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An analytical study of nonlinear double-diffusive convection in a porous medium under temperature/gravity modulation

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Abstract The article deals with nonlinear thermal instability problem of double-diffusive convection in a porous medium subjected to temperature/gravity modulation. Three types of imposed time-periodic boundary temperature (ITBT) are considered. The effect of imposed time-periodic gravity modulation (ITGM) is also studied in this problem. In the case of ITBT, the temperature gradient between the walls of the fluid layer consists of a steady part and a time-dependent periodic part. The temperature of both walls is modulated in this case. In the problem involving ITGM, the gravity field has two parts: a constant part and an externally imposed time-periodic part. Using power series expansion in terms of the amplitude of modulation, which is assumed to be small, the problem has been studied using the Ginzburg–Landau amplitude equation. The individual effects of temperature and gravity modulation on heat and mass transports have been investigated in terms of Nusselt number and Sherwood number, respectively. Further the effects of various parameters on heat and mass transports have been analyzed and depicted graphically.

Keywords Double-diffusive Convection · Non-linear stability analysis · Ginzburg–Landau equation · Temperature modulation · Gravity modulation

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List of Variables

Latin Symbols

A	Amplitude of streamline perturbation
d	Height of the fluid layer
Da	Darcy number $Da = \frac{K}{d^2}$
g	Acceleration due to gravity
$g_m(\tau)$	Modulation in gravity $g_m(\tau) = \epsilon^2\delta_2 Cos(\omega\tau)$
k_c	Critical wavenumber
Le	Lewis number, $Le = \frac{\kappa_T}{\kappa_S}$
Nu	Nusselt number
p	Reduced pressure
Pr	Prandtl number, $Pr = \frac{\nu}{\kappa_T}$
Ra_S	Solutal Rayleigh number, $Ra_S = \frac{\beta_S g \Delta S d^3}{\nu\kappa_T}$
Ra_T	Thermal Rayleigh number, $Ra_T = \frac{\beta_T g \Delta T d^3}{\nu\kappa_T}$
R_{0c}	Critical Rayleigh number
S	Solute concentration
ΔS	Solute difference across the fluid layer
Sh	Sherwood number
t	Time
T	Temperature
ΔT	Temperature difference across the fluid layer,
x,y,z	Space Co-ordinates

Greek Symbols

β_T	Coefficient of thermal expansion
β_S	Coefficient of solute expansion
δ^2	Horizontal wave number $k_c^2 + \pi^2$
δ_1	Amplitude of temperature modulation
δ_2	Amplitude of gravity modulation
ϵ	Perturbation parameter
γ	Heat capacity ratio $\frac{(\rho c_p)_m}{(\rho c_p)_f}$
κ_T	Effective thermal diffusivity in horizontal direction
κ_S	Effective thermal diffusivity in vertical direction
μ	Effective dynamic viscosity of the fluid
ν	Effective kinematic viscosity, $\left(\frac{\mu}{\rho_0}\right)$
ϕ	Porosity
Φ^*	Dimensionless amplitude of solutal perturbation
Φ	Solutal perturbation
ψ	Stream function
Ψ	Dimensionless amplitude of stream function
ρ	Fluid density
τ	Slow time $\tau = \epsilon^2 t$
Θ	Temperature perturbation
Θ^*	Dimensionless amplitude of temperature perturbation

Other Symbol

∇^2 $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Subscripts

b Basic state
c Critical
0 Reference value

Superscripts

$'$ Perturbed quantity
$*$ Dimensionless quantity
st Stationary

1 Introduction

Double-diffusive convection in porous media concerns instability in fluid-saturated porous media with two diffusing components like temperature and salt contributing to the instability in an opposing sense and with the components having unequal diffusivity coefficients. This problem has wide ranging applications in solidification of binary mixtures, migration of solutes in water-saturated soils, geophysical systems, electro-chemistry, and the migration of moisture through air contained in fibrous insulation [Ingham and Pop (1998, 2005); Vafai (2000, 2005); Nield and Bejan (2006) and Vadasz (2008)]. There are innumerable reported studies on double-diffusive convection in porous media[Patil and Rudraiah (1980); Griffith (1981); Chakrabarti and Gupta (1981); Rudraiah et al. (1982); Poulikakos (1986); Rudraiah and Malashetty (1986); Murray and Chen (1989); Rudraiah and Siddheshwar (1998); Kuznetsov and Nield (2008, 2010, 2011); Nield and Kuznetsov (2011), and references therein].

In most of the studies related to double-diffusive convection, steady temperature gradient is considered. However, it is not so in many practical problems. There are many situations of practical importance in which temperature gradient is a function of both space and time. This temperature gradient can be determined by solving the energy equation with suitable time-dependent thermal boundary conditions, and can be used as an effective mechanism to control the convective flow.

The study of Venezian (1969) or the Floquet theory has been extensively followed in the thermal convection problem in porous media when the boundary temperatures are time-periodic; Caltagirone (1976); Chhuon and Caltagirone (1979); Antohe and Lage (1996); Malashetty and Wadi (1999); Malashetty and Basavaraja (2002, 2003); Malashetty et al. (2006); Malashetty and Swamy (2007); Bhadauria (2007a,b); Bhadauria and Sherani (2008, 2010) and Bhadauria and Suthar (2009). However, there are only few studies available on the double-diffusive convection problem in porous media with temperature modulation of the boundaries (Malashetty and Basavaraja (2004); Bhadauria (2007c,d) and Bhadauria and Srivastava (2010)). It is apt to note here that temperature modulation may be considered in the double-diffusive convection problem with the basic-state solutal concentration being unaffected in a time-periodic way only when the cross-diffusion effects are not included.

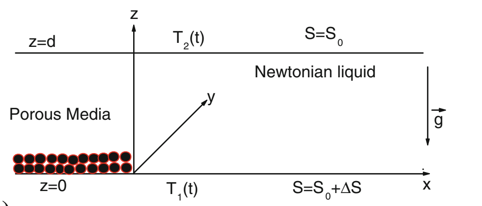
Another problem that leads to variable coefficients in the governing equations of the thermal and the double-diffusive convection problems in porous media is the one involving vertical time-periodic vibration of the system. This leads to the appearance of a modified gravity, collinear with actual gravity, in the form of a time-periodic gravity field perturbation and is known as gravity modulation or g-jitter in the literature [Malashetty and Padmavathi (1997); Rees and Pop (2002, 2001, 2003); Govender (2005a,b); Kuznetsov (2005, 2006a,b); Siddhavaram and Homsy (2006); Strong (2008a,b); Razi et al. (2009); Saravanan and Purusothaman (2009); Saravanan and Arunkumar (2010); Saravanan and Sivakumar (2010, 2011)) and references therein].

The studies thus far reviewed concern linear stability of the thermal or double-diffusive system in porous media in the absence/presence of temperature/gravity modulations, and hence address only questions on onset of convection. If one were to consider heat and mass transports in porous media in the presence of temperature/gravity modulations, then the linear stability analysis is inadequate and the nonlinear stability analysis becomes inevitable. There are no reported studies on this aspect of the problem. In the light of the above, we make a weakly nonlinear analysis of the problem using the Ginzburg–Landau equation and, in the process, quantify the heat and mass transports in terms of the amplitude governed by the Ginzburg–Landau equation.

