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Non-linear Convective Transport in a Binary Nanofluid Saturated Porous Layer

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Abstract In this article, we study double-diffusive convection in a horizontal porous medium saturated by a nanofluid, for the case when the base fluid of the nanofluid is itself a binary fluid such as salty water. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis, while the Darcy model is used for the porous medium. The thermal energy equations include the diffusion and cross-diffusion terms. The linear stability is studied using normal mode technique and for non-linear analysis, a minimal representation of the truncated Fourier series analysis involving only two terms has been used. For linear theory analysis, critical Rayleigh number has been obtained, while non-linear analysis has been done in terms of the Nusselt numbers.

Keywords Binary nanofluid · Porous medium · Natural convection · Horton–Roger–Lapwood problem

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List of Symbols

Latin Symbols

C	Solute concentration
D_B	Brownian diffusion coefficient
D_T	Thermophoretic diffusion coefficient
d	Dimensional layer depth
k_T	Effective thermal conductivity of porous medium
k_m	Thermal diffusivity of porous medium
K	Permeability
Le	Thermo-solutal Lewis number
Ln	Thermo-nanofluid Lewis number
N_A	Modified diffusivity ratio
N_B	Modified particle-density increment
N_{CT}	Soret parameter
N_{TC}	Dufour parameter
p	Pressure
g	Gravitational acceleration
Ra	Thermal Rayleigh-Darcy number
Rm	Basic density Rayleigh number
Rn	Nanoparticle concentration Rayleigh number
Rs	Solutal Rayleigh number
t	Time
T	Nanofluid temperature
T_c	Temperature at the upper wall
T_h	Temperature at the lower wall
$\mathbf{v}$	Nanofluid velocity
(x, y, z)	Cartesian coordinates

Greek Symbols

β_C	Solutal volumetric coefficient
β_T	Thermal volumetric coefficient
ε	Porosity
μ	Viscosity of the fluid
ρ_f	Fluid density
ρ_p	Nanoparticle mass density
γ	Thermal capacity ratio
ϕ	Nanoparticle volume fraction
ψ	Stream function
α	Wave number
ω	Frequency of oscillations

Subscripts

b	Basic solution
f	Fluid
p	Particle

Superscripts

* Dimensional variable
′ Perturbation variable

Operators

$\nabla^2 \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$
$\nabla_1^2 \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$

1 Introduction

"Nanofluids," a term first used by Choi (1995), refers to fluid suspensions of solid nano-sized particles in conventional heat transfer fluids. By conventional heat transfer liquids, we mean water, ethylene glycol, engine oils, etc. Nanoparticles can be metal particles such as those of Cu, Fe, Ag, or Au or metallic oxides or non-metallic oxide particles—Cuo, Al_2O_3, TiO, SiO, having dimensions in the range 1–100 nm. Significant features of nanofluids over base-fluids include enhanced thermal conductivity, greater viscosity, enhanced value of critical heat flux. Of these the most talked about is the enhanced thermal conductivity, a phenomenon which was first reported by Masuda et al. (1993). Choi (1999), and his team at Energy Technology Division, Argonne National Laboratory, supported by the US Department of Energy, reported the use of nanofluids in a wide variety of industries ranging from transportation, HVAC, and energy production and supply to electronics, textiles and paper production. All of these industries deal with heat transfer in some or the other way, and thus have a strong need for improved heat transfer mediums. This could possibly be nanofluids, because of some potential benefits over normal fluids—large surface area provided by nanoparticles for heat exchange, reduced pumping power due to enhanced heat transfer, minimal clogging, innovation of miniaturized systems leading to savings of energy and cost. Choi's results have been supported by other researchers from time to time. Eastman et al. (2001) reported an increase of 40% in the effective thermal conductivity of ethylene-glycol with 0.3% volume of copper nanoparticles of 10 nm diameter. Further 10–30% increase of the effective thermal conductivity in alumina/water nanofluids with 1–4% of alumina was reported by Das et al. (2003). These reports led Buongiorno and Hu (2005) to suggest the possibility of using nanofluids in advanced nuclear systems. Another recent application of the nanofluid flow is in the delivery of nano-drug as suggested by Kleinstreuer et al. (2008)

There have been many attempts to account for the reason for the observed abnormal behavior of nanofluids by Vadasz (2005, 2006), Eastman et al. (2004), and others. Of these, the efforts significant are being listed here. Eastman et al. (2004), accounted that despite several attempts, a satisfactory explanation for the abnormal enhancement in thermal conductivity and viscosity in nanofluids is yet to be found. Buongiorno (2006) conducted an extensive study of convective transport in nanofluids, but focused on explaining the further heat transfer enhancements observed during convective situations. Though several authors attribute the abnormal increase observed to dispersion of suspended nanoparticles, but, to this, Buongiorno, disagrees. He discards suspension, particle rotation, dispersion, and turbulence as significant agents for heat transfer enhancements, He went further to suggest a new model based on the mechanics of nanoparticles/base-fluid relative velocity. He took the absolute

velocity of nanoparticles to be the sum total of the base-fluid velocity and a relative velocity, (which he calls a slip velocity). Considering seven slip mechanisms—Inertia, Brownian diffusion, thermophoresis, diffusophoresis, Magnus effects, fluid drainage, and gravity settling, he concluded that in the absence of turbulent effects, Brownian diffusion, and thermophoresis dominate. Based on these two effects, he derived the conservation equations. Tzou (2008a) and Tzou (2008b) studied the onset of convection in a horizontal layer uniformly heated from below, for a nanofluid with the help of transport equations of Buongiorno, and found that as a result of Brownian motion and thermophoresis of nanoparticles, the critical Rayleigh number is to be much lower, by one to two orders of magnitude, as compared to that of an ordinary fluid. Kim et al. (2004, 2006, 2007) also investigated the onset of convection in a horizontal nanofluid layer and modified the three quantities, namely the thermal expansion coefficient, the thermal diffusivity and the kinematic diffusivity that appear in the definition of the Rayleigh number.

Convection in porous media finds its applications in modern science and engineering, including food and chemical processes, rotating machineries like nuclear reactors, petroleum industry, biomechanics and geophysical problems. Convection in porous medium has been studied by many authors including Horton and Rogers (1945); Lapwood (1948); Nield (1968); Rudraiah and Malashetty (1986); Murray and Chen (1989); Malashetty (1993); Vafai (2005); Nield and Bejan (2006); Bhadauria (2007a,b, 2008), to name a few. Since nanofluids are being looked upon as great coolants of the future, studies need to be conducted involving nanofluids in porous media and without it. Recently, Kuznetsov and Nield (2010a) studied the onset of thermal instability in a porous medium saturated by a nanofluid, using Brinkman model and incorporating the effects of Brownian motion and thermophoresis of nanoparticles. They found that the critical thermal Rayleigh number can be reduced or increased by a substantial amount, depending on whether the basic nanoparticle distribution is top-heavy or bottom-heavy, by the presence of the nanoparticles. The same Horton–Rogers–Lapwood Problem was investigated by Nield and Kuznetsov (2009a,b) for the Darcy Model. Kuznetsov and Nield (2010b,c) also studied local thermal non-equilibrium and flow past vertical plate for nanofluids. Agarwal et al. (2011); Bhadauria et al. (2011) studied the same problem of thermal instability in a rotating porous layer saturated by a nanofluid for top-heavy and bottom-heavy suspension for the Darcy model and for the Brinkman model for bottom-heavy suspension. Bhadauria and Agarwal (2011a,b) studied natural convection in a rotating porous layer saturated by a nanofluid using the Brinkman's model, and local thermal non-equilibrium effects in a nanofluid saturated porous layer using Brinkman's Model for linear and non-linear conditions. Also Agarwal and Bhadauria (2011) studied local thermal non-equilibrium effects on a nanofluid saturated porous layer using Darcy model for linear and non-linear conditions.

