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Magnetohydrodynamic Rotating Flow of a Generalized Burgers' Fluid in a Porous Medium with Hall Current

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Abstract This study concentrates on the unsteady magnetohydrodynamics (MHD) rotating flow of an incompressible generalized Burgers's fluid past a suddenly moved plate through a porous medium. Modified Darcy's law for generalized Burgers's fluid in a rotating frame has been used to model the governing flow problem. The closed form solution of the governing flow problem has been obtained by employing Laplace transform technique. The integral appearing in the inverse Laplace transform has been evaluated numerically. The influence of various parameters on the velocity profile has been delineated through several graphs and discussed in detail. It was found that the fluid is decelerated with increasing Hartmann number M and porosity parameter K. However, for large Hall parameter m, the real part of velocity decreases and the imaginary part of velocity increases.

Keywords Laplace transform · Generalized Burgers' fluid · Rotating MHD flow · Porous medium · Hall current

1 Introduction

The flow of non-Newtonian fluids through porous medium have been of considerable interest in the last few decades. This is mainly due to their several technological, industrial, geophysical and astrophysical applications such as geothermal energy extrusion, oil recovery, food processing, ground water flow, irrigation problems and the biophysical sciences where the human lungs for example are modeled as a porous medium (Reis et al. 2004; Kuwahara et al. 2009). Moreover, the flows of non-Newtonian fluids in the presence of a magnetic field

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have numerous applications in many areas such as the handling of biological fluids, liquid metals and alloys, plasma, mercury amalgams, blood, and electromagnetic propulsion. Many non-Newtonian fluid models are proposed in the literature. One of them which is the subclass of viscoelastic fluid called Burgers' fluid model (1935). This model has been used successfully to characterize diverse viscoelastic material for example food products such as cheese, soil etc. In geomechanics, Burger's fluid is introduced to study asphalt and asphalt mixes (Majidzadeh and Schweyer 1967; Gerritsen et al. 1978). Burgers' fluid model is also used in modeling of high temperature viscoelasticity of fine-grained polycrystalline olivine (Chopra 1997; Tan et al. 2001); in calculating the transient creep properties of the earth's mantle (Tromp and Mitrovica 1999), specifically related to the post-glacial uplift (Peltier et al. 1981; Yuen and Peltier 1982; Muller 1986; Rumpker and Wolf 1996; Jackson 2000); in the propagation of seismic waves in the interior of the earth (Anand et al. 2006) and for describing the mechanics of a coarse ligated plasma clot (Krishnan and Rajagopal 2003).

Few recent attempts considering the rotating flows of non-Newtonian fluids with magnetohydrodynamic and porosity effects are given in references Fakhar et al. (2006), Hayat et al. (2008a,b,c), Tiwari and Ravi (2009), and Abelman et al. (2009). These attempts further reduced when the effects of Hall current are considered Hayat et al. (2007a, 2008d,e). Asghar et al. (2005) studied the MHD transient flows of an Oldroyd-B fluid. Hayat et al. (2007b) extended their analysis from Oldroyd-B fluid to Burgers' fluid with consideration of porous medium and Hall current. In Hayat et al. (in press), studied the unsteady flow of a generalized Burgers' fluid and obtained the closed form solution using Laplace transform technique.

The objective of this work is to extend the analysis of Hayat et al. (in press) in three directions: (i) to study the MHD flow of a generalized Burgers' fluid, (ii) to include Hall current and (iii) to consider the flow in a porous medium using modified Darcy's law. The paper is structured as follows: Mathematical formulation is given in Sect. 2. Section 3 comprises solution expression in the transformed plane. Section 4 is reserved for results and discussion through several graphs to study the influence of pertinent parameters having effective control on the fluid motion. The paper ends with concluding remarks in Sect. 5.

