a square has 4 sides’, that ‘the diagonal of a square divides the square into two right triangles’, that ‘a right triangle has a right angle’, and so on. If the slave possesses this knowledge, then he will immediately identify these properties as such. The slave, like the mathematicians or the students of the previous examples, should have the geometrical experience of seeing *what he has to see*, that is to focus on the particular spatial and topological properties of the figure that are relevant to find the solution of the problem.

The slave then *understands* Fig. 6 as the figure that is able to dynamically correlate three different squares of three different sides and areas: he sees the figure as a large square which contains five smaller squares and at the same time eight small triangles. At the beginning, the slave has thought that the solution of the problem would have been a square of side 2. But now he sees that this is not the right solution, since the area of the square of side 2 is equivalent to the area of four squares of side 1. By contrast, what he needs is a square whose area is equivalent to the area of two squares of side 1, that is to the area of four triangles. Therefore, the solution to the problem is now evident: the square in the middle, rotated of 45° degrees, whose side is the diagonal of the square of side 1, contains four triangles. It is precisely at this point that I propose that intuition plays a role, since it is by intuition that the slave sees that the figure in Fig. 6 is *generic*, that is to say that it will be valid for squares in general, and not only for those particular squares. As a consequence, the conclusion *forces itself upon the slave as being general*. To clarify, this happens not because of some direct access the slave has to the truth of the conclusion, but because of its knowledge of the mathematical practice. Assumption (2) is rejected.

As Netz suggests for Greek mathematics, mathematical proofs constitute practical invariances, that is to say that proofs are learnt as such because they can be *repeated*.[18] Once the slave, who is a rational agent, has understood the figure, then he can *reproduce* it, in a non mechanical way and without ‘damages’. Figures, be them drawn on a piece of paper or on a blackboard, or shown on a computer screen, preserve the topological and geometrical properties of the space where they are placed. Moreover, they are *intentional* objects, that means that they are intended to have been drawn by the producer with a particular aim. For these reasons, the user must possess the cognitive control that allows the figure to be effective in relation to the problem.

There is one last thing to be noticed. The slave visualizes because he learns how to apply the instructions that he has learnt. If asked to describe his use of the figure, he would make the temporal order of the different construction steps explicit. That means that he would neither refer to some explicit and truth preserving syntactic rules nor to some visual properties of the figure, but to the message that the figure contains, which is about how to organize the space. The slave checks for the consistency between the process of arriving at his conclusion and the practice of Euclidean geometry. His mathematical knowledge is not the knowledge of single truths, but of a set of interconnected facts. His belief becomes stable only when the slave does not see it as an isolated belief, but as an element in a vast system of knowledge. At the end, the slave grasps by intuition and it is because of the figure that his conclusion is general.

What about being safeguarded against possible errors? We do not have to assume that the process I have just described is error-free; rather, there is a possibility to be misled by intuition and visualization, since both of them are defeasible. Nevertheless, it is their being intertwined with the rest of the shared system of knowledge, practices and procedures that serves as guarantee for their reliability. In particular, one must always check that (i) the hypotheses introduced are correct and consistent with the system of knowledge that is presupposed (checking for *pre-visual* errors), and that (ii) the visual medium does not introduce constraints of its own on the representation of the target area (checking for *post-visual* errors). Now, it can be surely argued—as Mancosu does—that this is an issue that can be settled by a detailed case by case analysis rather than a priori. Nonetheless, for the moment it suffices to realize that “after all, *mathematicians have been doing just that for more than two thousand years*”[19]: assumptions (1) and (2) in sect. 1 are abandoned, and the territory of mathematical activity reemerges in all its complexity.

5 Conclusions

In this article, I have discussed the relationship between mathematical intuition and mathematical visualization. I have argued that to give an account of their relationship, it is necessary to abandon the assumptions according to which (1) there is an opposition between visual and linguistic reasoning in mathematics and (2) mathematical intuition as well as mathematical visualization represent

[18] Netz (1999).

[19] Mancosu (2005, p. 26).

some direct cognitive access to mathematical objects, proofs or axioms.

If we disregard the received view of the proof-theoretic tradition and consider mathematical practice, then it will be clear that mathematical activity results from the interconnections between acquired knowledge and unstable beliefs: the system of mathematical knowledge is dynamic and constantly open to reconfiguration. In fact, Barwise and Etchemendy were pointing in the right direction: multimodality of reasoning is important for mathematical research and teaching. Nevertheless, to account for it and for the role it has in mathematical explanation and understanding, it is necessary to abandon traditional notions and to look back at what the history of mathematics teaches us.

If we consider examples taken from mathematical practice, we see that the appeal to visualization is not direct, because it strongly depends on expertise. Moreover, discovery by visualization is mediated by the intuition of the generality of the conclusions obtained by means of it. Finally, this process is not error-free. On the contrary it can be fallacious, at least in two senses. First, there can be pre-visual errors, if an erroneous hypothesis is made on how to draw the figure; secondly, there can be post-visual errors, if an erroneous hypothesis is made on properties of the figure which are not relevant to the mathematical problem. Nevertheless, intuition and visualization are interconnected parts of a vast web of knowledge that results in the learning and in the application of a mathematical practice. It is the preservation of these interconnections that allows for the intuition of the generality of some conclusion and the consequent stabilization of certain beliefs. If the new conclusion is not in line with other stable beliefs, then it will be necessary to retrace our steps and make ourselves aware of phenomena that we have been taking for granted. That is what we do once we check for the generality of the conclusions we have arrived at in the examples I have given.

The absorption of the techniques as well as more intuitive practices such as visualization are all controlled by expertise. There is nothing like mathematical intuition *ex nihilo*: it all depends on how much we are acquainted with the relations in our web of mathematical knowledge as well as on how experienced we are with mathematical manipulations.

Acknowledgments I thank Roberto Casati, Davide Crippa, Leon Horsten, John Mumma, and Mario Piazza for their useful suggestions on a first draft of the article. The research was supported by the European Community's Seventh Framework Program ([FP7/2007-2013] under a Marie Curie Intra-European Fellowship for Career Development, contract number no. 220686—DBR (Diagram-based Reasoning).

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