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Poroelasticity-III: Conditions on the Interfaces

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Abstract In this, the third part of our paper, we continue consideration of the major elements of the poroelastic theory which we started in Parts I and II (in Lopatnikov and Gillespie, Transp Porous Media, 84:471–492, 2010; Transp Porous Media, 89:475–486, 2011). This third part is devoted to considering the general interfacial conditions, consistent with the governing differential equations of the theory. Specifically, we will consider associated mass and momentum conservation laws. Because we developed the theory by construction, general boundary conditions obtained can be applied to the arbitrary interfaces: boundaries between different materials or, for example, moving interfaces of the shock fronts. We do not consider here the last group of conservation laws: the energy conservation laws, which we are going to introduce and investigate in the special part, devoted to the shock wave propagation. In the meantime, special attention is devoted to discussing the problem of "partial permeability" of the interfaces reflected in the literature. Particularly we show, that in the stationary case, the general theory allows only two conditions: either the interface is completely penetrable, or the interface is completely impenetrable. Thus, "partial permeability" solution always appears as only an approximation of an exact dynamic problem, which includes either thin low-permeable interfacial layer (with permeable boundaries), or a non-homogeneous boundary containing permeable and non-permeable patterns.

Keywords Poroelastics · Interfacial conditions · Conservation laws · Wave propagation

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1 Introduction

The first part of our paper was devoted to constructing our general theory of poroelastics (Lopatnikov and Gillespie 2010); in the second part (Lopatnikov and Gillespie 2011), we demonstrated that, contrary to Biot's approach, by using the framework of our theory, one can consistently define the equilibrium state of the poroelastic when it is, filled with fluid having an arbitrary equation of state. We devote this third part of our paper to considering the general interfacial conditions, consistent with the governing differential equations of the theory.

Biot (1956) published his popular two-part paper in which he introduced a linearized version of a poroelastic theory which was practically equivalent to the theory presented 12 years earlier by Frenkel (1944). However, neither Frenkel nor Biot, in his 1956 or later works (Biot 1962a,b), considered the problem of boundary conditions applicable to the theories they had developed. Nevertheless, interfaces are inevitably important elements of the practical reality these theories were designed to explain. Thus, "What interface conditions must be applied on the interfaces?" is one of the key questions for the mathematicians modeling the real world.

Deresiewicz (1960, 1961, 1962) and Deresiewicz and Skalak (1963) addressed the problem as applied to the boundary between two different penetrable poroelastic materials; Nikolaevskiy explored the more general theory (1970).

Deresiewicz and Skalak based their consideration on Biot's theory; thus, their conditions can be applied (if correct) only to solve linear problems. Bourbie at al. (1987) tried to use Hamilton's principle to prove the Deresiewicz–Skalak boundary conditions. However, their "proof" was based on the axiomatic introduction of the surface permeability of the interface between porous materials. It is unclear how one can introduce such surface permeability without physical contradictions and, if one can, under which conditions. As a result, a number of investigators have expressed doubts about the physical validity of the Deresiewicz–Skalak boundary conditions (e.g., the discussion by de la Cruz and Spanos (1989)). This concern is crucial to the validity of the poroelastic theory because, as Gurevich and Schoenberg (1999) stated, "Their concern, if justified, could throw into doubt all the theoretical and numerical results based on the interface conditions of Deresiewicz and Skalak, and, thus, needs to be addressed". Clearly, this issue has not yet been resolved.

In their 1999 paper, Gurevich and Schoenberg stated: "… it has long been known in mathematical physics that interface conditions at an internal discontinuity in a medium described by a linear system of partial differential equations can be derived from those equations if they are written for a general inhomogeneous medium."

These conditions cannot be introduced arbitrarily, but must follow naturally from the general theory, and thus be valid not only for linear but also for nonlinear perturbations and shock fronts, too.

The goal of our paper is to address this issue in the framework we developed as a general theory and derive general interfacial conditions which can be applied universally to boundaries between two media, as well as boundaries on moving fronts.

Similarly to notation accepted in our previous paper, we will refer to equations from the first part as (I-#) and from the second part (if necessary) as (II-#); the part of the paper will be denoted by Roman numbers, followed by the number of the equation using Arabic numbers.

2 Boundary Conditions and Conservation Laws

Boundary conditions can be obtained without any problems if some value satisfies the continuity equation. In that case, so long as there are no sources on the boundary, the normal component of the flow of this value though the boundary must be continuous and one cannot expect any jumps. A simple example comes from the mass conservation laws which lead to the obvious condition that maintains the continuity of the mass as it flows though the interface. Let us start from these laws, using the continuity equations for the solid and fluid phases. We do not consider here the law of energy conservation on the moving interfaces. This question is important for the theory of propagating a shock front; we are going to consider it separately in associated part of the paper.