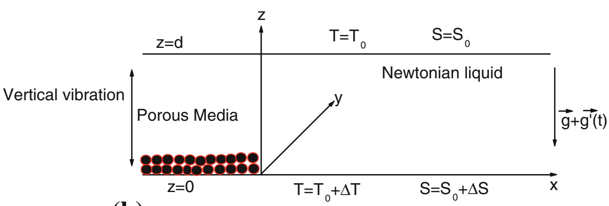
2 Governing Equations

We consider a two-component Newtonian fluid-saturated horizontal porous layer confined between two free-free boundaries at $z = 0$ and $z = d$. The layer is heated and salted from below. The configuration is as given in Fig. 1a. The fluid saturating the porous layer is considered to be Boussinesq, and thus the equations governing the flow are given by

$$\nabla.\overrightarrow{q} = 0, \tag{1}$$



(a) Physical configuration for the temperature modulation problem.



(b) Physical configuration for the gravity modulation problem.

Fig. 1 **a** Physical configuration for the temperature modulation problem. **b** Physical configuration for the gravity modulation problem

$$\frac{1}{\phi}\left[\frac{\partial \overrightarrow{q}}{\partial t}+\frac{1}{\phi}\left(\overrightarrow{q}.\nabla\right)\overrightarrow{q}\right]=-\frac{1}{\rho_0}\nabla p+\frac{\rho}{\rho_0}g\hat{k}-\frac{\nu}{K}\overrightarrow{q}+\nu\nabla^2\overrightarrow{q}, \tag{2}$$

$$\gamma\frac{\partial T}{\partial t}+(\overrightarrow{q}.\nabla)T=\kappa_T\nabla^2 T, \tag{3}$$

$$\phi\frac{\partial S}{\partial t}+(\overrightarrow{q}.\nabla)S=\kappa_S\nabla^2 S, \tag{4}$$

$$\rho=\rho_0[1-\beta_T(T-T_0)+\beta_S(S-S_0)]. \tag{5}$$

The constants and variables used in the above Eqs. 1–5 have their usual meanings and are given in the list of variables.

3 Mathematical Formulation for the Temperature Modulation Problem

The externally imposed surface temperature conditions considered in the problem are

$$\begin{aligned} T_1(t)&=T_0+\frac{\Delta T}{2}\left[1+\epsilon^2\delta_1 Cos\omega t\right] \qquad \text{at z = 0},\\ T_2(t)&=T_0-\frac{\Delta T}{2}\left[1-\epsilon^2\delta_1 Cos\left(\omega t+\phi\right)\right] \text{ at z = d}. \end{aligned} \tag{6}$$

where δ_1 represents the amplitude of modulation and ϵ is a quantity that indicates smallness in order of magnitude of modulation. ΔT is a small temperature that is modulated upon, ω is the modulation frequency, and ϕ is the phase angle.
Since we are not considering cross-diffusion terms, the walls of the liquid layer are assumed to be maintained at constant solute concentration as defined below

$$\begin{aligned} S&=S_0+\Delta S \quad \text{at z = 0},\\ &=S_0 \qquad\quad \text{at z = d}. \end{aligned} \tag{7}$$

The basic state of liquid is quiescent and is given by

$$\vec{q}_b=0,\ p=p_b(z,t),\ T=T_b(z,t),\ S=S_b(z),\ \rho=\rho_b(z,t), \tag{8}$$

$$\frac{\partial T_b}{\partial t}=\kappa_T\frac{\partial^2 T_b}{\partial z^2}, \tag{9}$$

$$\frac{d^2 S_b}{dz^2}=0, \tag{10}$$

$$\frac{\partial p_b}{\partial z}=-\rho_b g, \tag{11}$$

$$\rho_b=\rho_0\left[1-\alpha\left(T_b-T_0\right)+\beta\left(S_b-S_0\right)\right]. \tag{12}$$

The solution of the Eqs. 9–10, subject to the thermal and solutal boundary conditions (6–7), is given by

$$T_b\left(z,t\right)=T_S\left(z\right)+\epsilon^2\delta_1 Re\left\{T_3\left(z,t\right)\right\}, \tag{13}$$

$$S_b=S_0+\Delta S\left(1-\frac{z}{d}\right), \tag{14}$$

where

$$T_S(z) = T_0 + \frac{\Delta T}{2}\left(1 - \frac{2z}{d}\right), \tag{15}$$

$$T_3(z,t) = \left[a(\lambda)\, e^{\frac{\lambda z}{d}} + a(-\lambda)\, e^{\frac{-\lambda z}{d}}\right] e^{-i\omega t}, \tag{16}$$

$$\lambda^2 = -i\omega d^2/\kappa_T \quad \text{and} \quad a(\lambda) = \frac{\Delta T}{2}\frac{e^{\lambda\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}. \tag{17}$$

In the above equations, $T_S(z)$ is the steady temperature field, and T_3 is the oscillating part of T_b, while Re stands for the real part.
We assume finite amplitude perturbations on the basic state in the form:

$$\vec{q} = \vec{q}_b + \vec{q},\ T = T_b + \Theta',\ S = S_b + \Phi',\ p = p_b + p',\ \rho = \rho_b + \rho'. \tag{18}$$

Substituting Eq. 18 into Eqs. 1–5, we get the following equations:

$$\nabla.\vec{q'} = 0, \tag{19}$$

$$\frac{1}{\phi}\left[\frac{\partial \vec{q'}}{\partial t} + \frac{1}{\phi}\left(\vec{q'}.\nabla\right)\vec{q'}\right] = -\frac{1}{\rho_0}\nabla p' + (\beta_T\Theta' - \beta_S\Phi')g\hat{k} - \frac{\nu}{K}\vec{q'} + \nu\nabla^2\vec{q'}, \tag{20}$$

$$\gamma\frac{\partial\Theta'}{\partial t} + \left(\vec{q'}.\nabla\right)\Theta' + w'\frac{\partial T_b}{\partial z} = \kappa_T\nabla^2\Theta', \tag{21}$$

$$\phi\frac{\partial\Phi'}{\partial t} + \left(\vec{q'}.\nabla\right)\Phi' + w'\frac{dS_b}{dz} = \kappa_S\nabla^2\Phi', \tag{22}$$

$$\rho' = -\rho_0\left[\beta_T\Theta' - \beta_S\Phi'\right]. \tag{23}$$

We consider only two-dimensional disturbances in our study, and hence the stream function ψ may be introduced in the form:

$$u' = \frac{\partial\psi}{\partial z},\ w' = -\frac{\partial\psi}{\partial x}. \tag{24}$$

We eliminate the pressure term p' from Eq. 20 and then non-dimensionlize the equations following scales:

$$\psi = \kappa_T\Psi^*, (x,y,z) = d\left(x^*, y^*, z^*\right),\ \Theta = \Delta T\Theta^*, \quad \Phi = \Delta S\Phi^*, \quad t = \frac{d^2}{\kappa_T}.$$

The non-dimensional governing equations now have the form:

$$\left(\frac{1}{Pr}\frac{\partial\nabla^2}{\partial t} + \frac{1}{Da}\nabla^2 - \nabla^4\right)\Psi + Ra_T\frac{\partial\Theta}{\partial x} - Ra_S\frac{\partial\Phi}{\partial x} = \frac{1}{Pr}\frac{\partial(\Psi, \nabla^2\Psi)}{\partial(x,z)}, \tag{25}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\Theta - \frac{\partial\Psi}{\partial x}\frac{\partial T_b}{\partial z} = \frac{\partial(\Psi,\Theta)}{\partial(x,z)}, \tag{26}$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{Le}\nabla^2\right)\Phi + \frac{\partial\Psi}{\partial x} = \frac{\partial(\Psi,\Phi)}{\partial(x,z)}, \tag{27}$$

where $Pr = \frac{\nu}{\kappa_T}$, $Ra_T = \frac{\beta_T g\Delta T d^3}{\nu\kappa_T}$, $Ra_S = \frac{\beta_S g\Delta S d^3}{\nu\kappa_T}$, and $Le = \frac{\kappa_T}{\kappa_S}$.
In the above equations asterisks are dropped for simplicity. The non-dimensional basic temperature $T_b(z,t)$ which appears in the Eq. (26) can be obtained from Eq. (14) as

$$T_b(z,t) = T_0 + (1-z) + \epsilon^2\delta_1 F(z,t), \tag{28}$$

where

$$F(z,t) = Re\left[\left\{A(\lambda)e^{\lambda z} + A(-\lambda)e^{-\lambda z}\right\}e^{-i\Omega t}\right], \tag{29}$$

$$A(\lambda) = \frac{1}{2}\frac{\left(e^{-i\phi} - e^{-\lambda}\right)}{\left(e^{\lambda} - e^{-\lambda}\right)}; \qquad \lambda = (1-i)\sqrt{\frac{\Omega}{2}}.$$

To keep the time variation slow, we have rescaled the time t by using the time scale $\tau = \epsilon^2 t$.