For the preparation of nanofluids, instead of using a pure liquid as basefluid, when a binary liquid is used, it is termed as a binary nanofluid. These binary liquids can be salty water, ferrofluid etc. These binary nanofluids find their utility as a working fluid in absorption refrigeration, as a solution in electro or electroless plating and as a transfer medium in medical treatment (2006). Onset of convection in binary fluids has been studied by Kim et al. (2006) and Kuznetsov and Nield (2010d, 2011). In case of nanofluids, we come across two different Soret effects: one induced by the solute, while the other being contributed by the nanoparticles. So this problem is of a triple-diffusion type problem involving heat, the nanoparticles, and the solute. In this study, we tend to investigate the contribution of both the Soret effects in convection in binary nanofluids. Assuming that the nanoparticles being suspended in the binary nanofluid using either surfactant or surface charge technology, preventing the agglomeration and deposition of these on the porous matrix, in this article, we

study the linear and non-linear double-diffusive convection in a porous medium saturated by nanofluid, using Horton–Roger–Lapwood problem based on the Darcy's Model.

2 Governing Equations

We consider a porous layer saturated by a nanofluid, confined between two horizontal boundaries at $z = 0$ and $z = d$, heated from below and cooled from above. The boundaries are impermeable and perfectly thermally conducting. The porous layer is extended infinitely in x and y directions, and z axis is taken vertically upward with the origin at the lower boundary. T_h, C_h and T_c, C_c are the temperatures and solute concentrations at the lower and upper walls respectively, the former being greater. The conservation equations for the total mass, momentum, thermal energy, solute concentration and nanoparticles, come out to be as below. A detailed derivation of these has been dealt by Buongiorno (2006); Tzou (2008a,b), and Nield and Kuznetsov (2009a,b, 2010a,b);

$$\nabla \cdot \mathbf{v_D} = 0 \tag{1}$$

$$0 = -\nabla p - \frac{\mu}{K}\mathbf{v}_D + [\phi\rho_\mathrm{p} + (1-\phi)\rho(1 - \beta_\mathrm{T}(T_\mathrm{f} - T_\mathrm{c}) - \beta_\mathrm{C}(C_\mathrm{f} - C_\mathrm{c}))]g \tag{2}$$

$$(\rho c)_\mathrm{m}\frac{\partial T}{\partial t} + (\rho c)_\mathrm{f}\mathbf{v}_D \cdot \nabla T = k\nabla^2 T + \varepsilon(\rho c)_\mathrm{p}[D_\mathrm{B}\nabla\phi \cdot \nabla T + \frac{D_\mathrm{T}}{T_\mathrm{c}}\nabla T \cdot \nabla T] + \rho c D_\mathrm{TC}\nabla^2 C \tag{3}$$

$$\frac{\partial C}{\partial t} + \frac{1}{\varepsilon}\mathbf{v}_D \cdot \nabla C = D_{Sm}\nabla^2 C + D_\mathrm{CT}\nabla^2 T \tag{4}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon}\mathbf{v}_D \cdot \nabla \phi = D_\mathrm{B}\nabla^2\phi + \frac{D_\mathrm{T}}{T_\mathrm{c}}\nabla^2 T \tag{5}$$

where $\mathbf{v}_D = (u, v, w)$ is the fluid velocity. In these equations, ρ is the fluid density, ε is porosity, K is permeability. D_B and D_T denote the Brownian diffusion coefficient and thermophoretic diffusion respectively. We assume the flow to be slow to neglect an advective term and a Forchheimer quadratic drag term from the momentum equation. In the above equations, both Brownian transport and thermophoresis coefficients are taken to be time independent, in tune with the recent studies that neglect the effect of thermal transport attributed to the small size of the nanoparticles (as per recent arguments by Keblinski and Cahil 2005). Further, thermophoresis and Brownian transport coefficients are assumed to be temperature-independent due to the fact that the temperature ranges under consideration are not far away from the critical value, and the volume averages over a representative elementary volume.

Assuming the temperature, the solutal concentrations, and the volumetric fraction of the nanoparticles to be constant at the boundaries, we get the boundary conditions to be

$$\mathbf{v} = 0, \quad T = T_\mathrm{h}, \quad C = C_\mathrm{h}, \quad \phi = \phi_1 \quad at\ z = 0, \tag{6}$$

$$\mathbf{v} = 0, \quad T = T_\mathrm{c}, \quad C = C_\mathrm{c}\ \phi = \phi_0 \quad at\ z = d. \tag{7}$$

where ϕ_1 is greater than ϕ_0. The dimensionless variables are considered as given below:

$$(x^*, y^*, z^*) = (x, y, z)/d, \quad t^* = tk_\mathrm{T}/\gamma d^2$$

$$(u^*, v^*, w^*) = (u, v, w)d/k_\mathrm{T}, \quad p^* = pK/\mu k_\mathrm{T}$$

$$\phi^* = \frac{\phi - \phi_0}{\phi_1 - \phi_0} \quad T^* = \frac{T - T_\mathrm{c}}{T_\mathrm{h} - T_\mathrm{c}} \quad C^* = \frac{C - C_\mathrm{c}}{C_\mathrm{h} - C_\mathrm{c}}$$

where $k_{\mathrm{T}} = \dfrac{k_{\mathrm{m}}}{(\rho c)_{\mathrm{f}}}$, $\gamma = \dfrac{(\rho c_{\mathrm{p}})_{\mathrm{m}}}{(\rho c_{\mathrm{p}})_{\mathrm{f}}}$. The Eqs. 1–7, on non-dimesionalizing, take the form (after dropping the asterisk sign)

$$\nabla \cdot \mathbf{v} = 0 \tag{8}$$

$$0 = -\nabla p - \mathbf{v} - Rm\hat{e_z} + RaT\hat{e_z} - Rn\phi\hat{e_z} + (Rs/Le)C\hat{e_z} \tag{9}$$

$$\gamma \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla^2 T + \frac{N_{\mathrm{B}}}{Ln} \nabla\phi \cdot \nabla T + \frac{N_{\mathrm{A}} N_{\mathrm{B}}}{Ln} \nabla T \cdot \nabla T + N_{\mathrm{TC}} \nabla^2 C \tag{10}$$

$$\frac{\partial C}{\partial t} + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla C = \frac{1}{Le} \nabla^2 C + N_{\mathrm{CT}} \nabla^2 T \tag{11}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla \phi = \frac{1}{Ln} \nabla^2 \phi + \frac{N_{\mathrm{A}}}{Ln} \nabla^2 T_{\mathrm{f}} \tag{12}$$

$$\mathbf{v} = 0, \;\; T = 1, \;\; C = 1, \;\; \phi = 1 \; at \; z = 0, \tag{13}$$

$$\mathbf{v} = 0, \;\; T = 0, \;\; C = 0, \;\; \phi = 0 \; at \; z = 1. \tag{14}$$

Here

$Ln = \dfrac{k_{\mathrm{T}}}{D_{\mathrm{B}}}$ is the thermo-nanofluid Lewis number,

$Ra = \dfrac{\rho g \beta_{\mathrm{T}} K d (T_{\mathrm{h}} - T_{\mathrm{c}})}{\mu k_{\mathrm{T}}}$ is the thermal Rayleigh number,

$Rs = \dfrac{\rho g \beta_{\mathrm{C}} K d (C_{\mathrm{h}} - C_{\mathrm{c}})}{\mu D_{Sm}}$ is the solutal Rayleigh number,

$Rm = \dfrac{[\rho_{\mathrm{p}} \phi_0 + \rho(1 - \phi_0)] g K d}{\mu k_{\mathrm{T}}}$ is the basic density Rayleigh number,