2 Governing Equations

The unsteady flow of an incompressible generalized Burgers' fluid has been considered in this problem. The fluid fills the porous half space $z > 0$, over a rigid plate occupying xz-plane. The z-axis has been taken normal to the plate. Initially, we have assume that both the fluid and plate are at rest. After time $t = 0^+$, the fluid and the plate start rigid body rotation with the same angular velocity $\Omega = \Omega\hat{\mathbf{k}}$ ($\hat{\mathbf{k}}$ is a unit vector parallel to z-axis). A uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{k}}$ has been applied to the fluid. In addition, Hall current is taken into account. The electric field due to polarization of charges is taken zero (see Refs. Hayat et al. 2007b, 2008d,e; Asghar et al. 2005). The magnetic Reynolds number, i.e., the ratio of inertia forces to magnetic forces is taken small to neglect the effect of induced magnetic field. The flow of the fluid is created by sudden motion of the plate. The geometry of the problem is show in Fig. 1.

The governing equations for unsteady MHD flow with Hall current through a porous medium in a rotating frame are given as

$$\operatorname{div} \mathbf{V} = 0, \tag{1}$$

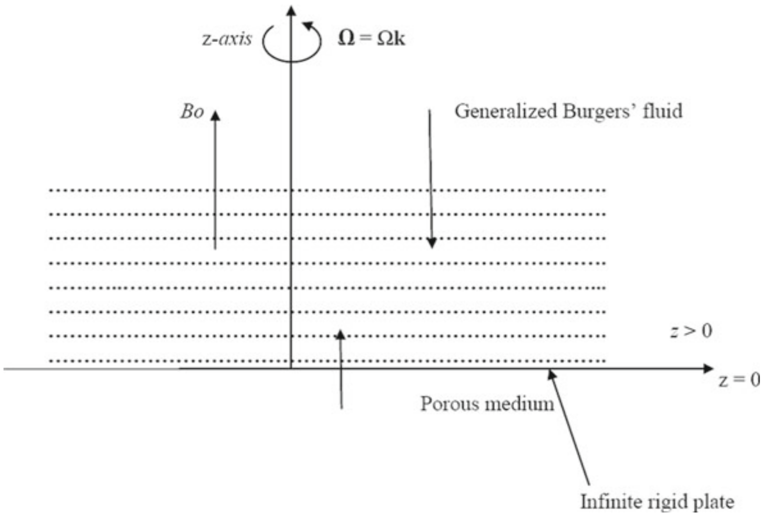

Fig. 1 Physical model and coordinate system

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}.\nabla)\,\mathbf{V}+\mathbf{2}\Omega\times\mathbf{V}+\Omega\times(\Omega\times\mathbf{r})\right)=\operatorname{div}\mathbf{T}+\mathbf{J}\times\mathbf{B}+\mathbf{R}, \tag{2}$$

where $\mathbf{V}=(u,v,w)$ is the velocity field, ρ is the fluid density, ∇ is the gradient operator, $\mathbf{R}$ is the Darcy's resistance, $\mathbf{r}$ is a radial vector with $r^2=x^2+y^2$. The Cauchy stress tensor $\mathbf{T}$ for an incompressible generalized Burgers' fluid (Krishnan and Rajagopal 2003) is given as

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S}, \tag{3}$$

where $\mathbf{I}$ is an identity tensor, and $\mathbf{S}$ is the extra stress tensor satisfying the following relation

$$\left(1+\lambda_1\frac{\delta}{\delta t}+\lambda_2\frac{\delta^2}{\delta t^2}\right)\mathbf{S}=\mu\left(1+\lambda_3\frac{\delta}{\delta t}+\lambda_4\frac{\delta^2}{\delta t^2}\right)\mathbf{A}_1, \tag{4}$$

in which μ is the dynamic viscosity, $\mathbf{A}_1$ is the first Rivilin and Ericken tensor given as

$$\mathbf{A}_1=\mathbf{L}+\mathbf{L}^T, \tag{5}$$

where $\mathbf{L}=\operatorname{grad}\mathbf{V}$, and T indicates the matrix transpose. Here, λ_i $(i=1,3)$ are the relaxation and retardation times, respectively, and λ_i $(i=2,4)$ are the material constants having the dimensions as the square of time multiplying the upper convected second-order derivative

$$\frac{\delta^2\mathbf{S}}{\delta t^2}=\frac{\delta}{\delta t}\left(\frac{\delta\mathbf{S}}{\delta t}\right)=\frac{\delta}{\delta t}\left(\frac{\delta\mathbf{S}}{\delta t}+(\mathbf{V}\cdot\nabla)\,\mathbf{S}-\mathbf{LS}-\mathbf{SL}^T\right). \tag{6}$$