Also the dimensionless form of the concentration gradient obtained from the Eq. 13 has been used to obtain the Eq. 27. Now, to study the stationary double-diffusive convection, we write the nonlinear Eqs. 25–27 in the matrix form as given below:

$$\begin{bmatrix} \frac{\epsilon^2}{Pr}\frac{\partial}{\partial\tau}\nabla^2 + \frac{1}{Da}\nabla^2 - \nabla^4 & Ra_T\frac{\partial}{\partial x} & -Ra_S\frac{\partial}{\partial x} \\ \left(1 - F'\delta_1\epsilon^2\right)\frac{\partial}{\partial x} & \epsilon^2\frac{\partial}{\partial\tau} - \nabla^2 & 0 \\ \frac{\partial}{\partial x} & 0 & \epsilon^2\frac{\partial}{\partial\tau} - \frac{1}{Le}\nabla^2 \end{bmatrix}\begin{bmatrix}\Psi \\ \Theta \\ \Phi\end{bmatrix} = \begin{bmatrix} \frac{1}{Pr}\frac{\partial\left(\Psi,\nabla^2\Psi\right)}{\partial(x,z)} \\ \frac{\partial(\Psi,\Theta)}{\partial(x,z)} \\ \frac{\partial(\Psi,\Phi)}{\partial(x,z)} \end{bmatrix}. \tag{30}$$

The boundary condition to solve Eqs. 25–27 are

$$\begin{aligned} \Psi &= 0 = \nabla^2\Psi, \quad \Theta = \Phi = 1 \quad \text{on z=0}, \\ \Psi &= 0 = \nabla^2\Psi, \quad \Theta = \Phi = 0 \quad \text{on z=1}. \end{aligned} \tag{31}$$

4 Weakly Nonlinear Stability Analysis and Heat Transport

We now introduce the following asymptotic expansion in Eq. 30:

$$Ra_T = R_{0c} + \epsilon^2 R_2 + \epsilon^4 R_4 + \ldots, \tag{32}$$

$$\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + \epsilon^3\Psi_3 + \ldots, \tag{33}$$

$$\Theta = \epsilon\Theta_1 + \epsilon^2\Theta_2 + \epsilon^3\Theta_3 + \ldots, \tag{34}$$

$$\Phi = \epsilon\Phi_1 + \epsilon^2\Phi_2 + \epsilon^3\Phi_3 + \ldots, \tag{35}$$

where R_{0c} is the critical value of the Rayleigh number at which the onset of convection takes place in the absence of temperature modulation. Substitute Eqs. 32–35 in Eqs. 30–31, we get the following system at the lowest order:

$$\begin{bmatrix} \frac{1}{Da}\nabla^2 - \nabla^4 & R_0\frac{\partial}{\partial x} & -R_S\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\nabla^2 & 0 \\ \frac{\partial}{\partial x} & 0 & -\frac{1}{Le}\nabla^2 \end{bmatrix}\begin{bmatrix}\Psi_1 \\ \Theta_1 \\ \Phi_1\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ 0\end{bmatrix}. \tag{36}$$

The solution of the lowest order system subject to the boundary conditions (31) is

$$\begin{aligned} \Psi_1 &= A\sin(k_c x)\sin(\pi z), \\ \Theta_1 &= -\frac{k_c}{\delta^2}A\cos(k_c x)\sin(\pi z), \\ \Phi_1 &= -\frac{k_c Le}{\delta^2}A\cos(k_c x)\sin(\pi z), \end{aligned} \tag{37}$$

where $\delta^2 = k_c^2 + \pi^2$. The critical value of the Rayleigh number and the corresponding wave number for the onset of stationary convection are as given below:

$$R_{0c} = \frac{\left(\mathrm{Da}^{-1} + k_c^2 + \pi^2\right)\left(k_c^2 + \pi^2\right)^2 + \mathrm{Le} Ra_S k_c^2}{k_c^2}, \tag{38}$$

$$k_c = \left[\sqrt{-\frac{1}{4\mathrm{Da}} - \frac{\pi^2}{4} + \frac{\sqrt{1 + 10\mathrm{Da}\pi^2 + 9\mathrm{Da}^2\pi^4}}{4\mathrm{Da}}}\right]. \tag{39}$$

At the second order, we have

$$\begin{bmatrix} \frac{1}{Da}\nabla^2 - \nabla^4 & R_0\frac{\partial}{\partial x} & -Ra_S\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\nabla^2 & 0 \\ \frac{\partial}{\partial x} & 0 & -\frac{1}{Le}\nabla^2 \end{bmatrix} \begin{bmatrix} \Psi_2 \\ \Theta_2 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \\ R_{23} \end{bmatrix}, \tag{40}$$

where

$$R_{21} = 0, \tag{41}$$

$$R_{22} = \frac{\partial \Psi_1}{\partial x}\frac{\partial \Theta_1}{\partial z} - \frac{\partial \Psi_1}{\partial z}\frac{\partial \Theta_1}{\partial x}, \tag{42}$$

$$R_{23} = \frac{\partial \Psi_1}{\partial x}\frac{\partial \Phi_1}{\partial z} - \frac{\partial \Psi_1}{\partial z}\frac{\partial \Phi_1}{\partial x}. \tag{43}$$

The second-order solution, subject to the boundary condition (31), can be obtained as follows:

$$\Psi_2 = 0, \tag{44a}$$

$$\Theta_2 = -\frac{k_c^2 A^2}{8\pi\delta^2}\sin(2\pi z), \tag{44b}$$

$$\Phi_2 = -\frac{k_c^2 Le^2 A^2}{8\pi\delta^2}\sin(2\pi z). \tag{44c}$$

The horizontally averaged Nusselt number, Nu, and Sherwood number, Sh, for the stationary double-diffusive convection(the mode considered in this problem) are given by

$$Nu(\tau) = \frac{\left[\frac{k_c}{2\pi}\int_0^{\frac{2\pi}{k_c}} (1 - z + \Theta_2)\, dx\right]_{z=0}}{\left[\frac{k_c}{2\pi}\int_0^{\frac{2\pi}{k_c}} (1 - z)\, dx\right]_{z=0}}, \tag{45}$$

$$Sh(\tau) = \frac{\left[\frac{k_c}{2\pi}\int_0^{\frac{2\pi}{k_c}} (1 - z + \Phi_2)\, dx\right]_{z=0}}{\left[\frac{k_c}{2\pi}\int_0^{\frac{2\pi}{k_c}} (1 - z)\, dx\right]_{z=0}}. \tag{46}$$

