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Nonlinear Electrohydrodynamic Stability of Two Superposed Streaming Finite Dielectric Fluids in Porous Medium with Interfacial Surface Charges

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Abstract A weakly nonlinear stability analysis of wave propagation in two superposed dielectric fluids streaming through porous media in the presence of vertical electric field producing surface charges is investigated in three dimensions. The method of multiple scales is used to obtain a dispersion relation for the linear problem and a nonlinear Klein–Gordon equation with complex coefficients describing the behavior of the perturbed system at the critical point of the neutral curve. In the linear case, we found that the system is always unstable for all physical quantities (including the dimension l), even in the presence of electric fields and porous medium, in the nonlinear case, novel stability conditions are obtained, and the effects of various parameters on the stability of the system are discussed numerically in detail.

Keywords Electrohydrodynamics · Multiple scales method · Nonlinear instability · Flows through porous media

1 Introduction

The instability of the plane interface between two superposed fluids moving with a relative horizontal velocity is called Kelvin–Helmholtz instability. Owing to its relevance to astrophysical, geophysical, and laboratory situations, it has been analyzed by several authors. The linear instability of the Kelvin–Helmholtz problem is discussed in Chandrasekhar's treatise (1961). The nonlinear development of the Kelvin–Helmholtz instability has been carried out by Drazin (1970), Hasimoto and Ono (1972) and Nayfeh and Saric (1972). The comprehensive nonlinear treatment of Weissman (1979) as well as other early studies on weakly nonlinear evolution of the amplitudes of linear fields in thermal convection, plane Poiseuille flow, a buckling problem in elasticity, and baroclinic flow have been organized into a comprehensive framework by Gibbon and McGuinness (1981).

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In the surface wave phenomena in nonconducting fluids where both gravity and surface tension are significant, the gravity and the capillary waves couple to form a single wave characterized by both effects. In contrast, conducting fluid flows in the presence of an external electric field have many different properties and various useful physical and engineering applications. When surface charges participate in the fluid motion, the resulting waves are referred to as electrohydrodynamic surface waves. The basic electrohydrodynamic phenomena include a spectrum of applications (Melcher 1963), such as the formation and coalescence of solid and liquid particles; the dielectrophoretic orientation and expulsion of liquids of zero gravity environments; electrogasdynamic high voltage and power generation; insulation research, atmospheric and cloud physics; physicochemical hydrodynamics, including membrane flows, heat, mass, and momentum transfer fluid mechanics; electro-optics of liquids and other phenomena useful for image processing, and electrohydrodynamics of biological system. The study of electrohydrodynamic Kelvin–Helmholtz instability of free surface charges, separating two semi-infinite dielectric streaming fluids and influenced by an electric field, has been treated extensively by Lyon (1962), Elshehawey (1986), and Mohamed and Elshehawey (1989), among others. For recent developments of the topic, see the monograph of Griffiths (2006), and the investigations of Castellanos (2003), Papageorgiou and Petropoulos (2004), Shankar and Sharma (2004), Tomar et al. (2007), Ugug and Aubry (2008), and Yecko (2009). In all the above-mentioned investigations, the medium is considered to be nonporous.

Flow through porous media has great interest for the past decades. It has many applications in various disciplines, such as agricultural, geophysical, ground water hydrology, chemical catalytic reactors, grain storage devices, and engineering. It is also of interest in petroleum industry in extracting pure petrol from crude oil. It is well known that Darcy's law is an empirical formula, which relates the pressure gradient, the bulk viscous resistance, and the gravitational force in a porous medium. However, in Darcy's law, the usual viscous term in the equation of motion is replaced by the resistive term $-(\mu/\lambda_1)\underline{\mathbf{v}}$, where μ is the fluid viscosity, λ_1 is the medium permeability, and $\underline{\mathbf{v}}$ is the Darcian (filter) velocity of the fluid. Much of the recent study on this topic is given by Nield and Bejan (1999), Ingham and Pop (1998), Vafai (2000), and Pop and Ingham (2001). In most previous studies on porous media, treatments based on Darcy's law and Forchheimer extended Darcy's law models have been considered. The electrohydrodynamics instability is sensitive to inclusion or exclusion of other viscous or inertial effects such as Brinkman or Forchheimer terms, and it has been studied linearly by El-Sayed (1997, 1998). The nonlinear studies of these two models are now of current research.