The velocity field for the present flow is defined as

$$\mathbf{V}=(u\,(z,t)\,,v\,(z,t)\,,w\,(z,t))\,, \tag{7}$$

where t is the time and u, v, and w are the velocity components in x, y and z-directions, respectively. The expression of $\mathbf{R}$ for generalized Burgers' fluid is given by

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\mathbf{R}=-\frac{\mu\phi}{k}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\mathbf{V}, \tag{8}$$

in which μ is the dynamic viscosityϕ $(0 < \phi < 1)$ is the porosity and $k > 0$ is the permeability of the porous medium.

In the presence of Hall current and in view of Maxwell's equations

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{B} = \mu_{\mathrm{m}} \mathbf{J}, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{9}$$

the usual Ohm's law modifies to

$$\mathbf{J} + \frac{\omega_{\mathrm{e}} \tau_{\mathrm{e}}}{\mathbf{B}} (\mathbf{J} \times \mathbf{B}) = \sigma [\mathbf{E} + \mathbf{V} \times \mathbf{B} + \frac{1}{e n_{\mathrm{e}}} \nabla p_{\mathrm{e}}], \tag{10}$$

called the generalized Ohm's law. Here **B** is the total magnetic field, **J** is the current density, **E** is the total electric field, μ_{m} is the magnetic permeability, ω_{e} is the cyclotron frequency of electrons, τ_{e} is the electron collision time, e is the electron charge and p_{e} is the electron pressure and n_{e} is the number density of electrons, σ is the finite electrical conductivity of the fluid. It should be noted that $\omega_{\mathrm{e}} \tau_{\mathrm{e}} \sim o(1)$, $\omega_{\mathrm{i}} \tau_{\mathrm{i}} \ll 1$($\omega_{\mathrm{i}}$ and τ_{i} are respectively cyclotron frequency and collision time for ions) and ion-slip and thermoelectric effects are not included in the present case.

Introducing the following dimensionless quantities

$$\bar{F} = \frac{F}{U_0} = \frac{u + iv}{U_0}, \bar{Z} = \frac{Z U_0}{\nu}, \bar{t} = \frac{t U_0^2}{\nu}, \bar{\Omega} = \frac{\Omega \nu}{U_0^2}, M^2 = \frac{\delta B_0^2 \nu}{\rho U_0^2}$$
$$\frac{1}{K} = \frac{\phi \nu^2}{k U_0^2}, \bar{\lambda}_i = \frac{\lambda_i U_0^2}{\nu} \,(i = 1, 3)\,, \bar{\lambda}_i = \frac{\lambda_i U_0^4}{\nu^2} \,(i = 2, 4)\,, \tag{11}$$

and using Eq. 7, the continuity equation is automatically satisfied and in view of Eqs. 2–6 and Eqs. 8–10, the equation of motion in dimensionless form after dropping the bar signs becomes

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial F}{\partial t} + 2i\Omega F\right) + \frac{M^2}{1 - im} \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) F$$
$$= \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \frac{\partial^2 F}{\partial z^2} - \frac{1}{K} \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) F, \tag{12}$$

in which ν is the kinematic viscosity, U_0 is the constant plate velocity and $m = \omega_{\mathrm{e}} \tau_{\mathrm{e}}$ is Hall parameter. The appropriate initial and boundary conditions are

$$F(z, 0) = \frac{\partial F(z, 0)}{\partial t} = \frac{\partial^2 F(z, 0)}{\partial t^2} = 0, z > 0, \tag{13}$$
$$F(0, t) = 1, F(\infty, t) = 0, t > 0. \tag{14}$$