One must note here that $F(z,t)$ is effective at $O(\epsilon^2)$ and affects $Nu(\tau)$ and $Sh(\tau)$ through $A(\tau)$ as shown later. Substituting Eqs. 44a and b in Eqs. 45 and 46 and simplifying, we get

$$Nu(\tau) = 1 + \frac{k_c^2}{4\delta^2}[A(\tau)]^2, \tag{47}$$

$$Sh(\tau) = 1 + \frac{k_c^2 Le^2}{4\delta^2}[A(\tau)]^2. \tag{48}$$

At the third order, we have

$$\begin{bmatrix} \frac{1}{Da}\nabla^2 - \nabla^4 & R_0\frac{\partial}{\partial x} & -Ra_S\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\nabla^2 & 0 \\ \frac{\partial}{\partial x} & 0 & -\frac{1}{Le}\nabla^2 \end{bmatrix}\begin{bmatrix} \Psi_3 \\ \Theta_3 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \\ R_{33} \end{bmatrix}, \tag{49}$$

$$R_{31} = -\frac{1}{Pr}\frac{\partial}{\partial \tau}(\nabla^2\Psi_1) - R\frac{\partial\Theta_1}{\partial x}, \tag{50}$$

$$R_{32} = \frac{\partial\Psi_1}{\partial x}\frac{\partial\Theta_2}{\partial z} + \delta_2 F(z,t)\frac{\partial\Psi_1}{\partial x} - \frac{\partial\Theta_1}{\partial\tau}, \tag{51}$$

$$R_{33} = \frac{\partial\Psi_1}{\partial x}\frac{\partial\Phi_2}{\partial z} - \frac{\partial\Phi_1}{\partial\tau}. \tag{52}$$

Substituting Ψ_1, Θ_1, Φ_1, Θ_2, and Φ_2 from Eqs. 38 and 44 into Eqs. (50–52), we get

$$R_{31} = \left[\frac{\delta^2}{Pr}\frac{\partial A(\tau)}{\partial\tau} - \frac{Rk_c^2}{\delta^2}A(\tau)\right]\sin(k_c x)\sin(\pi z), \tag{53}$$

$$\begin{aligned} R_{32} = &-\frac{k_c^3 A^3(\tau)}{4\delta^2}\cos(k_c x)\sin(\pi z)\cos(2\pi z) + \delta_2 F(z,t)\,k_c(x)\,A(\tau)\cos(k_c x)\sin(\pi z) \\ &+ \frac{k_c}{\delta^2}\frac{dA(\tau)}{d\tau}\cos(k_c x)\sin(\pi z), \end{aligned} \tag{54}$$

$$R_{33} = -\frac{k_c^3 Le^2 A^3(\tau)}{4\delta^2}\cos(k_c x)\sin(\pi z)\cos(2\pi z) + \frac{k_c Le}{\delta^2}\frac{dA(\tau)}{d\tau}\cos(k_c x)\sin(\pi z). \tag{55}$$

Now applying the solvability condition for the existence of the third order solution, we get the Ginzburg–Landau equation in the form:

$$\left[\frac{\delta^2}{Pr} + \frac{R_0 k_c^2}{\delta^4} - \frac{R_S k_c^2 Le^2}{\delta^4}\right]\frac{dA(\tau)}{d\tau} = \frac{k_c^2}{\delta^2}[R_2 - 2R_0\delta_2 I]A(\tau) - \frac{k_c^4}{8\delta^4}\left[R_0 - R_S Le^3\right]A^3(\tau). \tag{56}$$

where $I = \int_{z=0}^{1} F(z,\tau)\sin^2(\pi z)\,dz$.
The solution of Eq. (56), subject to the initial condition $A(0) = a_0$, where a_0 is a chosen initial amplitude of convection, can be obtained by using any Runge-Kutta method. In calculations we may assume $R_2 = R_0$, to keep the parameters to the minimum.

We now move on to consider the effect of gravity modulation on double-diffusive convection in the absence of temperature modulation.

5 Mathematical Formulation for the Gravity Modulation Problem

When the physical configuration of the double-diffusive convection problem in porous media is vibrated time-periodically in the z-direction, then the effective gravity takes the form:

$$\overrightarrow{g}\,(t) = g_0\left[1+\epsilon^2\delta_2\cos\left(\omega t\right)\right]\hat{k}, \tag{57}$$

where g_0 is the mean gravity, δ_2 is the small amplitude of gravity modulation, ω is the frequency, and t is the time. The physical configuration is given in the Fig. 1b.

The governing equations for the problem are same as Eqs. 1–5 but with $\overrightarrow{g}$ given by Eq. 57.

Since the basic temperature in the present case is considered to be steady, it is given by

$$T_b = 1 - z, \tag{58}$$

which satisfies the following equations:

$$\frac{d^2T_b}{dz^2} = 0. \tag{59}$$

Following the analysis as in the previous section, the non-dimensional form of the perturbation equations, on using Eq. (57), take the form:

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t}\nabla^2 + \frac{1}{Da}\nabla^2 - \nabla^4\right)\Psi + Ra_T(1+g_m)\frac{\partial\Theta}{\partial x} - Ra_S(1+g_m)\frac{\partial\Phi}{\partial x} = \frac{1}{Pr}\frac{\partial(\Psi,\nabla^2\Psi)}{\partial(x,z)}, \tag{60}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\Theta + \frac{\partial\Psi}{\partial x} = \frac{\partial\left(\Psi,\Theta\right)}{\partial\left(x,z\right)}, \tag{61}$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{Le}\nabla^2\right)\Phi + \frac{\partial\Psi}{\partial x} = \frac{\partial\left(\Psi,\Phi\right)}{\partial\left(x,z\right)}, \tag{62}$$

where $g_m = \epsilon^2\delta_2 cos(\omega t)$ and the parameters are as defined in the previous two sections. We use the time variation only at slow time scale $\tau = \epsilon^2 t$, and thus $g_m(\tau)$ is taken to be

$$g_m\left(\tau\right) = \epsilon^2\delta_2 Cos\left(\Omega\tau\right), \quad \text{where} \quad \Omega = \frac{\omega}{\epsilon^2}. \tag{63}$$

Now following the analysis of the previous section, we get the Ginzburg–Landau equation in the form

$$\left[\frac{\delta^2}{Pr} + R_0\frac{k_c^2}{\delta^4} - \frac{k_c^2}{\delta^4}Le^2 Ra_S\right]\frac{dA}{d\tau} - \left[R_0\left(\frac{R_2}{R_0} + \delta_2\cos(\Omega\tau)\right)\frac{k_c^2}{\delta^2} - \frac{k_c^2}{\delta^2}Le Ra_S\delta_2\cos(\Omega\tau)\right]A$$
$$+\frac{k_c^2}{8\delta^2}[R_0 - Ra_S Le^3]A^3 = 0 \tag{64}$$

The solution of Eq. 64 is obtained in the same way as in the previous section.