2.1 Mass Conservation Law

First, let us consider the interfacial conditions following from the continuity equations, Eqs. I-109 and I-110 from Part I of the paper:

$$\frac{\partial\left(1-\varphi\right)\rho_{\mathrm{S}}}{\partial t}+\frac{\partial\left(1-\varphi\right)\rho_{\mathrm{S}}V_{\mathrm{S}i}}{\partial X_i}=0 \tag{1}$$

$$\frac{\partial\varphi\rho_{\mathrm{f}}}{\partial t}+\frac{\partial\varphi\rho_{\mathrm{f}}V_{\mathrm{f}i}}{\partial X_j}=0. \tag{2}$$

Let us consider the jump of material parameters, moving in space in direction x at the velocity D.

As the continuity equations have a divergent form, one immediately has two equations of conservation for solid and fluid mass flows:

$$\left(1-\varphi^{(1)}\right)\rho_{\mathrm{S}}^{(1)}\left(V_{\mathrm{S}x}^{(1)}-D\right)=\left(1-\varphi^{(2)}\right)\rho_{\mathrm{S}}^{(2)}\left(V_{\mathrm{S}x}^{(2)}-D\right) \tag{3}$$

$$\varphi^{(1)}\rho_{\mathrm{f}}^{(1)}\left(V_{\mathrm{f}x}^{(1)}-D\right)=\varphi^{(2)}\rho_{\mathrm{f}}^{(2)}\left(V_{\mathrm{f}x}^{(2)}-D\right), \tag{4}$$

where index (1) indicates the value of the appropriate parameter in front of the propagating jump, and index (2) indicates the value of the appropriate parameter after the jump.

In particular case where the boundary lies between two materials, one can put:

$$D=V_{\mathrm{S}x}^{(1)}\equiv V_{\mathrm{S}x}^{(2)}. \tag{5}$$

Condition (5) can be reformulated as (or, one can consider it as the condition of impermeability of the interface for solids):

$$\xi_{\mathrm{S}x}^{(1)}=\xi_{\mathrm{S}x}^{(2)}, \tag{6}$$

where $\xi_{\mathrm{S}x}^{(\alpha)}$ is the normal component of the displacement of the solid phase on the different sides of interface.

In this form, Eq. 3 is identically satisfied and Eq. 4 obtains the form:

$$\phi^{(1)}\rho_{\mathrm{f}}^{(1)}\left(V_{\mathrm{f}x}^{(1)}-V_{\mathrm{S}x}\right)=\phi^{(2)}\rho_{\mathrm{f}}^{(2)}\left(V_{\mathrm{f}x}^{(2)}-V_{\mathrm{S}x}\right). \tag{7}$$

One can immediately see that in absence of the constant fluid flow, the linearized form of Eq. 7 is similar to the first Deresiewicz–Skalak interfacial condition.

2.2 Condition of Total Momentum Conservation

Using continuity equations (I-106–I-107) of the paper[1], one can present momentum conservation equations in the form:

$$\frac{\partial \Pi_{Si}}{\partial t} + \frac{\partial \left(\Pi_{Si} V_{Sj} - \Sigma_{Sij}\right)}{\partial X_j} + F_{\mathrm{int}i} + \hat{\alpha}\left(\phi, t\right)\left(V_{Si} - V_{fi}\right) = 0 \quad (8)$$

$$\frac{\partial \Pi_{fi}}{\partial t} + \frac{\partial \Pi_{fi} V_{fj} + P_f \delta_{ij}}{\partial X_j} - F_{\mathrm{int}i} + \hat{\alpha}\left(\phi, t\right)\left(V_{fi} - V_{Si}\right) = 0. \quad (9)$$

Here we use the definitions of momentum from Eqs. I-74 and I-78. Also, we suggested that external fields are absent and, following the dissipation function analyses presented in Lopatnikov and Cheng (2004), we explicitly introduced a visco-inertial attenuation term (which also includes the effect of added mass).