$Rn = \dfrac{(\rho_{\mathrm{p}} - \rho)(\phi_1 - \phi_0) g K d}{\mu k_{\mathrm{T}}}$ is the nanoparticle concentration Rayleigh number,

$N_{\mathrm{A}} = \dfrac{D_{\mathrm{T}} (T_{\mathrm{h}} - T_{\mathrm{c}})}{D_{\mathrm{B}} T_{\mathrm{c}} (\phi_1 - \phi_0)}$ is the modified diffusivity ratio,

$N_{\mathrm{B}} = \dfrac{\varepsilon (\rho c)_{\mathrm{p}} (\phi_1 - \phi_0)}{(\rho c)_{\mathrm{f}}}$ is the modified particle density increment,

$Le = \dfrac{k_{\mathrm{T}}}{D_S}$ is the thermo-solutal Lewis number,

$N_{\mathrm{TC}} = \dfrac{D_{\mathrm{TC}} (C_{\mathrm{h}} - C_{\mathrm{c}})}{k_{\mathrm{T}} (T_{\mathrm{h}} - T_{\mathrm{c}})}$ is the Dufour parameter,

$N_{\mathrm{CT}} = \dfrac{D_{\mathrm{CT}} (T_{\mathrm{h}} - T_{\mathrm{c}})}{k_{\mathrm{T}} (C_{\mathrm{h}} - C_{\mathrm{c}})}$ is the Soret parameter.

3 Basic Solution

The quantities at the basic state vary only in the z direction as we assume the nanofluid to be at rest at the basic state, and will be given by

$$\mathbf{v} = 0, \quad p = p_{\mathrm{b}}(z), \quad T = T_b(z), \quad C = C_b(z), \quad \phi = \phi_{\mathrm{b}}(z) \tag{15}$$

Substituting Eq. 15 in Eqs. 11–14, we get

$$\frac{\mathrm{d}^2 T_b}{\mathrm{d}z^2} + \frac{N_\mathrm{B}}{Ln}\frac{\mathrm{d}\phi_\mathrm{b}}{\mathrm{d}z}\frac{\mathrm{d}T_b}{\mathrm{d}z} + \frac{N_\mathrm{A} N_\mathrm{B}}{Ln}\left(\frac{\mathrm{d}T_b}{\mathrm{d}z}\right)^2 + N_\mathrm{TC}\frac{\mathrm{d}^2 C_b}{\mathrm{d}z^2} = 0 \tag{16}$$

Kuznetsov and Nield (2010a) showed by using an order of magnitude analysis that the second and third terms in Eq. 16 are small, hence we have:

$$\frac{\mathrm{d}^2 T_b}{\mathrm{d}z^2} = 0, \quad \frac{\mathrm{d}^2 C_b}{\mathrm{d}z^2} = 0, \quad \frac{\mathrm{d}^2 \phi_\mathrm{b}}{\mathrm{d}z^2} = 0, \tag{17}$$

The boundary conditions for solving Eq. 17 are obtained from Eqs. 13 and 14 as:

$$T_b = 1,\ C_\mathrm{b} = 1, \quad \phi_\mathrm{b} = 1, \quad \text{at z} = 0, \tag{18}$$

$$T_\mathrm{b} = 0,\ C_\mathrm{b} = 0, \quad \phi_\mathrm{b} = 0, \quad \text{at z} = 1. \tag{19}$$

The remaining solution $p_\mathrm{b}(z)$ at the basic state can be obtained by integrating Eq. (9) for p_b after substituting T_b, C_b and ϕ_b from Eq. (17). Solving Eq. (17), subject to conditions (18) and (19), we obtain:

$$T_\mathrm{b} = 1 - z \tag{20}$$

$$C_\mathrm{b} = 1 - z \tag{21}$$

$$\phi_\mathrm{b} = 1 - z. \tag{22}$$

4 Stability Analysis

Imposing perturbations on the basic state as given below:

$$\mathbf{v} = \mathbf{v}',\ p = p_\mathrm{b} + p',\ T = T_\mathrm{b} + T',\ C = C_\mathrm{b} + C',\ \phi = \phi_\mathrm{b} + \phi'. \tag{23}$$

For simplicity, we consider the case of two dimensional rolls, assuming all physical quantities to be independent of y. Substituting the above expression (23) in Eqs. 9–12, and using the expressions (20–22), eliminating the pressure and introducing the stream function we obtain:

$$\nabla_1^2 \psi - \alpha\left[Ra\frac{\partial T}{\partial x} - Rn\frac{\partial \phi}{\partial x} + \frac{Rs}{Le}\frac{\partial C}{\partial x}\right] = 0 \tag{24}$$

$$\gamma\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial x} = \nabla_1^2 T + N_\mathrm{TC}\nabla_1^2 T + \frac{\partial(\psi, T)}{\partial(x, z)} \tag{25}$$

$$\frac{\partial C}{\partial t} + \frac{1}{\varepsilon}\left(\frac{\partial \psi}{\partial x}\right) = \frac{1}{Le}\nabla_1^2 C + N_\mathrm{CT}\nabla_1^2 + \frac{1}{\varepsilon}\frac{\partial(\psi, C)}{\partial(x, z)} \tag{26}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon}\left(\frac{\partial \psi}{\partial x}\right) = \frac{1}{Ln}\nabla_1^2 \phi + \frac{N_\mathrm{A}}{Ln}\nabla_1^2 T + \frac{1}{\varepsilon}\frac{\partial(\psi, \phi)}{\partial(x, z)} \tag{27}$$

We solve Eqs. 24–27 subjecting them to stress-free, isothermal, iso-solutal, iso-nanoconcentration boundary conditions:

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = T = C = \phi = 0 \quad \text{at z} = 0, 1. \tag{28}$$

Then using normal mode technique, the expressions for the Rayleigh numbers for stationary and oscillatory convection and the frequency of oscillation, ω, are given by

$$Ra^{\mathrm{st}} = RsN_{\mathrm{CT}} - \frac{Rn\delta^2 N_{\mathrm{A}}}{Ln} + \left(\frac{\delta^2}{\alpha^2} + \frac{RnLe}{\varepsilon\delta^2} - \frac{Rs}{\varepsilon\delta^2}\right)\left[\frac{\varepsilon\delta^2(1 - N_{\mathrm{CT}}N_{\mathrm{TC}}Le)}{\varepsilon - LeN_{\mathrm{TC}}}\right] \tag{29}$$

$$Ra^{\mathrm{osc}} = \frac{Rs\delta^4 N_{\mathrm{CT}}}{\delta^4 + \omega^2 Le^2} - \frac{Rn\delta^2 N_{\mathrm{A}}}{Ln} + \left(\frac{\delta^2}{\alpha^2} + \frac{T_6\delta^2}{\delta^4 + \omega^2 Le^2}\right)\left(\frac{T_1T_3 - T_4\omega^2}{T_3^2 - \omega^2 Le^2\varepsilon^2}\right) + \left(\frac{\omega^2 T_6 Le}{\delta^4 + \omega^2 Le^2}\right)\left(\frac{T_5 - \omega^2\gamma\varepsilon^2 Le^2}{T_3^2 - \omega^2 Le^2\varepsilon^2}\right) \tag{30}$$

$$\omega^2 = \frac{-X_2 + \sqrt{X_2^2 - 4X_1X_3}}{2X_1} \tag{31}$$

where

$$X_1 = \gamma\varepsilon Le^4\frac{\delta^2}{\alpha^2}$$

$$X_2 = \frac{\delta^6\gamma\varepsilon Le^2}{\alpha^2} - Rs\delta^2\varepsilon^2 Le^3 N_{\mathrm{CT}} - T_6T_4Le + T_6\delta^2\gamma\varepsilon^2 Le^2 - T_5Le^2\frac{\delta^2}{\alpha^2}$$

$$X_3 = Rs\delta^2 N_{\mathrm{CT}}LeT_3^2 + T_1T_6T_3Le - T_5T_6\delta^2 - T_5\frac{\delta^2}{\alpha^2}$$

and

$$\delta^2 = \pi^2 + \alpha^2,\ T_1 = \varepsilon\delta^4(1 - LeN_{\mathrm{CT}}N_{\mathrm{TC}}),\ T_2 = \varepsilon\delta^2(\gamma + Le),\ T_3 = \delta^2(\varepsilon - LeN_{\mathrm{TC}}),$$
$$T_4 = \varepsilon Le(\gamma T_3 - T_2),\ T_5 = T_2T_3 - \varepsilon T_1Le,\ T_6 = \frac{1}{\varepsilon}(RnLe - Rs)$$