6 Results and Discussion

Before embarking on the discussion of the results, we make some comments on the following aspects of the problem:

1. The need for nonlinear stability analysis,

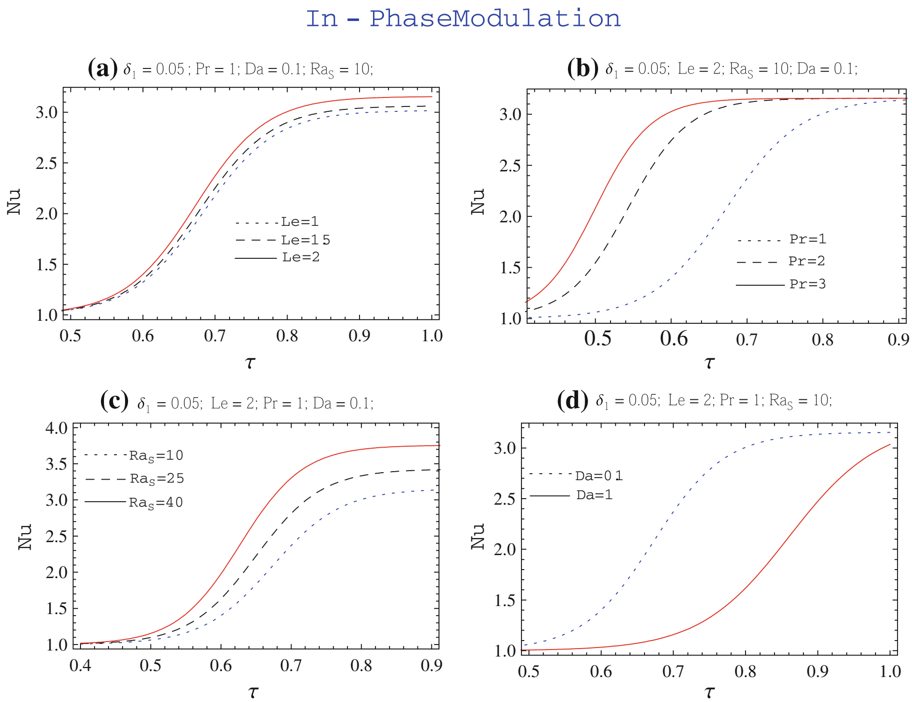


Fig. 2 In-phase modulation: Nu versus τ for different values of **a** Le, **b** Pr, **c** Ra_s, **d** Da and **e** δ_1

2. The relation of the problem to a real application, and
3. The selection of all dimensionless parameters utilized in computations.

As mentioned in the last paragraph of the introduction, it is imperative that a nonlinear study is made if one wants to quantify heat and mass transports which the linear stability theory is unable to do so.

External regulation of convection is important in the study of double-diffusive convection in porous media. The objective of this article is to consider two such candidates, namely temperature/gravity modulations for either enhancing or inhibiting convective heat transport as is required by a real application.

The parameters that arise in the problem are Pr, Le, Ra_S, Da, ϕ, δ_1, and δ_2, and these influence the convective heat and mass transports. The first four parameters relate to the fluid and the structure of the porous medium, and the last three concern the two external mechanisms of controlling convection. The medium has high porosity, and hence the Brinkman Lapwood model is considered for the conservation of the linear momentum. Further, owing to high porosity the two viscosities with the Darcy and Brinkman terms are taken to be the same. The values thus considered for Pr and Le are the same as that usually considered for a clear two component fluid. Positive values of Ra_S are considered, and in such a case, one gets positive values of Ra_T, and these signify the assumption of a situation in which we have cool fresh water overlying warm salty water. In the absence of cross-diffusion, this situation is conducive for the appearance of salt-fingers, which arises in a stationary regime of onset of convection. Owing to the assumption of high porosity medium, the values considered for Da are 0.01 and 0.1. Because the small amplitude modulations are considered, the values of

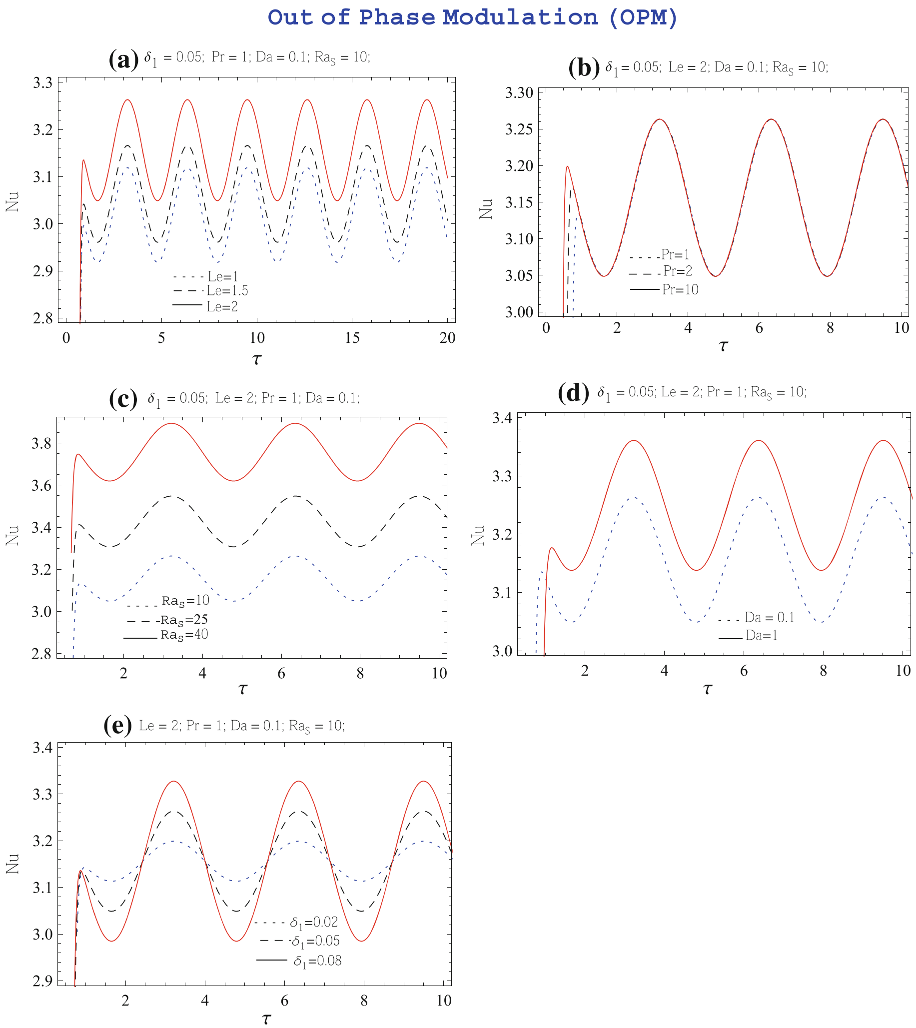


Fig. 3 Out-phase modulation (OPM): Nu versus τ for different values of **a** Le, **b** Pr, **c** Ra_s, **d** Da and **e** δ_1

δ_1 and δ_2 lie between 0 and 0.5. Further, the modulation of the boundary temperature and the vertical time-periodic fluctuation of gravity are assumed to be of low frequency. At low range of frequencies, the effect of frequencies on onset of convection as well as on heat transport is minimal. This assumption is required to ensure that the system does not pick up oscillatory convective mode at onset due to modulation in a situation that is conductive otherwise to stationary mode. It is important at this stage to consider the effect of Le, Ra_S, Da, and δ_1 on the onset of convection. This has been reported by many investigators earlier who found that

1. $[Ra_{Tc}]_{Le=0} < [Ra_{Tc}]_{Le\neq 0}$,
2. $[Ra_{Tc}]_{Ra_S=0} < [Ra_{Tc}]_{Ra_S\neq 0}$,
3. $[Ra_{Tc}]_{Da=\infty} < [Ra_{Tc}]_{Da<<1}$, and
4. $[Ra_{Tc}]_{\delta_i=0.02} > [Ra_{Tc}]_{\delta_i=0.05} > [Ra_{Tc}]_{\delta_i=0.08}$, $(i = 1, 2)$,

for both temperature and gravity modulations.