Summation of these equations leads us to the total momentum conservation law:

$$\left(\frac{\partial \left(\Pi_{Si} + \Pi_{fi}\right)}{\partial t}\right) + \frac{\partial \left(\Pi_{Si} V_{Sj} + \Pi_{fi} V_{fi}\right) - \left(\Sigma_{Sij} - P_i \delta_{ij}\right)}{\partial X_j} = 0. \quad (10)$$

As Eq. 10 has a divergent form, one again comes to the obvious conditions of conservation for the components of total momentum:

$$\begin{aligned} &\Pi_{Si}^{(1)}\left(V_{Sx}^{(1)} - D\right) + \Pi_{Si}^{(1)}\left(V_{fx}^{(1)} - D\right) - \Sigma_{Six}^{(1)} + P_f^{(1)}\delta_{ix} \\ &\quad = \Pi_{Si}^{(2)}\left(V_{Sx}^{(2)} - D\right) + \Pi_{fi}^{(2)}\left(V_{fx}^{(2)} - D\right) - \Sigma_{Six}^{(2)} + P_f^{(2)}\delta_{ix} \end{aligned} \quad (11)$$

Equation 11 reflects the force equilibrium of the piece of interface:

$$\mathbf{F}_{N_1} + \mathbf{F}_{N_2} = 0. \quad (12)$$

The components of the forces are:

$$\begin{aligned} F_{N_1 i} &= \left(\Pi_{Si}^{(1)}\left(V_{Sj}^{(1)} - DN_{1j}\right) + \Pi_{fi}^{(1)}\left(V_{fj}^{(1)} - DN_{1j}\right) - \Sigma_{Sij}^{(1)} + P_f^{(1)}\delta_{ij}\right) N_{1j} \\ &\equiv \Pi_{Si}^{(1)}\left(V_{Sx}^{(1)} - D\right) + \Pi_{fi}^{(1)}\left(V_{fx}^{(1)} - D\right) - \Sigma_{Six}^{(1)} + P_f^{(1)}\delta_{ix} \end{aligned} \quad (13)$$

$$\begin{aligned} F_{N_2 i} &= \left(\Pi_{Si}^{(1)}\left(V_{Sj}^{(1)} - DN_{1j}\right) + \Pi_{fi}^{(1)}\left(V_{fj}^{(1)} - DN_{1j}\right) - \Sigma_{Sij}^{(1)} + P_f^{(1)}\delta_{ij}\right) N_{2j} \\ &\equiv -\left(\Pi_{Si}^{(1)}\left(V_{Sx}^{(1)} - D\right) + \Pi_{fi}^{(1)}\left(V_{fx}^{(1)} - D\right) - \Sigma_{Six}^{(1)} + P_f^{(1)}\delta_{ix}\right). \end{aligned} \quad (14)$$

Equation 11 represents three equations for the continuity of the normal and tangential components of the solid momentum.

In the general case, Eq. 11 for the tangential component of the momentum has the form:

$$\begin{aligned} &\Pi_{Si}^{(1)}\left(V_{Sx}^{(1)} - D\right) + \Pi_{Si}^{(1)}\left(V_{fx}^{(1)} - D\right) - \Sigma_{Six}^{(1)} \\ &\quad = \Pi_{Si}^{(2)}\left(V_{Sx}^{(2)} - D\right) + \Pi_{fi}^{(2)}\left(V_{fx}^{(2)} - D\right) - \Sigma_{Six}^{(2)}; \quad i = y, z. \end{aligned} \quad (15)$$

In the particular case of the boundary between two materials, due to condition (5), one has:

$$\Pi_{Si}^{(1)}\left(V_{fx}^{(1)} - D\right) - \Sigma_{Six}^{(1)} = \Pi_{fi}^{(2)}\left(V_{fx}^{(2)} - D\right) - \Sigma_{Six}^{(2)}; \quad i = y, z. \quad (16)$$

[1] As stated in the last paragraph of Sect. 1, we will use the notation (I-#) to refer to the equations found in Part I of the paper.

If one can neglect the hydrodynamic component of stress (as in case of linearized governing equations), these conditions can be simplified:

$$\Sigma_{\mathrm{S}ix}^{(1)} = \Sigma_{\mathrm{S}ix}^{(2)}; \quad i = y, z. \tag{17}$$

For the normal component of the force, in general case, one has:

$$\begin{aligned} &\Pi_{\mathrm{S}x}^{(1)}\left(V_{\mathrm{S}x}^{(1)} - D\right) + \Pi_{\mathrm{S}x}^{(1)}\left(V_{\mathrm{f}x}^{(1)} - D\right) - \Sigma_{Sxx}^{(1)} + P_{\mathrm{f}}^{(1)} \\ &= \Pi_{\mathrm{S}x}^{(2)}\left(V_{\mathrm{S}x}^{(2)} - D\right) + \Pi_{\mathrm{f}x}^{(2)}\left(V_{\mathrm{f}x}^{(2)} - D\right) - \Sigma_{xx}^{(2)} + P_{\mathrm{f}}^{(2)}. \end{aligned} \tag{18}$$