It is quite obvious from Eq. 31 that oscillatory convection is possible only when

$$X_2^2 - 4X_1X_3 > 0 \tag{32}$$

To perform a local non-linear stability analysis, we take the following Fourier expressions:

$$\psi = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin(m\alpha x)\sin(n\pi z) \tag{33}$$

$$T = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}(t)\cos(m\alpha x)\sin(n\pi z) \tag{34}$$

$$\phi = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}(t)\cos(m\alpha x)\sin(n\pi z) \tag{35}$$

$$C = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{mn}(t)\cos(m\alpha x)\sin(n\pi z) \tag{36}$$

Further, we take the modes (1,1) for stream function, and (0,2) and (1,1) for temperature, solute concentration and nanoparticle concentration, to get

$$\psi = A_{11}(t)\sin(\alpha x)\sin(\pi z) \tag{37}$$

$$T = B_{11}(t)\cos(\alpha x)\sin(\pi z) + B_{02}(t)\sin(2\pi z) \tag{38}$$

$$\phi = C_{11}(t)\cos(\alpha x)\sin(\pi z) + C_{02}(t)\sin(2\pi z) \tag{39}$$

$$C = D_{11}(t)\cos(\alpha x)\sin(\pi z) + D_{02}(t)\sin(2\pi z) \tag{40}$$

where the amplitudes $A_{11}(t)$, $B_{11}(t)$, $B_{02}(t)$, $C_{11}(t)$, $C_{02}(t)$, $D_{11}(t)$, and $D_{02}(t)$ are functions of time and are to be determined. Substituting Eqs. 37–40 in Eqs. 24–27, taking the orthogonality condition with the eigenfunctions associated with the considered minimal model, we get

$$A_{11}(t) = \frac{\alpha}{\delta^2}\left[RnC_{11}(t) - RaB_{11}(t) - \frac{Rs}{Le}D_{11}(t)\right] \tag{41}$$

$$\frac{\mathrm{d}B_{11}(t)}{\mathrm{d}t} = -\frac{1}{\gamma}\{\delta^2 B_{11}(t) + \alpha A_{11}(t) + \pi\alpha A_{11}(t)B_{02}(t) + \delta^2 N_{\mathrm{CT}}D_{11}(t)\} \tag{42}$$

$$\frac{\mathrm{d}B_{02}(t)}{\mathrm{d}t} = \frac{1}{2\gamma}\left\{\alpha\pi A_{11}(t)B_{11}(t) - 8\pi^2 B_{02}(t) - 8\pi^2 N_{\mathrm{TC}}D_{02}(t)\right\} \tag{43}$$

$$\frac{\mathrm{d}C_{11}(t)}{\mathrm{d}t} = -\left\{\frac{\alpha\pi}{\varepsilon}A_{11}(t)C_{02}(t) + \frac{\alpha}{\varepsilon}A_{11}(t) + \delta^2\left[\frac{C_{11}(t)}{Ln} + \frac{N_{\mathrm{A}}}{Ln}B_{11}(t)\right]\right\} \tag{44}$$

$$\frac{\mathrm{d}C_{02}(t)}{\mathrm{d}t} = \frac{1}{2}\left\{\frac{\alpha\pi}{\varepsilon}A_{11}(t)C_{11}(t) - 8\pi^2\left[\frac{C_{02}(t)}{Ln} + \frac{N_{\mathrm{A}}}{Ln}B_{02}(t)\right]\right\} \tag{45}$$

$$\frac{\mathrm{d}D_{11}(t)}{\mathrm{d}t} = -\left\{\frac{\alpha\pi}{\varepsilon}A_{11}(t)D_{02}(t) + \frac{\alpha}{\varepsilon}A_{11}(t) + \delta^2\left[\frac{D_{11}(t)}{Le} + N_{\mathrm{CT}}B_{11}(t)\right]\right\} \tag{46}$$

$$\frac{\mathrm{d}D_{02}(t)}{\mathrm{d}t} = \frac{1}{2}\left\{\frac{\alpha\pi}{\varepsilon}A_{11}(t)D_{11}(t) - 8\pi^2\left[\frac{D_{02}(t)}{Ln} + \frac{N_{\mathrm{A}}}{Ln}B_{02}(t)\right]\right\} \tag{47}$$

The above system of simultaneous autonomous ordinary differential equations is solved numerically using Runge–Kutta–Gill method. Further the six-mode differential Eqs. 42–47 has an interesting property in phase space:

$$\frac{\partial \dot{B_{11}}}{\partial B_{11}} + \frac{\partial \dot{B_{02}}}{\partial B_{02}} + \frac{\partial \dot{C_{11}}}{\partial C_{11}} + \frac{\partial \dot{C_{02}}}{\partial C_{02}} + \frac{\partial \dot{D_{11}}}{\partial D_{11}} + \frac{\partial \dot{D_{02}}}{\partial D_{02}} = -\left[\delta^2 + 4\pi^2 + \frac{\delta^2}{Ln} + \frac{4\pi^2}{Ln} + \frac{\delta^2}{Le} + \frac{4\pi^2}{Le}\right] < 0 \tag{48}$$

which indicates that the system is dissipative and bounded. One may also conclude that the trajectories of the Eqs. 41–47 will be confined to the finiteness of the ellipsoid. Thus, the effect of the parameters Rn, Ln, N_{A}, Rs, $Le\varepsilon_{\mathrm{p}}$, N_{CT}, N_{TC} on the trajectories is to attract them to a set of measure zero, or to a fixed point to say.

5 Heat and Nanoparticle Concentration Transport

The thermal Nusselt number, $Nu_{\mathrm{f}}(t)$ is defined as

$$Nu_{\mathrm{f}}(t) = \frac{\text{Heat transport by (conduction + convection)}}{\text{Heat transport by conduction}} = 1 + \left[\frac{\int_0^{2\pi/\alpha_{\mathrm{c}}} \left(\frac{\partial T}{\partial z} \right) \mathrm{d}x}{\int_0^{2\pi/\alpha_{\mathrm{c}}} \left(\frac{\partial T_{\mathrm{B}}}{\partial z} \right) \mathrm{d}x} \right]_{z=0} \tag{49}$$

Substituting expressions (20) and (38) in Eq. 49, we get

$$Nu_{\mathrm{f}}(t) = 1 - 2\pi B_{02}(t) \tag{50}$$

Similarly the nanoparticle concentration Nusselt number, $Nu_{\phi}(t)$, and the solute concentration Nusselt number, $Nu(\text{solute})$ are found to be:

$$Nu_{\phi}(t) = (1 - 2\pi C_{02}(t)) + N_{\mathrm{A}}(1 - 2\pi B_{02}(t)) \tag{51}$$

$$Nu_{C}(t) = (1 - 2\pi D_{02}(t)) + N_{\mathrm{CT}}(1 - 2\pi B_{02}(t)) \tag{52}$$

6 Results and Discussion

In Fig. 1, we draw linear stability curves showing the stationary and oscillatory modes of convections. In the figure, we see that the region of over stability over-lies the region of damped oscillations, i.e., at the start of instability, stationary or aperiodic pattern of motion prevails. Therefore, we can say that the *Principle of Exchange of stabilities* is valid (Chandrashekhar 1961; Drazin and Reid 1981). This can be explained as; since stationary motion is prevalent at the onset of convection, the restoring forces provoked are not strong enough to prevent the system from tending away from equilibrium.