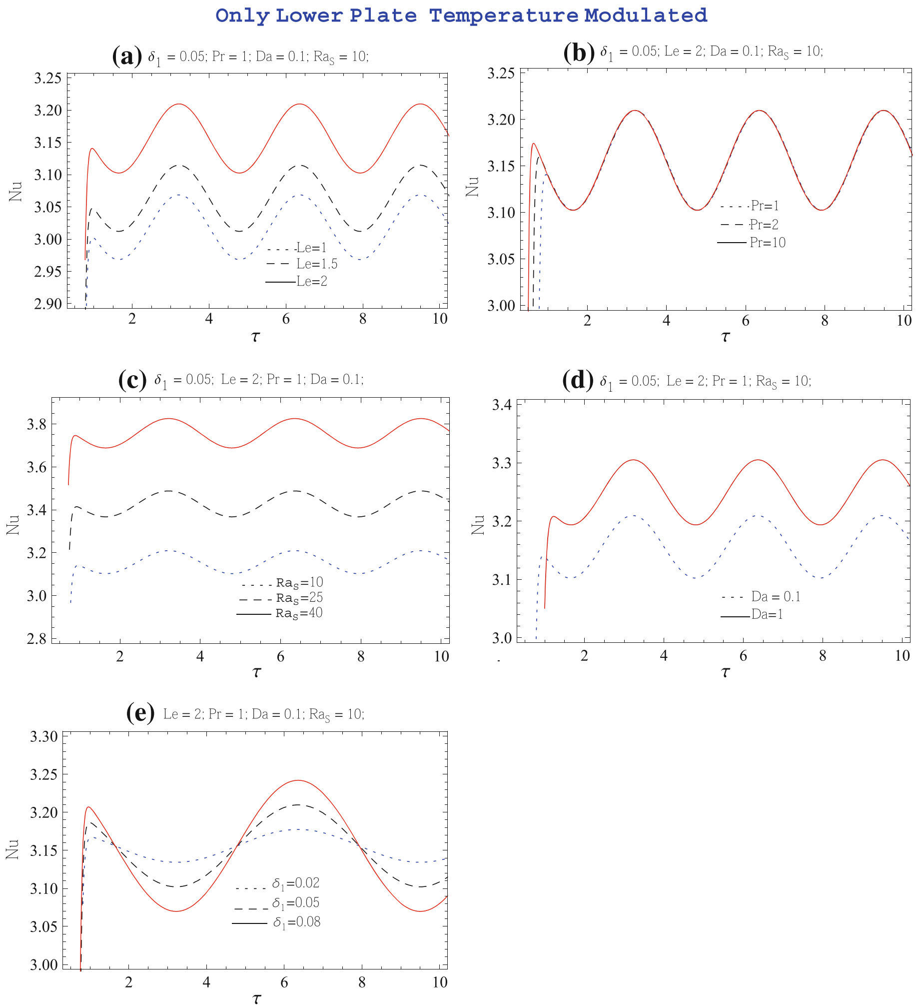


Fig. 4 Only lower plate temperature modulated: Nu versus τ for different values of **a** Le, **b** Pr, **c** Ra_s, **d** Da and **e** δ_1

For the considered convective mode, Pr has no say on the onset of convection, and Le has to be greater than unity for stationary onset. The above observations serve as a guideline for the computations carried out in this article.

We have considered two types of modulation, as follows:

1. Temperature modulation and
2. Gravity modulation

We consider individual effects of these on double-diffusive convection in a porous medium that arises when heat and salt make opposing contributions and for $\kappa_T \neq \kappa_S$. Direct mode is preferred in unmodulated case when $\frac{\kappa_S}{\kappa_T} < 1$, and Hopf mode other wise. We concentrate on the modulated problem for only the direct mode. The focus in this article is essentially on the effect of modulation on heat and mass transports. In both the modulated problems considered, the Ginzburg–Landau equation is non-autonomous. We first discuss the results

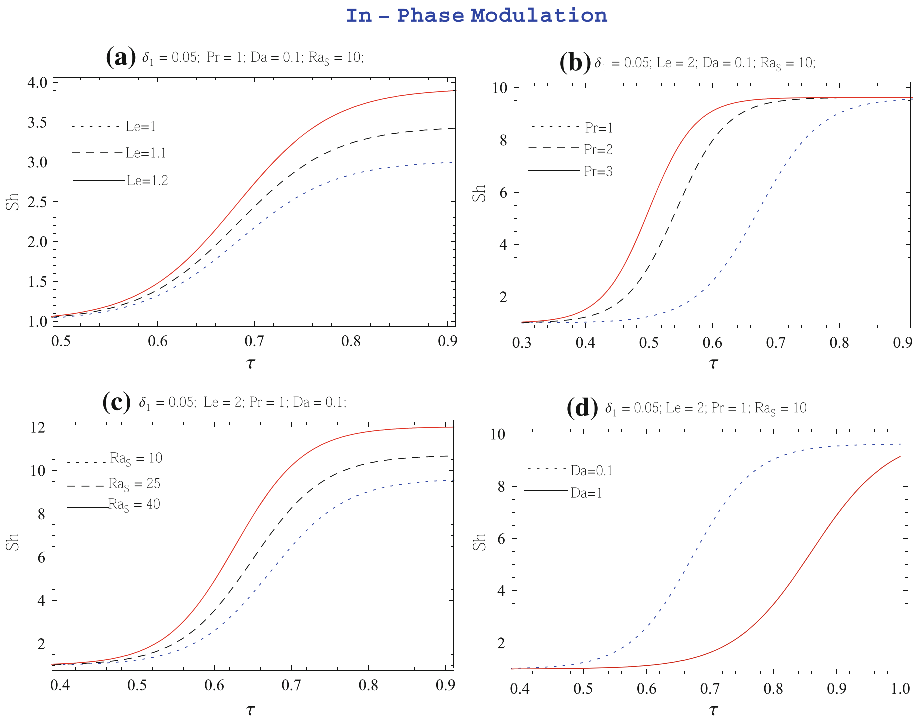


Fig. 5 In-phase modulation: Sh versus τ for different values of **a** Le, **b** Pr, **c** Ra_S, **d** Da and **e** δ_1

on temperature modulation and then that on gravity modulation.
We consider the following three types of temperature modulation:

1. In-phase modulation (IPM) ($\phi = 0$),
2. Out-of-phase modulation (OPM) ($\phi = \pi$), and
3. Modulation of only the lower boundary (MOLB) ($\phi = -i\infty$).

The effect of each type of modulation on heat and mass transports is shown in Figs. 2,3,4, 5,6, and 7. Figure 2a–d concerning IPM shows that Nu increases with individual and collective increases in Lewis number Le, Prandtl number Pr, solutal Rayleigh number Ra_S, and Darcy number Da. The Nu versus τ curves start with $Nu = 1$, signifying the initial conduction state. As time progresses, the value of Nu increases, thus showing that convective regime is in place and then finally the curves of Nu level off at long times. This result is seen since the amplitude of temperature modulation is quite small. The above patterns in the variation of Pr, Le, and Ra_S are also seen in the case of OPM (see Fig. 3a–e and MOLB (see Fig. 4a–e while the patterns of Nu with the variation of Da is opposite(see Fig. 3d and 4d. In the OPM and MOLB cases, however, the Nu versus τ curves are oscillatory. From these figures, we find the following general result:

$$Nu^{IPM} < Nu^{MOLP} < Nu^{OPM}$$

Figure 3e shows the effect of amplitude of temperature modulation on Nu in the case of OPM. It is obvious from the figure that