3 Solution of the Problem

Applying the Laplace transform to Eq. 12 and using the initial conditions (13), the equation in the transformed (z, q) plane is given as

$$\frac{\mathrm{d}^3 \tilde{F}(z, q)}{\mathrm{d}z^3} - \frac{K\left(q + 2i\Omega + \lambda_1 q^2 + 2i\Omega\lambda_1 q + \lambda_2 q^3 + 2i\Omega\lambda_2 q^2\right)(1 - im) + K M^2\left(1 + \lambda_1 q + \lambda_2 q^2\right) + (1 - im)\left(1 + \lambda_3 q + \lambda_4 q^2\right)}{K(1 - im)\left(1 + \lambda_3 q + \lambda_4 q^2\right)}$$
$$\times \tilde{F}(z, q) = 0, \tag{15}$$

where

$$\tilde{F}(z,q) = \mathcal{L}\{F(z,t)\} = \int_0^\infty e^{-st} F(z,t)\,\mathrm{d}t.$$

The corresponding boundary conditions become

$$\tilde{F}(0,q) = \frac{1}{q} \quad \text{and} \quad \tilde{F}(z,q) \to 0 \text{ as } z \to \infty. \tag{16}$$

Equation (15) in view of Eq. 16 yields

$$\tilde{F}(z,q) = \frac{1}{q}\exp[-\alpha z], \tag{17}$$

where

$$\alpha = \sqrt{\frac{K\left(q + 2i\Omega + \lambda_1 q^2 + 2i\Omega\lambda_1 q + \lambda_2 q^3 + 2i\Omega\lambda_2 q^2\right)(1-im) + KM^2\left(1 + \lambda_1 q + \lambda_2 q^2\right) + (1-im)\left(1 + \lambda_3 q + \lambda_4 q^2\right)}{K(1-im)\left(1 + \lambda_3 q + \lambda_4 q^2\right)}} \tag{18}$$

and $\mathrm{Re}(\alpha) > 0$.

4 Graphical Results and Discussion

The inverse Laplace transform of Eq. 17 has been computed by software Mathematica. The results have been displayed graphically for various parameters of interest. The effects of these parameters especially Hartmann number M, permeability of porous medium K, Hall parameter$m$ and dimensionless time t on the velocity profile for (a) real and (b) imaginary parts have been studied.

It is clear from Fig. 2 that with an increase in M, both the real and imaginary parts of velocity decrease. Further boundary layer thickness also decreases with an increase in M. However, the magnitude of the boundary layer thickness for the imaginary part is quite large when compared with the real part of velocity. It is due to the fact that the application of transverse magnetic field results a resistive type force (called Lorentz force) similar to drag

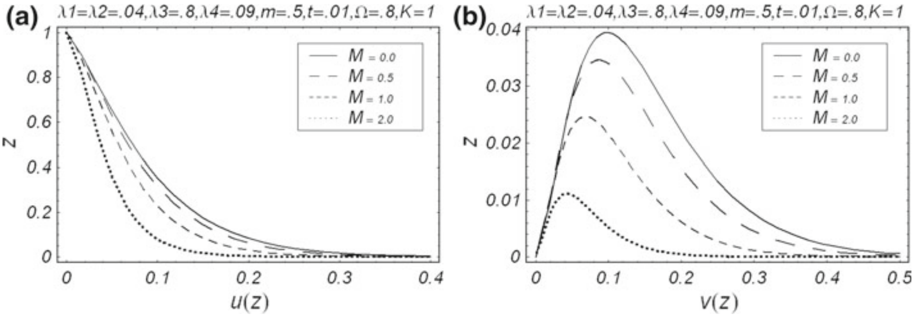

Fig. 2 Velocity profiles showing the effect of M (G. Burgers' fluid)