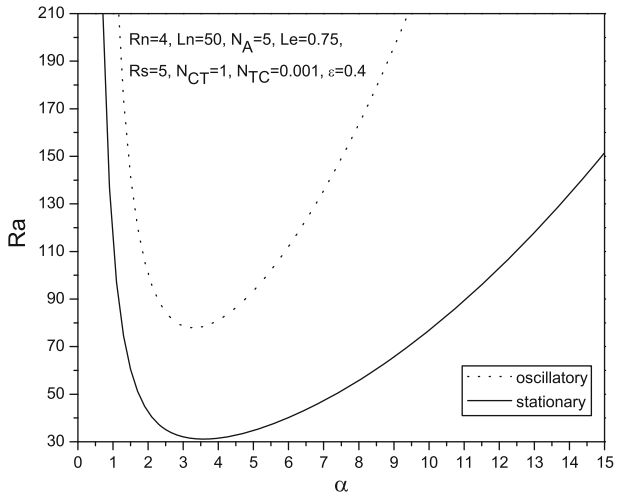

Fig. 1 Linear stability curve showing stationary and oscillatory modes of convection

Figure 2 shows the effect of various parameters on the neutral stability curves for $Rn = 4$, $Ln = 50$, $N_A = 5$, $Rs = 5$, $Le = 0.75$, $N_{CT} = 1$, $N_{TC} = 0.001$ and $\varepsilon = 0.4$, with a variation in one of these parameters. From our calculations we found that thermo-nanofluid Lewis number Ln, thermo-solutal Lewis number Le, Soret parameter N_{CT}, Dufour parameter N_{TC} and porosity ε have a stabilizing effect on the system, i.e., an increase in their value increases the critical Rayleigh number for stationary convection thus delaying the onset of convection. On the other hand, nanoparticle Rayleigh number Ra, solutal Rayleigh number Rs, and modified diffusivity ratio N_A have destabilizing effects on the system. However, we have included only three graphs in Fig. 2 corresponding to the parameters N_{CT}, N_{TC} and Rs.

In Fig. 3a, b, we have shown the variation of critical thermal Rayleigh number Ra_{cr} for stationary convection with respect to the nanoparticle concentration Rayleigh number Rn. From the figures, we observe that for small values of Rn, the value Ra_{cr} is high, which decreases on increasing nanoparticle concentration Rayleigh number. From Fig. 3a, b, we find that on increasing the values of Soret parameter N_{CT} and Dufour parameter N_{TC} the value of Ra_{cr} increases. However, from Fig. 3c, we see that on increasing the value of solutal Rayleigh number Rs the value of Ra_{cr} decreases. Results were also calculated for other varameters and it was found that on increasing the values of thermo-nanofluid Lewis number Ln and thermo-solutal Lewis number Le the value of Ra_{cr} increases while it decreases on increasing the modified diffusivity ratio N_A.

Figure 4a–c display the variation of critical thermal Rayleigh number Ra_{cr} with respect to the thermo-nanofluid Lewis number Ln. It is observed from the figure that for small values of Ln, we have smaller values of Ra_{cr} which increase with increasing thermo-nanofluid Lewis number Ln. Also it is evident from the Fig. 4a, b that on increasing the values of Soret parameter N_{CT} and Dufour parameter N_{TC} the value of Ra_{cr} increases, however, it decreases on increasing the value of solutal Rayleigh number Rs.

Figure 5a–c depict the variation of critical thermal Rayleigh number Ra_{cr} with the modified diffusivity ratio N_A. From Fig. 5, it is clear that corresponding to small values of N_A, we have large value of Ra_{cr} which decreases with increasing modified diffusivity ratio N_A. Further the effect of various parameters is found to be the same as in the Figs. 3 and 4.

The nature of Nusselt numbers, Nu^{ϕ}, $Nu(solute)$ and Nu, as a function of time t, for $Rn = 4$, $Ln = 50$, $N_A = 5$, $Ra = 4$, $Rs = 20$, $Le = 0.75$, $N_{CT} = 0.75$, $N_{TC} = 0.001$, $\varepsilon = 0.4$ and $\gamma = 2.5$ with a variation in the value of one of these parameters, is shown in Figs. 6, 7, and 8, respectively. These figures indicate that initially, when time is small, there occurs large scale oscillations in the values of Nu^{ϕ}, $Nu(solute)$ and Nu indicating an unsteady rate of nanoparticle concentration, solute concentration and heat transfer. As time passes by, these approach to steady values, corresponding to a near conduction instead of convection stage.

It is to be noted in these figures that the value of Nu^{ϕ} starts from 6, Nu(solute) starts from 1.75 and the value of Nu starts from 1. The expressions for these are

$$\begin{aligned} Nu_f(t) &= 1 - 2\pi B_{02}(t) \\ Nu_{\phi}(t) &= (1 - 2\pi C_{02}(t)) + N_A(1 - 2\pi B_{02}(t)) \\ Nu_C(t) &= (1 - 2\pi D_{02}(t)) + N_{CT}(1 - 2\pi B_{02}(t)) \end{aligned}$$

At $t = 0$, the values of $B_{02}(t)$, C_{02} and $D_{02}(t)$ are all zero corresponding to conduction stage, resulting in these value of Nu^{ϕ}, $Nu(solute)$ and Nu, as we take $N_A = 5$ and $N_{CT} = 0.75$ for our calculations.

It is seen from Figs. 6, 7, and 8 that the effect of the parameters concentration Rayleigh number Rn, thermo-nanofluid Rayleigh number Ln, modified diffusivity ratio N_A, solutal Rayleigh number Rs, is to increase the extent of oscillations in the figures. The vibrations

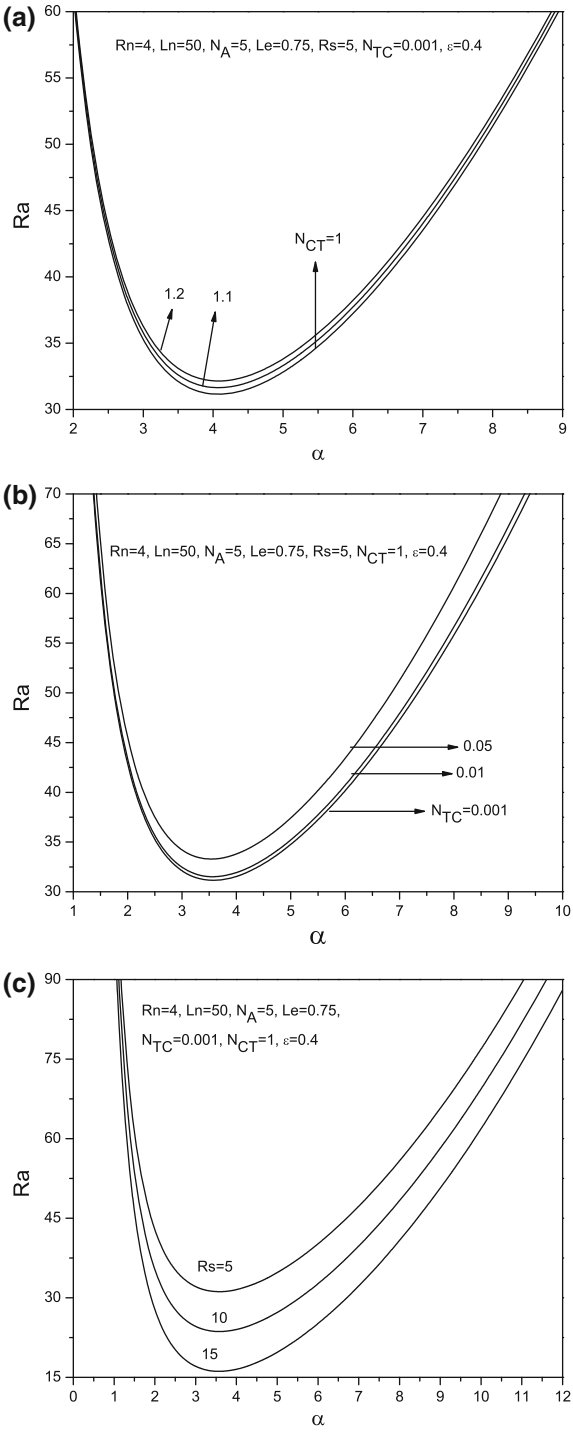

Fig. 2 Neutral stability curves for different values of **a** N_{CT}, **b** N_{TC}, **c** Rs