In the particular case of the interface between two materials, due to Eq. 5 one has:

$$\Pi_{\mathrm{S}x}^{(1)}\left(V_{\mathrm{f}x}^{(1)} - D\right) - \Sigma_{Sxx}^{(1)} + P_{\mathrm{f}}^{(1)} = \Pi_{\mathrm{f}x}^{(2)}\left(V_{\mathrm{f}x}^{(2)} - D\right) - \Sigma_{xx}^{(2)} + P_{\mathrm{f}}^{(2)}. \tag{19}$$

If one can neglect the hydrodynamic part of pressure and the dynamics of the internal degrees of freedom, one can present Eq. 19 as:

$$-\left(1 - \varphi^{(1)}\right)\sigma_{xx}^{(1)} + \varphi^{(1)} p_{\mathrm{f}}^{(1)} = -\left(1 - \varphi^{(2)}\right)\sigma_{xx}^{(2)} + \varphi^{(2)} p_{\mathrm{f}}^{(2)}. \tag{20}$$

Equation 20 clearly shows that Eq. 19 represents a balance of average normal forces acting on the interfacial surface from different sides. Equations 16 and 19 also represent the Deresiewicz–Skalak's boundary condition: continuity of the total normal and tangential stress tractions.

3 Condition on the Interface for the Momentum of the Fluid

In contrast to the equation for total momentum, which has a divergent form, the equations for partial momentums of phases, Eqs. 8 and 9, are not conservative. The presence of interaction forces provides an exchange of momentum between the phases, due either to viscous friction or to the non-homogeneity of the material. It means that the conditions defining conservation of momentum for fluids and/or solids have a more complex nature and need special consideration.

We saw that for tangential components of force, proper conditions are given by Eqs. 15, 16 or 17, depending on the case, Thus the question is only a matter of defining conditions for σ_{xx} or fluid pressure p_{f}. It is more convenient to find conditions just for fluid pressure p_{f}.

To determine the appropriate interface condition, we are going to use Eq. 9. As we mentioned, Eq. 9 does not have a divergent form.

Let us consider again the jump of material parameters, moving in space in direction x at velocity D. One has:

$$\frac{\partial \Pi_{\mathrm{f}i}\left(V_{\mathrm{S}j} - DN_j\right) + P_{\mathrm{f}}\delta_{ij}}{\partial X_j} = F_{\mathrm{int}i} - \hat{\alpha}\left(\varphi, t\right)\left(V_{\mathrm{f}i} - V_{\mathrm{S}i}\right). \tag{21}$$

Let consider a narrow, rectangular pattern of porous materials containing the jump (Fig. 1) and volumetric integral of Eq. 21 over this volume. One has:

$$\int\limits_{\Omega} \frac{\partial \Pi_{\mathrm{f}i}\left(V_{\mathrm{S}j} - DN_j\right) + P_{\mathrm{f}}\delta_{ij}}{\partial X_J} d\Omega = \int\limits_{\Omega} \left(F_{\mathrm{int}i} - \hat{\alpha}\left(\phi, t\right)\left(V_{\mathrm{f}i} - V_{\mathrm{S}i}\right)\right) \mathrm{d}\Omega. \tag{22}$$

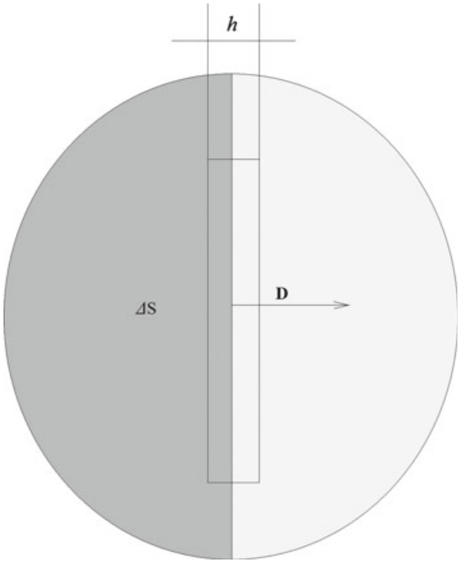

Fig. 1 Moving jump of the material parameters

Taking the limit of an infinitely small thickness of the layer and using Gauss' theorem, one has:

$$\Pi_{\mathrm{f}i}^{(2)}\left(V_{\mathrm{f}x}^{(2)}-D\right)-\Pi_{\mathrm{f}i}^{(1)}\left(V_{\mathrm{f}x}^{(1)}-D\right)+\left(P_{\mathrm{f}}^{(2)}-P_{\mathrm{f}}^{(1)}\right)\delta_{ix}$$
$$=\lim_{h\to 0}\frac{1}{\Delta S}\int_{\Omega}\left(F_{\mathrm{int}i}-\hat{\alpha}\left(\phi\left(\mathbf{X},t\right),t\right)\left(V_{\mathrm{f}i}-V_{\mathrm{S}i}\right)\right)\mathrm{d}\Omega. \tag{23}$$