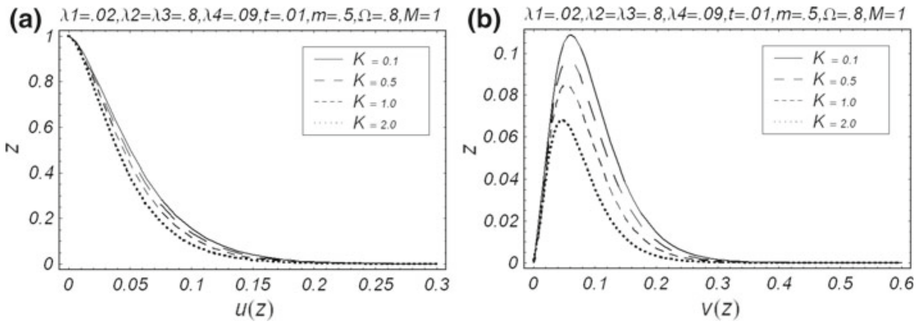

Fig. 3 Velocity profiles showing the effect of K (G. Burgers' fluid)

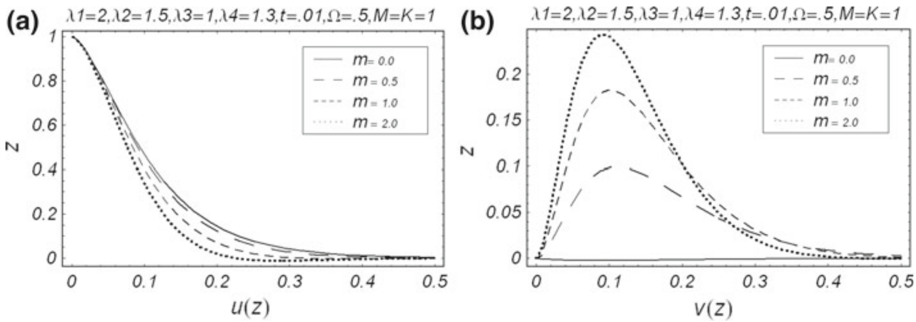

Fig. 4 Velocity profiles showing the effect of m (G. Burgers' fluid)

force and upon increasing the values of M, the drag force increases which leads to the deceleration of the flow. It is also observed from this figure that the real part of velocity decay early when compared with imaginary part of velocity. Figure 3 illustrates the effect of K, on the velocity profile which is similar to that of M. The characteristics of M and K on the fluid motion is similar to that observed in references (Fakhar et al. 2006; Hayat et al. 2008a). It is noted that the boundary layer thickness is comparatively shorter than the velocity profile for M. Moreover, it is found that the velocity profile reached to the steady state solution quite early when compared with the velocity profile for M. The influence of Hall parameter m on the velocity profile is shown in Fig. 4. We observed that with an increase in m, the real part of velocity decreases but quite opposite behavior was observed for the imaginary part of velocity. The velocity profile increases with increasing values of m. Physically this phenomenon is due to the fact that with increasing values of m, the effective conductivity of the fluid decreases which reduces the magnetic damping force on the velocity.

The variation of velocity for the dimensionless time t is shown in Fig. 5. It was observed that the velocity is a decreasing function of time. This is true in the sense that with increasing time, the viscosity might increase as the shear rate increases causes the fluid motion to slow down. However, the time required to reach the steady state for the real part of velocity is smaller than imaginary part of velocity. Indeed, the required time to reach the steady state also depends on the other material constants such as λ_i $(i = 1 - 4)$, M, K, Ω, and m. From this graph, it is clearly seen that boundary layer thickness for the real part of velocity is shorter when compared to the imaginary part of velocity. As a special case, the effects of M, K, and m on the velocity profiles for Burgers' and Oldroyd-B fluids have been displayed in Figs.

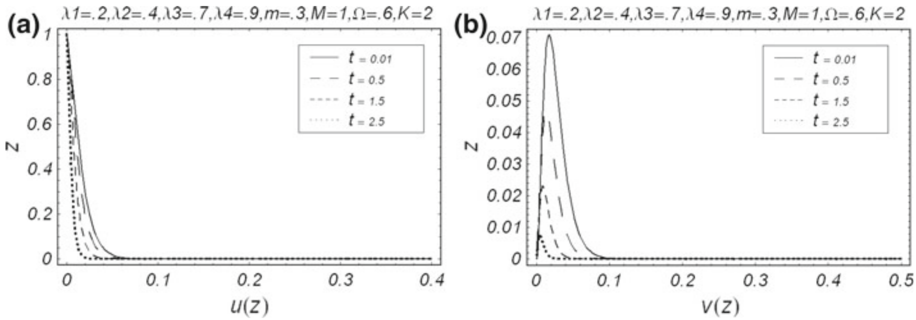

Fig. 5 Velocity profiles showing the effect of t (G. Burgers' fluid)