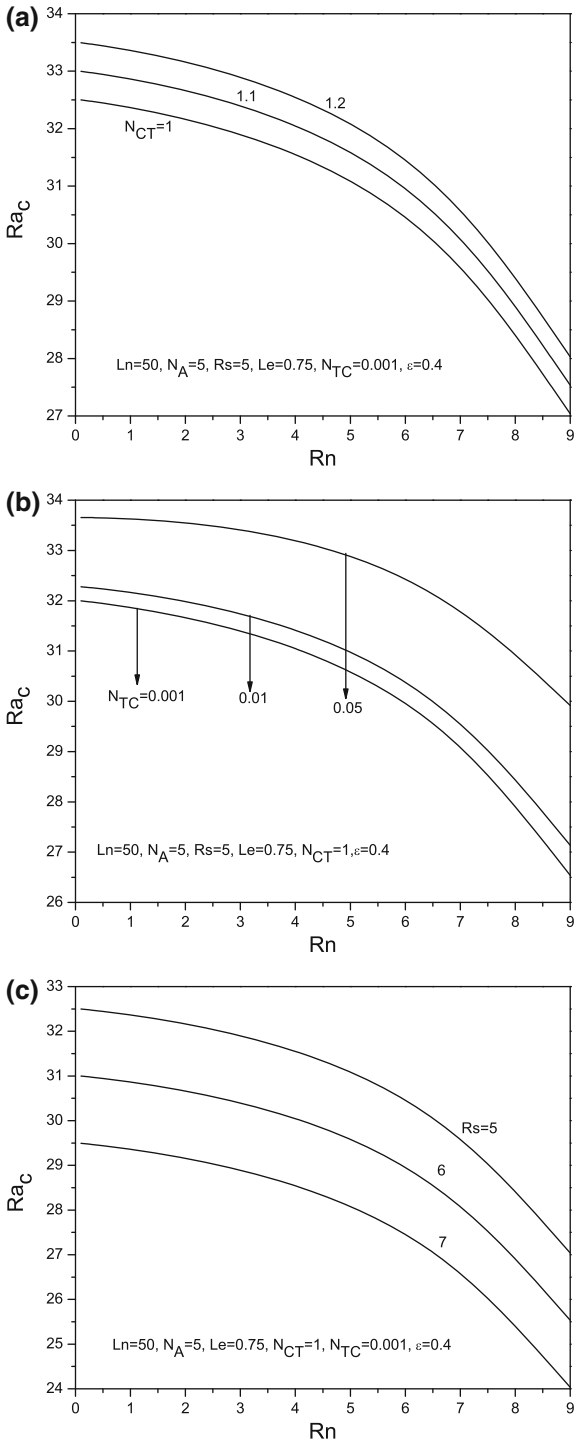

Fig. 3 Variation of critical Rayleigh number, Ra_c, with concentration Rayleigh number, Rn, for different values of **a** N_{CT}, **b** N_{TC}, **c** Rs

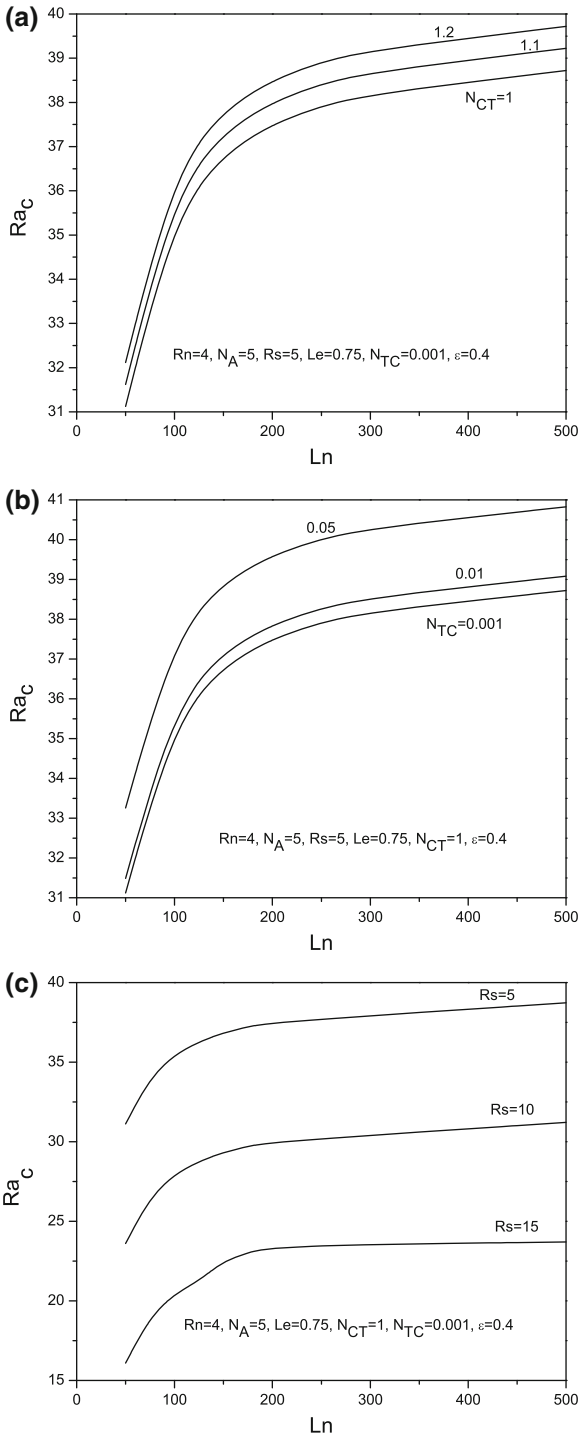

Fig. 4 Variation of critical Rayleigh number, Ra_c, with Lewis number, Ln, for different values of **a** N_{CT}, **b** N_{TC}, **c** Rs