It is convenient to introduce the operator value $\hat{A}=\frac{\hat{\alpha}(\phi(\mathbf{X},t),t)}{\varphi\rho_{\mathrm{f}}}$ and present the second term in the right-hand part of Eq. 23 as:

$$\lim_{d\to 0}\frac{1}{\Delta S}\int_{\Omega}\hat{\alpha}\left(\phi\left(\mathbf{X},t\right),t\right)\left(V_{\mathrm{f}i}-V_{\mathrm{S}i}\right)\mathrm{d}\Omega\equiv\lim_{d\to 0}\frac{1}{\Delta S}\int_{\Omega}\hat{A}\left(\phi\left(\mathbf{X},t\right),t\right)\varphi\rho_{\mathrm{f}}\left(V_{\mathrm{f}i}-V_{\mathrm{S}i}\right)\mathrm{d}\Omega$$
$$=\left[\lim_{h\to 0}\int\hat{A}\left(\phi\left(\mathbf{X},t\right),t\right)\mathrm{d}x\right]\cdot\left(\varphi\rho_{\mathrm{f}}\left(V_{\mathrm{f}i}-V_{\mathrm{S}i}\right)\right)_{\Gamma}, \tag{24}$$

where we took into account the continuity of the fluid flowing though the boundary and marked the value of the fluid flow though the interface as $\left(\varphi\rho_{\mathrm{f}}\left(V_{\mathrm{f}i}-V_{\mathrm{S}i}\right)\right)_{\Gamma}$.

3.1 Static Equilibrium. Boundary Between Two Materials

Static equilibrium, by definition, is the state in which a system can remain for an indefinitely long time. It means that: $D=\Pi_{\mathrm{f}i}=V_{\mathrm{f}i}=V_{\mathrm{S}i}=\dot{\phi}=\dot{\rho}_{\mathrm{f}}=0$.

If $\hat{A}\left(\phi\left(\mathbf{X},t\right),t\right)$ in not a singular operator-function of x, $\lim_{h\to 0}\int\hat{A}\left(\phi\left(\mathbf{X},t\right),t\right)\mathrm{d}x$ is obviously equal to a zero operator. By taking into account the definitions of momentum flow and interaction force, in the stationary case one has:

$$\left(\varphi_2 p_{\mathrm{f}}^{(2)}-\varphi_1 p_{\mathrm{f}}^{(1)}\right)\delta_{ix}=\lim_{d\to 0}\frac{1}{\Delta S}\int_{\Omega}p_{\mathrm{f}}\frac{\partial\varphi}{\partial X_i}\mathrm{d}\Omega. \tag{25}$$

Let us consider the x-component of Eq. 25. One has:

$$\left(\varphi_2 p_{\mathrm{f}}^{(2)} - \varphi_1 p_{\mathrm{f}}^{(1)}\right) = \lim_{h\to 0} \int \left(\frac{\mathrm{d}\varphi p}{\mathrm{d}x} - \varphi \frac{\mathrm{d}p}{\mathrm{d}x}\right) \mathrm{d}h. \tag{26}$$

Meanwhile:

$$\lim_{h\to 0} \int \left(\frac{\mathrm{d}\varphi p}{\mathrm{d}x} - \varphi \frac{\mathrm{d}p}{\mathrm{d}x}\right) \mathrm{d}x \equiv \left(\varphi_2 p_{\mathrm{f}}^{(2)} - \varphi_1 p_{\mathrm{f}}^{(1)}\right) - \lim_{h\to 0} \int \varphi \frac{\mathrm{d}p_{\mathrm{f}}}{\mathrm{d}x} \mathrm{d}\Omega. \tag{27}$$

Comparing Eq. 26 with Eq. 27, one has:

$$\lim_{h\to 0} \int\limits_{\Omega} \phi \frac{\mathrm{d}p_{\mathrm{f}}}{\mathrm{d}x} \mathrm{d}\Omega = 0 \tag{28}$$

It is easy to see that Eq. 28 can be satisfied only if p_{f} remains constant on the jump of properties.

In the static case, one cannot expect the pressure of fluid *on the penetrable boundary* to jump. Thus, in the stationary case, the conservation of momentum condition is reduced to the continuity of pressure condition:

$$p_{\mathrm{f}}^{(1)} = p_{\mathrm{f}}^{(2)}. \tag{29}$$

However, if $\hat{A}\left(\phi\left(\mathbf{X}, t\right), t\right)$ is on the boundary, a singular function of x, the limit of the integral $\lim\limits_{h\to 0} \int \hat{A}\left(\phi\left(\mathbf{X}, t\right), t\right) \mathrm{d}x$, could be not-equal to zero. This case separates into two conditions.