On the other hand, the Kelvin–Helmholtz instability for flow in porous media has attracted little attention in the scientific literature. Sharma and Spanos (1982) investigated the instability of the plane interface between two uniform superposed and streaming fluids through porous media. A linear theory of Kelvin–Helmholtz instability for parallel flow in porous media was introduced by Bau (1982). El-Sayed (1998) studied the instability of two superposed viscous streaming fluids through porous media under normal electric field with Darcian and Forchheimer flows. For recent reviews about the subject of linear electrohydrodynamic flows in porous media, see the studies of El-Sayed (2006), El-Sayed et al. (2009), and Wang and Pan (2008), among others. Studies on nonlinear stability theory have received greater interest in recent years. The study of the nonlinear interfacial instability has received a considerable number of contributions in porous media, e.g., Mohamed et al. (2002) investigated the nonlinear gravitational stability of streaming in an electrified viscous flow through porous media, Moatimid and El-Dib (2004) studied the nonlinear Kelvin–Helmholtz instability of Oldroydian viscoelastic fluid in porous media. The nonlinear electrohydrodynamic stability of capillary–gravity waves on the interface between two semi-infinite dielectric fluids under the effect of a vertical electric field in the presence of surface charges is investigated by El-Dib and Moatimid (2004). Recently, El-Sayed et al. (2009) investigated the nonlinear electrohydrodynamic stability of two superposed dielectric bounded fluids streaming through porous media in the presence of a horizontal electric field in three dimensions.

In this article, we formulate the general interfacial problem for two superposed dielectric fluids of finite depths moving through porous media in the presence of vertical electric fields with interfacial surface charges, then the linear and nonlinear analysis using multiple scales expansion are carried out. This problem, to the best of our knowledge, has not been investigated yet. The stability criteria are obtained for both linear and nonlinear problems. We obtained the dispersion relation in linear case, and a nonlinear Klein–Gordon equation with complex coefficients describing the behavior of the perturbed system near the critical point of the neutral curve in nonlinear case, and they are discussed analytically and numerically. The obtained results are listed in Sect. 5 in view of the effects of various physical parameters on the stability of the considered system.

2 Formulation and Boundary Conditions

We consider the finite amplitude three-dimensional capillary–gravity stokes wave propagation on the interface $z = 0$ which represents the equilibrium state. The disturbed interface $z = \eta(x, y, t)$ separates the two incompressible dielectric fluids in relative horizontal motion with uniform depths. The upper fluid occupies the region $\eta(x, y, t) < z < h_2$, and having density ρ_2, velocity U_2, dielectric constant ε_2, viscosity coefficient μ_2, and bounded from above by the plane $z = h_2$ with zero potential, while the lower fluid occupies the region $-h_1 < z < \eta(x, y, t)$, and having density ρ_1 ($< \rho_2$), velocity U_1, dielectric constant ε_1, viscosity coefficient μ_1, and bounded from below by the plane $z = -h_1$ which is raised to electric potential V_0. As a result of the potential difference between the planes, both the lower and upper fluids are subjected to uniform normal electric fields to the interface (E_{01} and E_{02}, respectively) in the z-direction. The two media are considered to be porous in which λ_1 and m denote, respectively, the medium permeability and the porosity of porous medium. The solid material in the porous domain is considered as nonconducting. The acceleration due to gravity g acts in the negative z-direction. The surface tension force T is taken into account between the two fluids. No induced electrical or magnetic fields exist in this study. The interface is represented by $F(x, y, z, t) = z - \eta(x, y, t) = 0$, for which the outward normal vector $\mathbf{n}$ is written as

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \left[1 + \eta_x^2 + \eta_y^2\right]^{-\frac{1}{2}} \ (-\eta_x, -\eta_y, 1) \tag{1}$$

The equations governing the motion of an incompressible fluid through a porous medium, where the porosity and the permeability of the medium are taking into account are (El-Sayed 1997)

$$\frac{\rho}{m}\left[\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{m}(\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p - \rho g \mathbf{e}_z - \frac{\mu}{\lambda_1}\mathbf{v} \tag{2}$$

and the equation of continuity will be

$$\nabla \cdot \mathbf{v} = 0 \tag{3}$$

where $\mathbf{v}$ is the fluid velocity, p is the pressure, and $\mathbf{e}_z$ is the unit vector in the z-direction. Assuming the flow of the fluids to be irrotational, and then there are velocity potentials $\Phi_j(x, y, z, t)$ such that $\mathbf{v}_j = U_j\mathbf{e}_x + \nabla\Phi_j$, $j = 1, 2$, where $\mathbf{v}_j$ is the total fluid velocities, and $\mathbf{e}_x$ is the unit vector in the x-direction. For incompressible fluids, the potential Φ_j $(j = 1, 2)$ satisfies Laplace's equation, i.e.,

$$\nabla^2\Phi_j = 0, (j = 1, 2) \quad \text{for} \; -h_1 < z < \eta(x, y, t) \quad \text{and} \quad \eta(x, y, t) < z < h_2 \tag{4}$$

Since the system is stressed by normal electric fields, we shall assume that they allow for the presence of surface charges at the surface of separation, such that $\varepsilon_1 E_{01} \neq \varepsilon_2 E_{02}$. We shall also assume that the quasi-static approximation is valid, and hence the electric field $\mathbf{E}$ is irrotational. Thus, the electrical equations are

$$\nabla \times \mathbf{E} = \mathbf{0} \quad \text{and} \quad \nabla \cdot (\varepsilon\mathbf{E}) = \mathbf{0} \tag{5}$$