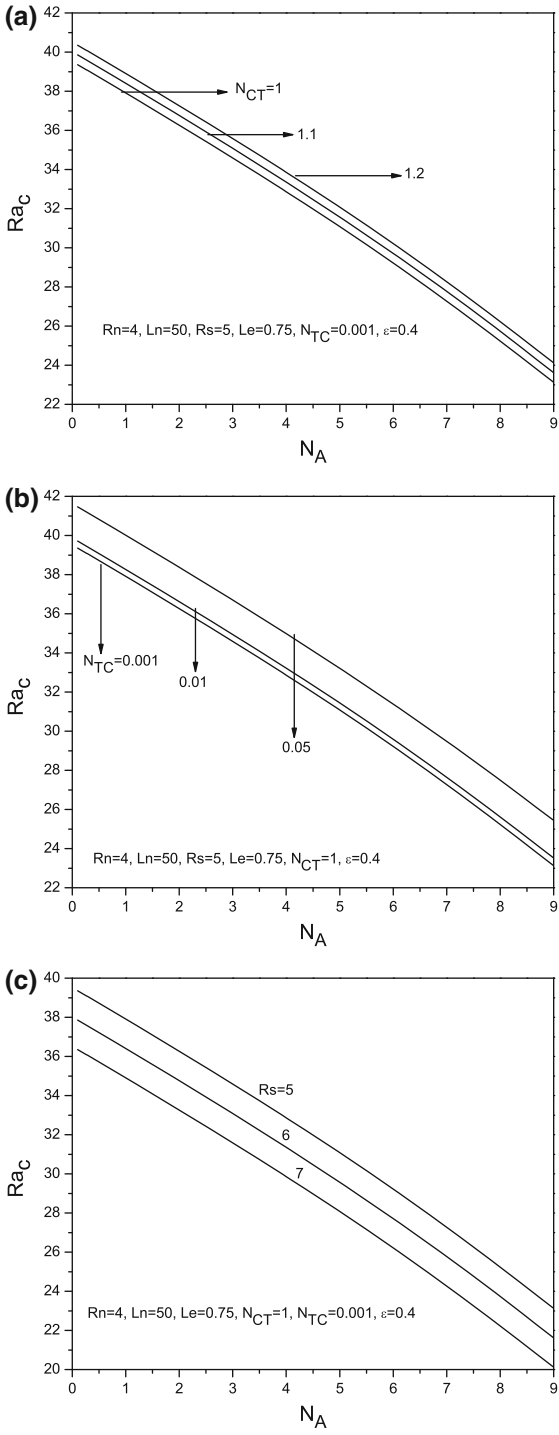

Fig. 5 Variation of critical Rayleigh number, Ra_c, with modified diffusivity ratio, N_A, for different values of **a** N_{CT}, **b** N_{TC}, **c** Rs

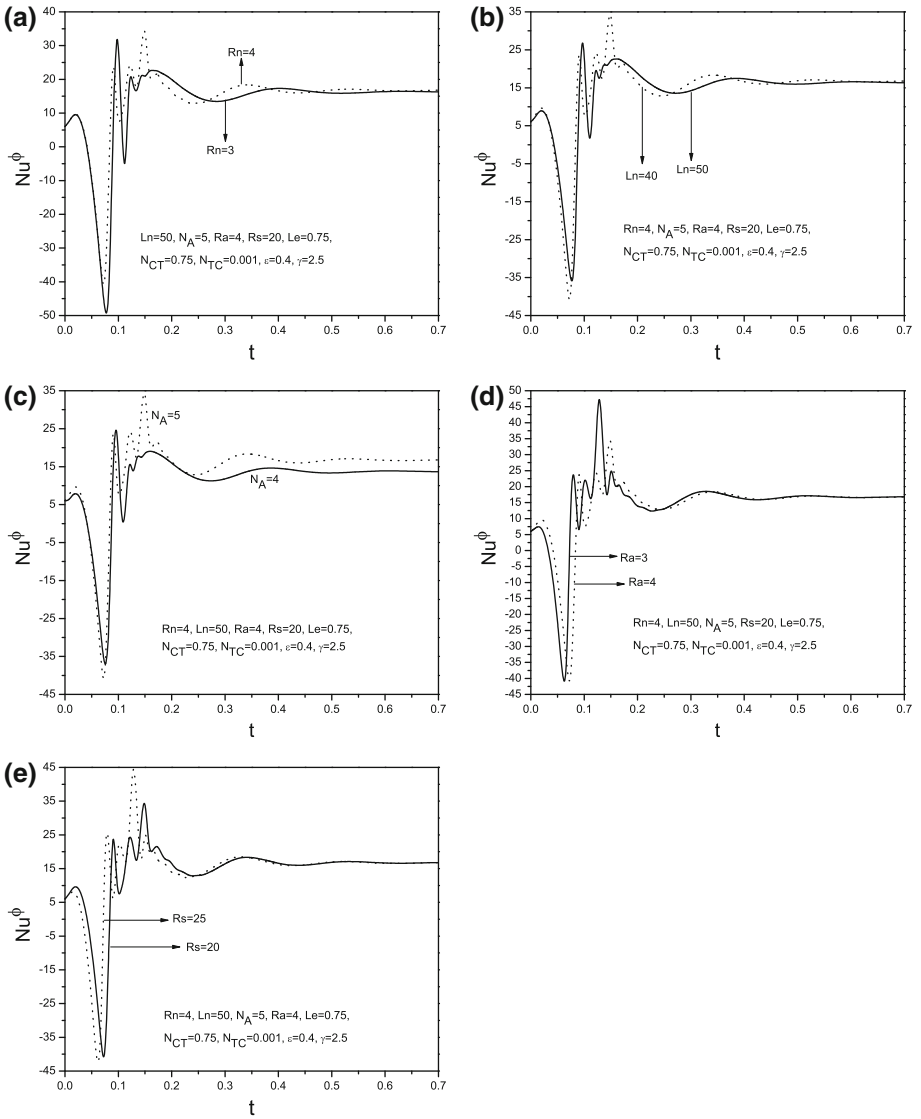

Fig. 6 Variation of concentration Nusselt number Nu^{ϕ} with time t for different values of **a** Rn, **b** Ln, **c** N_{A}, **d** Ra, and **e** Rs

are enhanced on increasing the values of these parameters. This implies that an increase in their value increases the rate of nanoparticle concentration, rate of solute concentration and the amount of heat transferred. The results corresponding to Soret parameter N_{CT} and thermal capacity ratio γ were also obtained and found to have the similar effects as the above parameters. While it is to be noted that the effect of the parameters thermal Rayleigh number Ra (Figs. 6d, 7, 8d), thermo-solutal Lewis number Le, Dufour parameter N_{TC} is to decrease the vibrations, i.e., to inhibit the rate of nanoparticle concentration, solute concentration and the heat transfer. Whereas, the effect of porosity ε, is to decrease the rate of nanoparticle concentration but to enhance the rate of solute concentration transfer and heat transfer.