In the first case, $\lim\limits_{h\to 0} \int \hat{A}\left(\phi\left(\mathbf{X}, t\right), t\right) \mathrm{d}x$ is finite. As a result, in the stationary case, because $V_{\mathrm{f}i} = V_{\mathrm{S}i}$, the product $\left(\varphi\rho_{\mathrm{f}}\left(V_{\mathrm{f}i} - V_{\mathrm{S}i}\right)\right)_\Gamma \lim\limits_{h\to 0} \int \hat{A}\left(\phi\left(\mathbf{X}, t\right), t\right) \mathrm{d}x$ is equal to zero and the condition in Eq. 29 would still be valid.

In second case, $\lim\limits_{h\to 0} \int \hat{A}\left(\phi\left(\mathbf{X}, t\right), t\right) \mathrm{d}x$ is infinite and the product $\left(\varphi\rho_{\mathrm{f}}\left(V_{\mathrm{f}i} - V_{\mathrm{S}i}\right)\right)_\Gamma \lim\limits_{h\to 0} \int \hat{A}\left(\phi\left(\mathbf{X}, t\right), t\right) \mathrm{d}x$ in equilibrium is indefinite. Thus, the integral $\lim\limits_{h\to 0} \int_\Omega \phi \frac{\mathrm{d}p_{\mathrm{f}}}{\mathrm{d}x} \mathrm{d}\Omega$ also cannot be uniquely defined and $p_{\mathrm{f}}^{(1)} - p_{\mathrm{f}}^{(2)}$ can obtain an arbitrary value. It means that the boundary is impenetrable and pressure on it sides can arbitrarily differ.

This consideration has important consequences: only two discrete boundary conditions can be accepted in the static limit: either a condition of perfect permeability of the surface Eq. 29, or a condition of perfect impermeability:

$$\left[V_{\mathrm{f}i} - V_{\mathrm{S}i}\right]_\Gamma = 0. \tag{30}$$

Taking in account that the idea of "imperfect hydraulic" contact has been accepted in a number of publications (de la Cruz and Spanos 1989; Guiroga-Goode and Carcione 1996; Gurevich and Schoenberg 1999; Rasolofosaon 1988), this problem must be considered in more detail. We will return to this subject later in this paper.

3.2 Condition on the interface in the dynamic case

The dynamic case is significantly more complicated. Let consider a simplified form of the interaction between force and momentum, where we ignored the terms, related to storage of momentum in the internal degrees of freedom (I-A9,I-A10). In this case, the interaction force still has the same form as in the static case: $F_{\mathrm{int}} = p_{\mathrm{f}} \frac{\partial\varphi}{\partial X_i}$.

In accordance with the continuity equations, if the jump of $(V_{fi} - V_{Si})$ is finite and *if there is no singularity of* $\hat{\alpha}(\varphi, t)$ *on the interface*, the associated part of the integral will equal zero. In this case, the integral in Eq. 23 has the form:

$$I_i = \lim_{h\to 0} \frac{1}{\Delta S} \int\limits_{\Omega} p_f \frac{\partial \varphi}{\partial X_i} d\Omega = \lim_{h\to 0} \frac{1}{\Delta S} \int\limits_{\Omega} \frac{\partial \varphi p_f}{\partial X_i} d\Omega - \lim_{h\to 0} \frac{1}{\Delta S} \int\limits_{\Omega} \varphi \frac{\partial p_f}{\partial X_i} d\Omega. \quad (31)$$

Under the suggestions used, the orthogonal to the interface component of Eq. 23 obtains the form:

$$\Pi_{fx}^{(2)} \left(V_{fx}^{(2)} - D\right) - \Pi_{fx}^{(1)} \left(V_{fx}^{(1)} - D\right) = - \lim_{h\to 0} \frac{1}{\Delta S} \int\limits_{\Omega} \varphi \frac{dp_f}{dx} d\Omega. \quad (32)$$

There are two problems here. First, in contrast to the stationary case, one cannot make the statement that the right-hand part in Eq. 32 is equal to zero, because the left-hand part of Eq. 32 is not necessarily equal to zero. Second, the limit of the integral in the right-hand part of Eq. 32 is not uniquely defined, because the values of both φ and $\frac{\partial p}{\partial X}$ can be expected to jump and the product of the jumps is not well-defined. Thus, the limit in the right-hand part of Eq. 32 will depend on the distribution porosity and pressure with the (infinitely) thin transient layer. However, one can see that if it is possible to ignore the hydrodynamic pressure in comparison to the static pressure of the fluid (and if there is no singularity of $\hat{\alpha}(\varphi, t)$), the static boundary conditions of Eq. 29 would still be valid, with the associated accuracy.