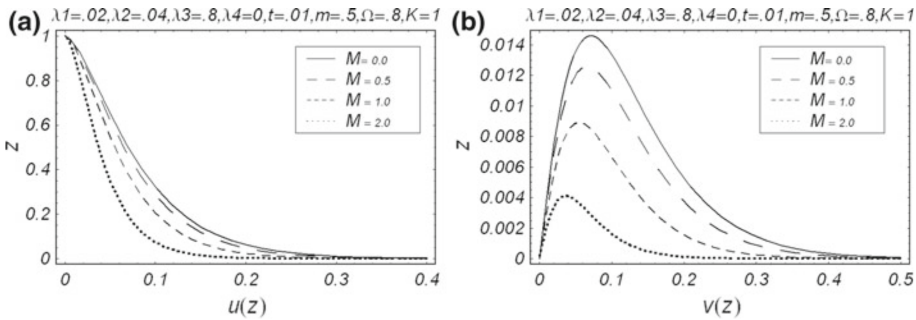

Fig. 6 Velocity profiles showing the effect of M (Burgers' fluid)

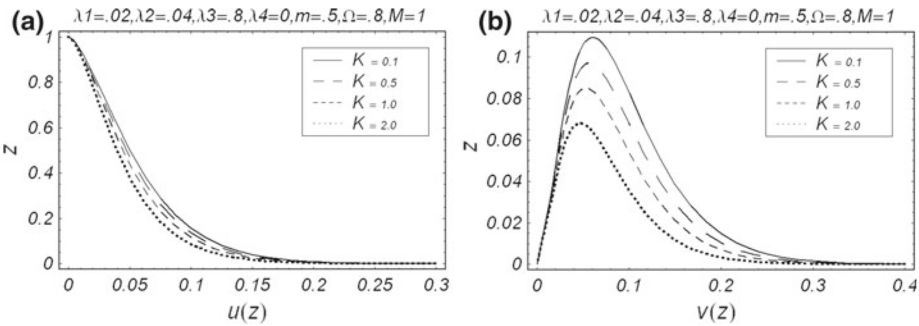

Fig. 7 Velocity profiles showing the effect of K (Burgers' fluid)

6, 7, 8, 9, 10, and 11. As expected, the variations of velocity with increasing values of M, K, and m for both type of fluids are similar to that of generalized Burgers' fluid. However, these observations are not similar quantitatively. It is due to the different values of chosen parameters. Moreover, in case of Burgers' fluid, for larger values of Hall parameter m, the real part of velocity increases whereas the magnitude of imaginary part of velocity decreases. This effect is quite opposite for Oldroyd-B fluid.

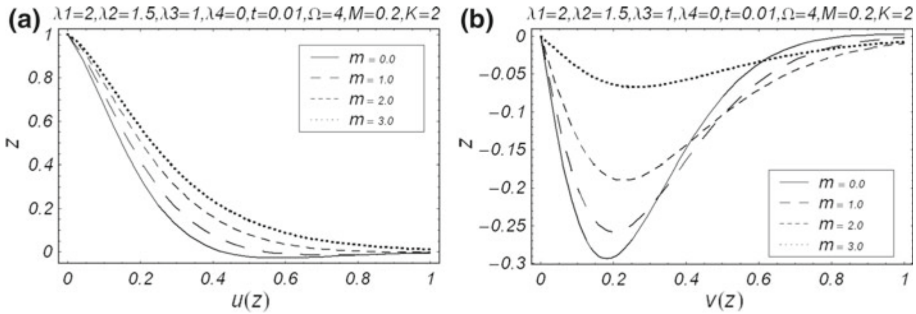

Fig. 8 Velocity profiles showing the effect of m (Burgers' fluid)