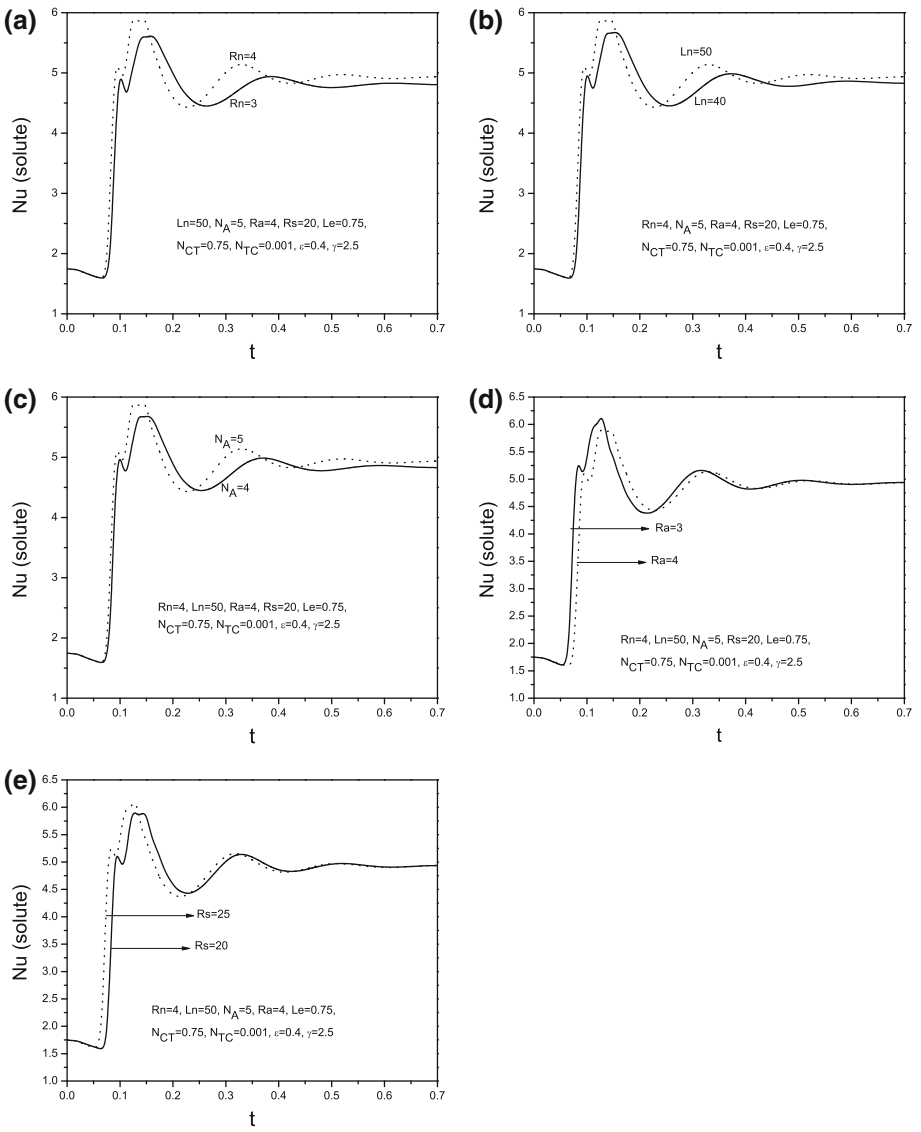

Fig. 7 Variation of solute Nusselt number Nu(solute) with time t for different values of **a** Rn, **b** Ln, **c** N_A, **d** Ra, and **e** Rs

Figure 9 shows the comparison of the value of Rayleigh number Ra for nanofluid with ordinary fluid in double-diffusive convection. It is observed that the critical Rayleigh number is lower in case of nanofluids as compared to ordinary fluids. This abnormality can be attributed to the two new parameters, Soret and Dufour parameters, along with the modified diffusivity ratio, arising due to the triple diffusion condition being encountered in binary nanofluids. These two parameters lower the Rayleigh number for stationary convective transport in nanofluids.

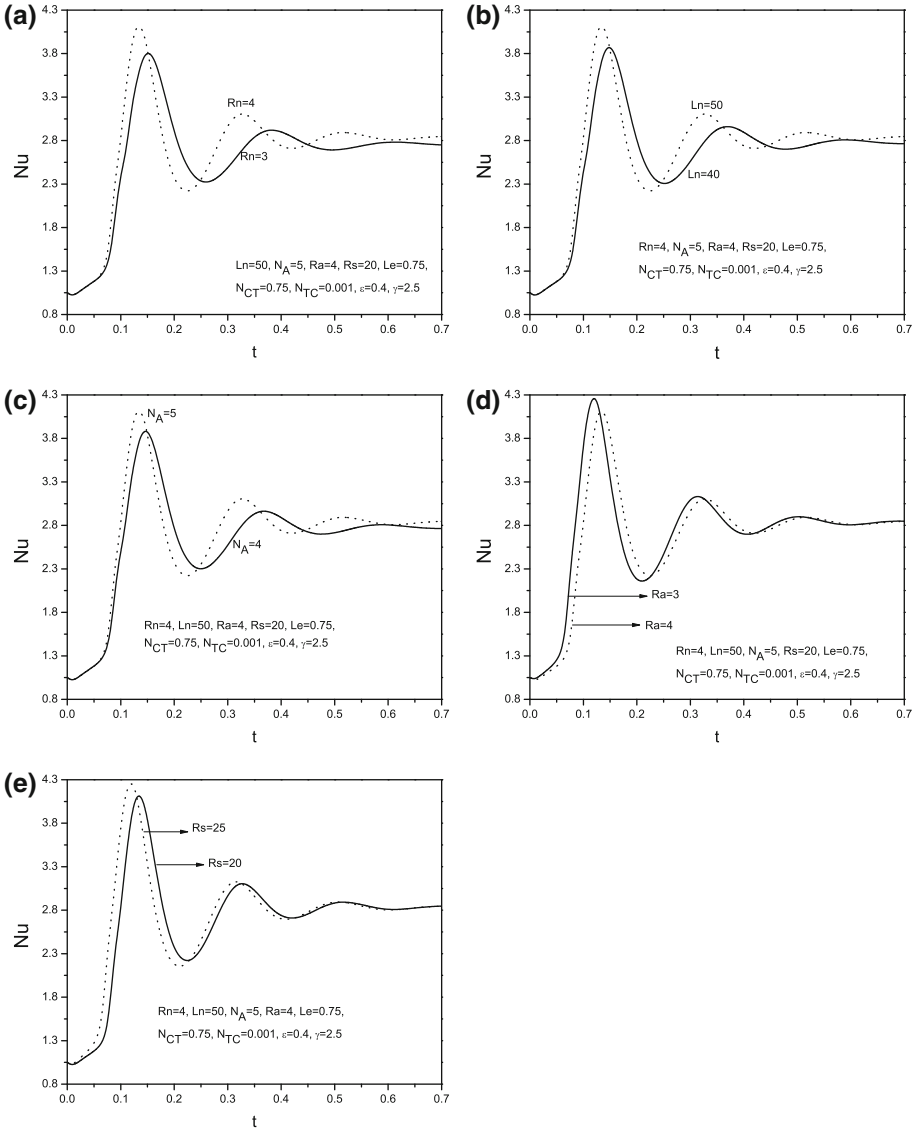

Fig. 8 Variation of thermal Nusselt number Nu(solute) with time t for different values of **a** Rn, **b** Ln, **c** N_A, **d** Ra, and **e** Rs

7 Conclusions

We considered linear stability analysis in a horizontal porous medium saturated by a nanofluid, heated from below and cooled from above, using Darcy model which incorporates the effect of Brownian motion along with thermophoresis, under non-equilibrium conditions. Further bottom-heavy suspension of nanoparticles has been considered. Linear analysis has been made using normal mode technique. Then the effect of various parameters on the onset of thermal instability has been found. The results have been presented graphically. We draw the following conclusions:

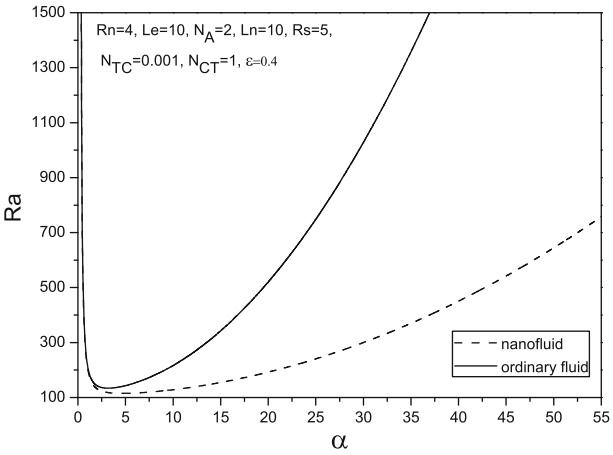

Fig. 9 Comparison of the value of Rayleigh number *Ra* for nanofluid with ordinary fluid in double-diffusive convection

1. The effect of the Ln, Le, N_{CT}, N_{TC}, ε is to stabilize the system, while N_{A}, Rs tend to destabilize the system.
2. The effect of time on Nusselt numbers is found to be oscillatory, when t is small. However, when time t becomes very large Nusselt number approaches the steady value.

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