If one cannot consider the left-hand part in Eq. 32 as negligibly small, for example in the case of shock-front motion, no general conclusions about the value of integral in the right-hand part of Eq. 32 can be made. Instead, one must introduce some micro-model of the transition layer and calculate the integral directly.

The simplest model to suggest regarding the distribution of porosity and pressure within the interfacial layer is to suggest that both are distributed linearly:

$$\varphi(x) = \varphi_1 + \frac{\varphi_2 - \varphi_1}{d} x \quad (33)$$

$$p_f(x) = p_{f1} + \frac{p_{f2} - p_{f1}}{d} x. \quad (34)$$

Using these ansatz functions, one can explicitly calculate the integral and the limit in Eq. 32. One has:

$$\lim_{h\to 0} \frac{1}{\Delta S} \int\limits_{\Omega} \varphi \frac{dp_f}{dX_i} d\Omega = \lim_{h\to 0} \int\limits_0^d \left(\varphi_1 + \frac{\varphi_2 - \varphi_1}{d} x\right) \frac{p_{f2} - p_{f1}}{d} x dx$$
$$= \frac{1}{2} (\varphi_2 - \varphi_1) \left(p_{f2} - p_{f1}\right) \quad (35)$$

Thus, the interfacial condition of Eq. 32 obtains the form:

$$\Pi_{fx}^{(2)} \left(V_{fx}^{(2)} - D\right) - \Pi_{fx}^{(1)} \left(V_{fx}^{(1)} - D\right) = -\frac{1}{2} (\varphi_2 - \varphi_1) \left(p_{f2} - p_{f1}\right) \quad (36)$$

The condition in Eq. 36 perfectly matches the results obtained by us earlier: in the stationary case or if one can ignore the hydrodynamic pressure, one immediately comes to the condition in Eq. 29. However, one has to remember that we obtained the interfacial condition in Eq. 36 by using a model, and use the result with care.

Let consider now the behavior of the tangential components of the fluid.

For the tangential component, the interfacial conditions have the form:

$$\Delta\left(\Pi_{\mathrm{f}i}\left(V_{\mathrm{f}x}-D\right)+P_{\mathrm{f}}\delta_{ix}\right)=-\lim_{h\to 0}\frac{1}{\Delta S}\int\limits_{\Omega}p_{\mathrm{f}}\frac{\partial\varphi}{\partial X_i}\mathrm{d}\Omega;\quad i\neq x. \tag{37}$$

The difference between the conditions in Eqs. 32 and 37 is that the partial derivatives with respect to the tangential coordinates are present in Eq. 37, and therefore the type of integration used with Eq. 32 is not possible.

However, one can suggest that the characteristic length of the porosity changes along the tangential directions is significantly larger than the thickness of the transitory layer itself; thus, the right-hand part of Eq. 37 is sufficiently small.

After taking advantage of these suggestions, the tangential conditions on the interface obtain a simple form:

$$\Delta\left(\Pi_{\mathrm{f}i}\left(V_{\mathrm{f}x}-D\right)+P_{\mathrm{f}}\delta_{ix}\right)=0;\quad i\neq x. \tag{38}$$

3.3 The Problem of the "Imperfect" Permeability

The considerations in Sects. 3.1 and 3.2 of this paper show that in all events, if the surface layer is not impermeable in principle, the jump of pressure on the interface is the dynamic problem: for slow enough processes, the difference between pressures on the opposite sides of the interface has to tend to zero. Thus, for permeable boundaries, the correct question about partially impenetrable interface is: "What is the characteristic time for equalizing the pressures on opposite sides of the boundary?"

To answer this question, one must introduce a realistic model of the interfacial layer and, in the general case of a perturbation of arbitrary amplitude, to solve the appropriate nonlinear problem. We will consider this problem in more detail elsewhere, because the solution requires some additional ideas which we have not yet introduced.

However, we can state at this point that one can represent the interfacial layer as an additional porous layer of finite thickness with penetrable boundaries, and sufficiently low permeability. Recall that in accordance with Frenkel's and Biot's considerations, two longitudinal waves exist. It is easy to understand that the characteristic time to establish equilibrium over this interfacial layer will approximately equal to the time for a wave of second type (with low frequencies, below the frequency of the Biot diffusion type) to penetrate the depth of the layer:

$$\tau\sim\frac{l^2}{D_{\mathrm{p}}}, \tag{39}$$

where l is the thickness of the interfacial layer, and D_{p} is its coefficient of pieso-conductivity, equal to approximately:

$$D_{\mathrm{p}}=\frac{k}{\mu\left(\varphi\beta_{\mathrm{f}}+\left(1-\varphi\right)\beta_{\mathrm{S}}\right)}, \tag{40}$$

where k is the permeability of the layer, β_{f} and β_{S} are compressibility of the fluid and solid matrix, respectively, and μ is the dynamic viscosity of the fluid.