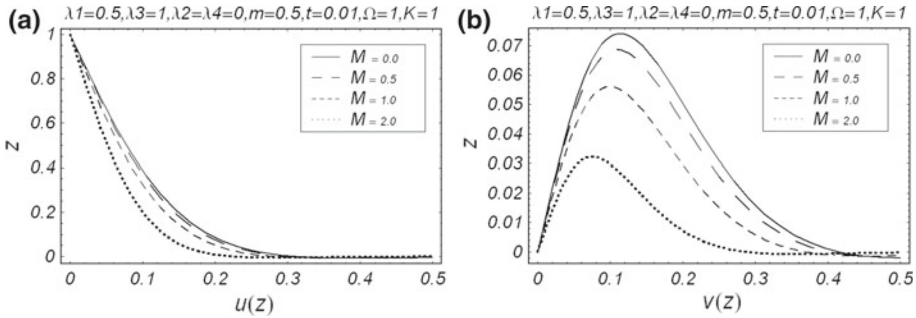

Fig. 9 Velocity profiles showing the effect of M (Oldroyd-B fluid)

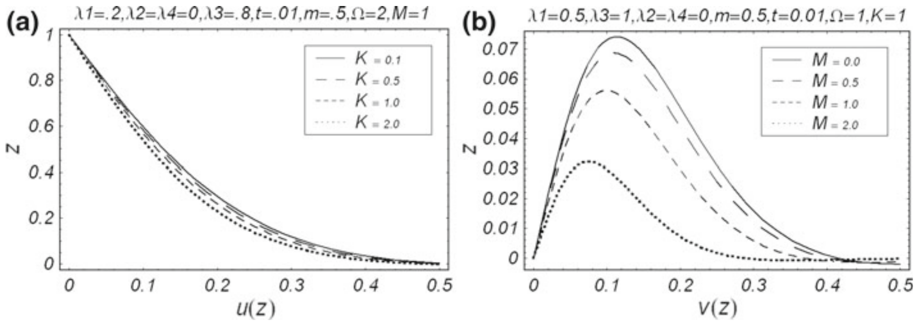

Fig. 10 Velocity profiles showing the effect of K (Oldroyd-B fluid)

5 Concluding Remarks

In this article, we have obtained the exact solution for MHD rotating flow of a generalized Burgers' fluid. The governing equation has been solved for the closed form solution by using Laplace transform technique. Graphical results have been displayed for the real and imaginary parts of velocity. The required time to reach the steady state has been determined using graphical illustrations. It has been observed that this time decreases or increases with variations of the involved material constants and effect the decay of the transient flow. The obtained results for hydrodynamic fluid in a non-porous space are in good match with those of Hayat et al. (in press) and those for Burgers' and Oldroyd-B fluids which ensure

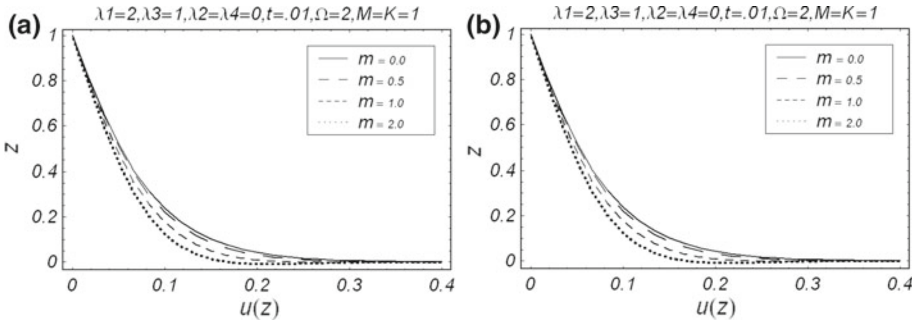

Fig. 11 Velocity profiles showing the effect of m (Oldroyd-B fluid)

the accuracy of the model and results obtained here. It has been further observed that these results are in close agreement in qualitative sense. However, further inspection revealed that these results are different quantitatively.

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