$$Nu/_{\delta_1=0.02} < Nu/_{\delta_1=0.05} < Nu/_{\delta_1=0.08}.$$

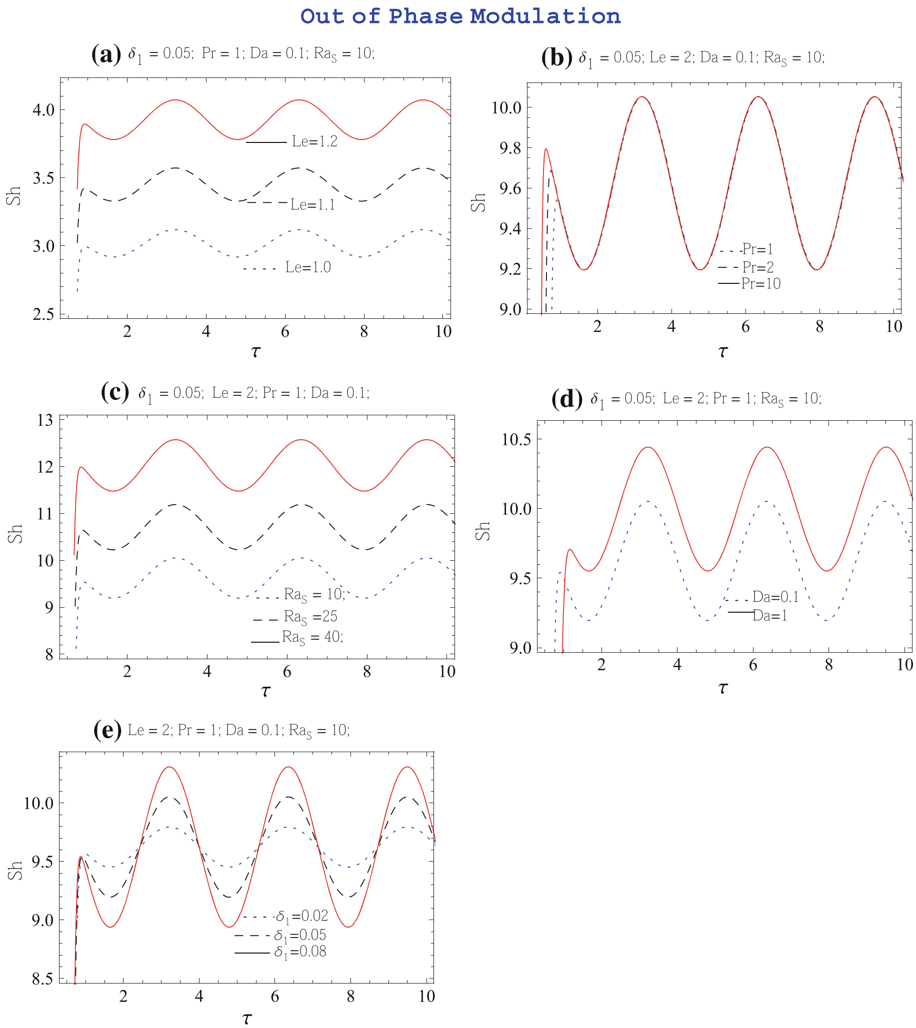


Fig. 6 Out of phase modulation: Sh versus τ for different values of **a** Le, **b** Pr, **c** Ra_s, **d** Da and **e** δ_1

Further from Fig. 4e, we find a similar effect in the case of MOLP also. The various parameters' influence on Nusselt number seem fine when seen together with the results on their influence on Ra_{Tc} discussed earlier.

Figures 5a–d,6a–e, and 7a–e depict the fact that Sherwood number variations with Pr, Le, Ra_S, and Darcy number Da are similar to what was seen with Nu in Figs. 2a–d, 3a–e, and 4a–e. The results on Sh in Figs. 6e and 7e result in some general results on Sh, similar to those on Nu. They are

$$Sh^{IPM} < Sh^{MOLP} < Sh^{OPM},$$
$$Sh/_{\delta_1=0.02} < Sh/_{\delta_1=0.005} < Sh/_{\delta_1=0.08}.$$

When seen in conjunction with the results on Ra_{Tc}, the above observation reiterates such a finding.

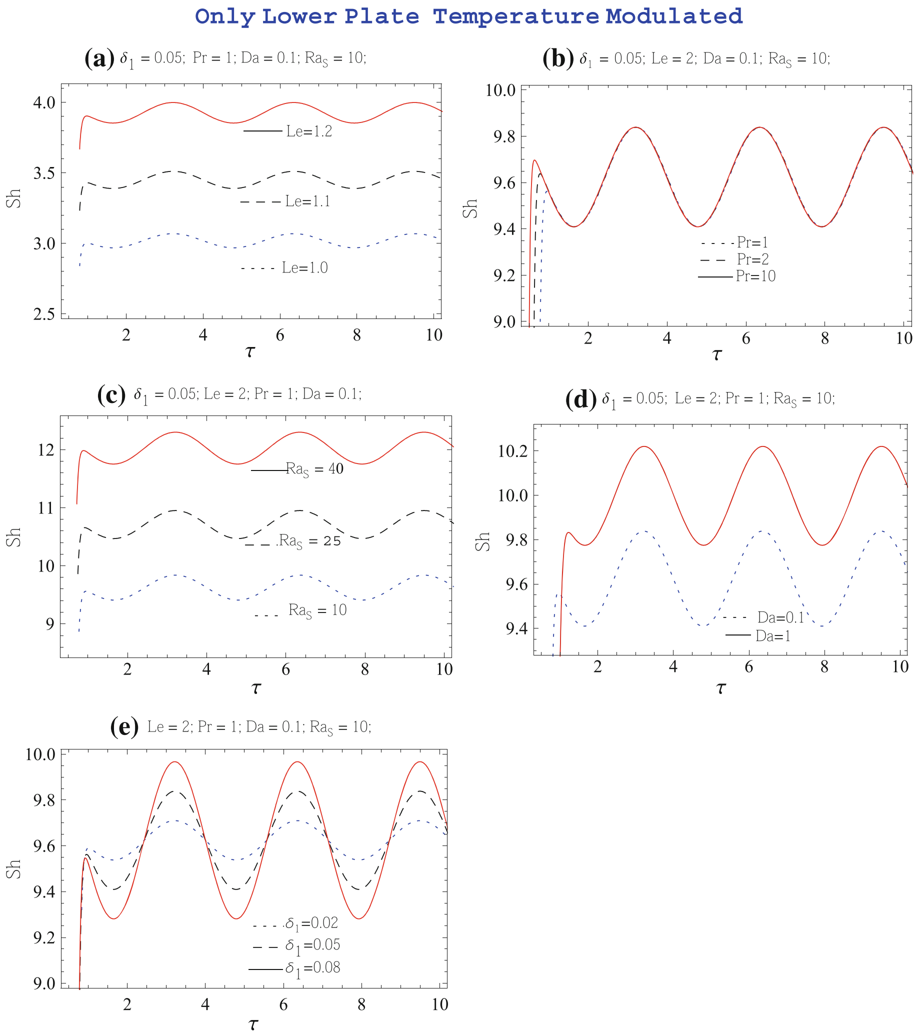


Fig. 7 Only lower plate temperature modulated: Sh versus τ for different values of **a** Le, **b** Pr, **c** Ra_s, **d** Da and **e** δ_1

Now we discuss the results corresponding to gravity modulation. Figs. 8a–e and 9a–e reveal that the variations of Nu and Sh with Pr, Le, Ra_S, and Da are similar to that of temperature modulation. Further, we find that the effect of Pr on Nu and Sh is only felt at short times. The effect of amplitude of gravity modulation δ_2 on Nu and Sh is similar to that of temperature modulation.