Keeping in mind its application for seismic prospecting, let us suggest for estimating purposes that $k\sim 9.86\times 10^{-16}\,\mathrm{m}^2$ (1 millidarcy), (which one can consider as typical for the clay layer), porosity $\varphi=0.1$, compressibility of the solid matrix is $\beta_{\mathrm{S}}\sim 3\times 10^{-10}\,\mathrm{Pa}^{-1}$, compressibility of water $\beta_{\mathrm{f}}=4.4\times 10^{-10}\,\mathrm{Pa}^{-1}$, and viscosity of water $\mu\sim 10^{-3}\,\mathrm{Pa\,s}$, giving one for the pieso-conductivity coefficient:

$$D_{\mathrm{p}} = \frac{k}{\mu\left(\varphi\beta_{\mathrm{f}} + (1-\varphi)\,\beta_{\mathrm{S}}\right)} = \frac{9.86\times10^{-17}}{10^{-3}\left(0.1\times4.4\times10^{-10}+0.9\times3\times10^{-10}\right)}$$
$$= 3.14\times10^{-3}\,\mathrm{m}^2\,\mathrm{s}^{-1} \quad (41)$$

Suggesting that the thickness of the interfacial layer is equal to 1 mm, the characteristic size of the grain of the material, one has that the characteristic time for such an interfacial layer is $\sim 3.18\times10^{-4}$ s, which matches to characteristic frequency $f_\tau \sim 300$ Hz. This ansatz calculation means that one could consider this interfacial layer to be absolutely impermeable for all waves with frequency higher than 100 Hz and completely permeable for waves with frequencies of 100 Hz and lower. These figures provide some insight into the natural conditions under which accounts of the partial dynamic permeability of the interfacial layers can make sense.

Another possible model of the partially penetrable boundary layer assumes the interfacial surface with given concentrations of permeable and impermeable patterns. This problem leads to a completely different type of consideration: the scattering and refraction of waves on a surface with non-homogeneous boundary conditions (Fig. 2).

In the framework of this model, the question is: can we homogenize boundary conditions of this type so that Eq. 31 will be valid?

It is obvious, again, that in the static limit, this interface is either completely penetrable, or completely impenetrable and one can to speak about "partial" permeability of such a surface layer only in the dynamic sense. This partial permeability will be the result of solving the scattering problem on a non-homogeneous boundary. We will consider this problem elsewhere.

However, the major conclusion is that discussions in the literature of "partially permeable" boundary conditions are always the result of some approximation of the dynamic considerations of a complete problem. In the static case, the poroelasticity equations allow only two boundary conditions: the boundary is either penetrable, and thus, there is no jump of pressure as fluids cross the interfacial boundary, or the boundary is impenetrable and no "partial permeability" can be introduced.

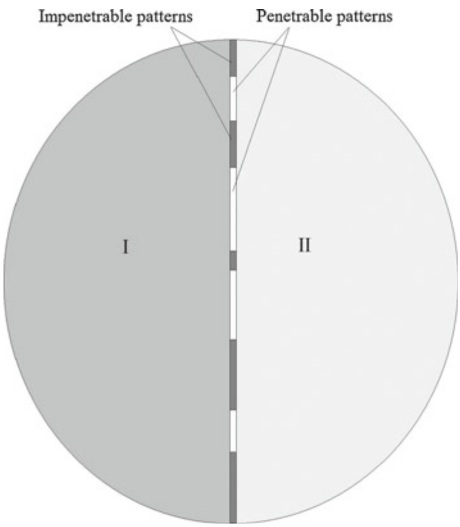

Fig. 2 The model of the interface, based on distributed penetrable and impenetrable patterns

4 Conclusion

The part is devoted to the consistent consideration of the general interfacial conditions, compatible with the governing differential equations of the theory. Because we developed theory by construction, the general, derived boundary conditions can be applied to the arbitrary interfaces.

We have devoted special attention to a discussion of the problem reflected in the literature of “partial permeability” of the interfaces. We show that in the stationary case, the general theory allows only two conditions: the interface is either completely penetrable or completely impenetrable. Thus, “partial permeability” always only appears as an approximation of an exact dynamic problem that includes either a thin, low-permeable interfacial layer (with permeable boundaries), or a non-homogeneous boundary containing permeable and non-permeable patterns.

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