7 Conclusions

The effects of temperature and gravity modulations on weak nonlinear double-diffusive convection in a viscous liquid-saturated porous medium are studied using Ginzburg–Landau

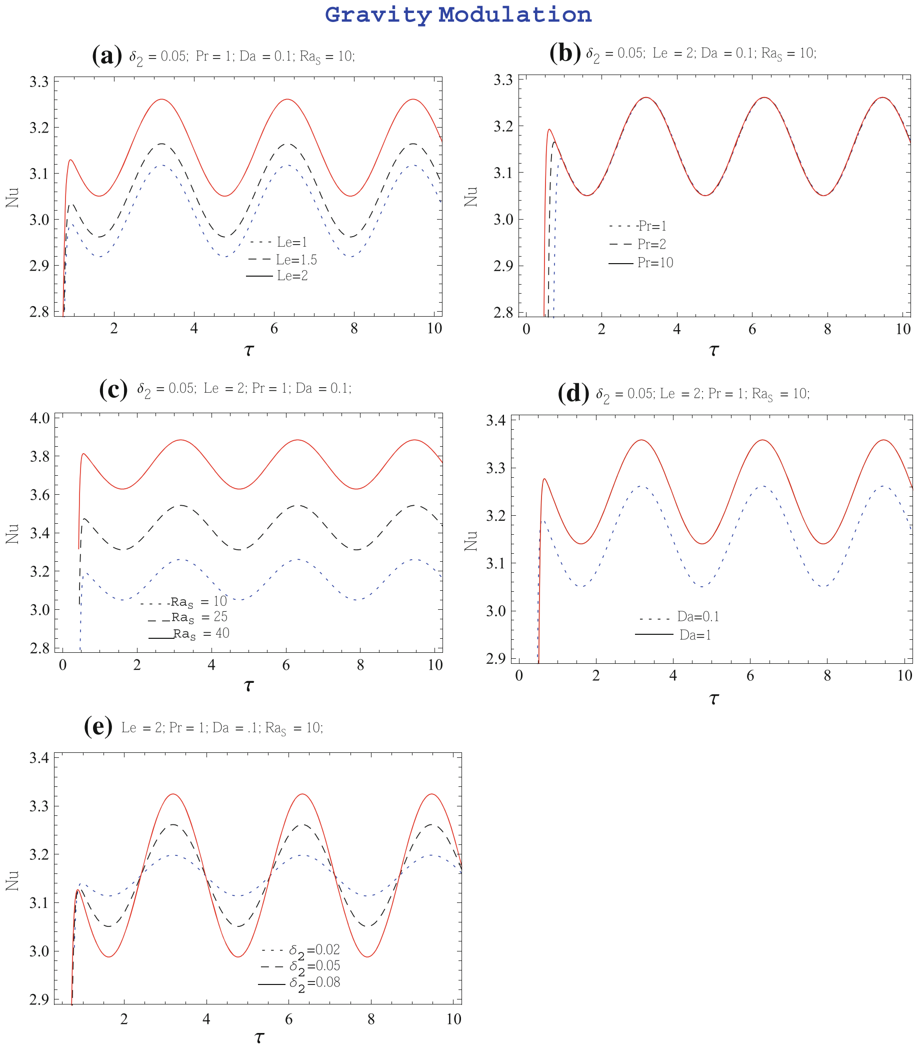


Fig. 8 Gravity modulation: Nu versus τ for different values of **a** Le, **b** Pr, **c** Ra_s, **d** Da and **e** δ_2

equation. Onset criteria for double-diffusive convection in both the cases are derived analytically. The following conclusions are drawn:

1. Effects of increasing δ_1, δ_2, Ra_S, Le, Pr and Da are found to increase Nu and Sh, thus increasing heat and mass transfers in all the three cases.
2. Effect of increasing Da is to decrease the value of Nu for IPM while Nu increases on increasing Da in the other two cases of temperature modulation.
3. In the case of IPM, the values of Nu and Sh increase steadily for intermediate value of time t; however, they become constant when t is large.
4. In the cases of OPM and MOLP, the nature of Nu and Sh remain oscillatory.

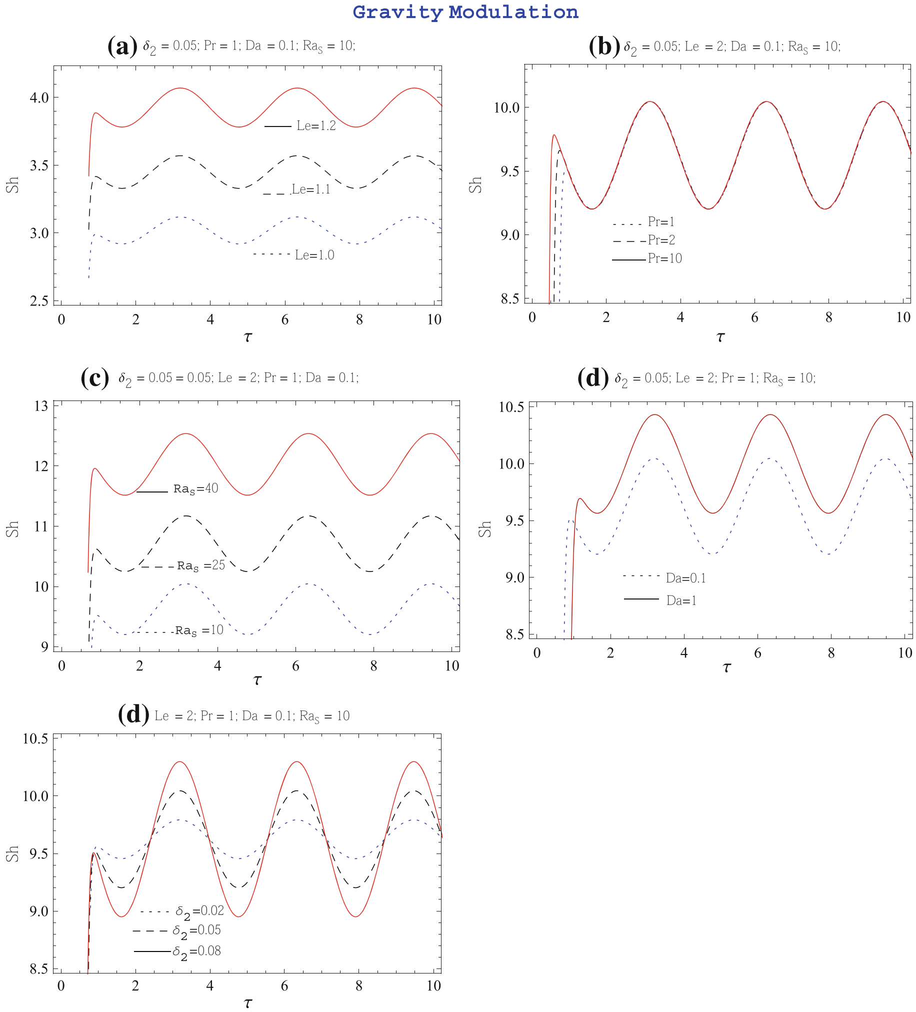


Fig. 9 Gravity modulation: Sh versus τ for different values of **a** Le, **b** Pr, **c** Ra_s, **d** Da and **e** δ_2

5. The values of Nu and Sh for MOLP are greater than those in IPM but smaller than those in OPM. Thus, OPM can be used for enhanced heat transport and IPM for inhibiting heat transport.
6. The effect of gravity modulation is similar to that of temperature modulation found in OPM and MOLP cases.

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