Therefore, the electric field can be expressed in terms of electrostatic potentials $\Psi(x, y, z, t)$, i.e., $\mathbf{E} = -\nabla\Psi$ such that the total electric fields can be written as

$$\mathbf{E}_j = E_{0j}\mathbf{e}_z - \nabla\Psi_j, \quad j = 1, 2 \tag{6}$$

It follows from Eq. 5 that the electrostatic potentials also satisfy Laplace's equation, i.e.,

$$\nabla^2\Psi_j = 0, (j = 1, 2) \quad \text{for} \; -h_1 < z < \eta(x, y, t) \quad \text{and} \quad \eta(x, y, t) < z < h_2 \tag{7}$$

Note that, in the presence of free surface charges on the interface, Eq. 5 should be replaced by $\nabla \cdot (\varepsilon\mathbf{E}) = Q$. Free charges Q are present due to different electrophysical properties at the fluids. Since free charges will only be present at the interfaces, then in the bulk, Eq. 5 are valid, and they have to be solved to obtain the electric field distribution in the analytical domain. The solutions for the potentials Φ_j and Ψ_j $(j = 1, 2)$ should satisfy the following boundary conditions

1. The normal fluid velocities vanish on the rigid boundaries $z = -h_1$ and $z = h_2$, i.e.,

$$\frac{\partial\Phi_j}{\partial z} = 0 \quad \text{on}\; z = (-1)^j h_j, \quad j = 1, 2 \tag{8}$$

2. The tangential components of the electric fields vanish on the rigid boundaries $z = -h_1$ and $z = h_2$, i.e.,

$$\frac{\partial\Psi_j}{\partial x} = \frac{\partial\Psi_j}{\partial y} = 0 \quad \text{on}\; z = (-1)^j h_j, \quad j = 1, 2 \tag{9}$$

3. The kinematic condition, i.e., the condition that the interface is moving with the fluid, leads to

$$m\frac{\partial\eta}{\partial t} - \frac{\partial\Phi_j}{\partial z} + \frac{\partial\eta}{\partial x}\left(U_j + \frac{\partial\Phi_j}{\partial x}\right) + \frac{\partial\eta}{\partial y}\frac{\partial\Phi_j}{\partial y} = 0 \quad \text{at}\; z = \eta(x, y, t), \quad j = 1, 2 \tag{10}$$

4. The tangential component of the electric field is supposed to be continuous at the interface, and thus leads to

$$\begin{aligned} \|\Psi_x\| + \eta_x\|\Psi_z\| - \eta_x\|E_0\| &= 0 \\ \|\Psi_y\| + \eta_y\|\Psi_z\| - \eta_y\|E_0\| &= 0 \quad \text{at}\; z = \eta(x, y, t) \\ \eta_x\|\Psi_y\| - \eta_y\|\Psi_x\| &= 0 \end{aligned} \tag{11}$$

where $\|*\|$ represents the jump across the interface.

5. The interfacial tangential and normal stress tensor components are balanced at the dividing surface of the system, then the condition satisfied by the tangential stress leads to the following components

$$\frac{2}{m}\left\|\mu\left\{\eta_y\left(\eta_x\Phi_{xz}+\eta_y\Phi_{yz}+\Phi_{yy}-\Phi_{zz}\right)+\eta_x\Phi_{xy}-\Phi_{yz}\right\}\right\| \\ -\left\|\varepsilon(\Psi_z-E_0)\left[\Psi_y+\eta_y(\Psi_z-E_0)\right]\right\| \\ +\left\|\varepsilon(\eta_x\Psi_x+\eta_y\Psi_y)(\Psi_y-\eta_yE_0)\right\|=0 \tag{12}$$

$$\frac{2}{m}\left\|\mu\left\{\eta_x\left(\eta_x\Phi_{xz}+\eta_y\Phi_{yz}+\Phi_{xx}-\Phi_{zz}\right)+\eta_y\Phi_{xy}-\Phi_{xz}\right\}\right\| \\ -\left\|\varepsilon(\Psi_z-E_0)\left[\Psi_x+\eta_x(\Psi_z-E_0)\right]\right\| \\ +\left\|\varepsilon(\eta_x\Psi_x+\eta_y\Psi_y)(\Psi_x-\eta_xE_0)\right\|=0 \tag{13}$$

$$\frac{2}{m}\left\|\mu\left\{\left(\eta_x^2-\eta_y^2\right)\Phi_{xy}+\eta_x\eta_y\left(\Phi_{yy}-\Phi_{xx}\right)+\left(\eta_y\Phi_{zx}-\eta_x\Phi_{yz}\right)\right\}\right\| \\ -\left\|\varepsilon(\Psi_z-E_0)(\eta_x\Psi_y-\eta_y\Psi_x)\right\|=0 \tag{14}$$

while the condition satisfied by the normal stress leads to

$$\frac{1}{2m^2}\left\{2\left\|\rho\left(m\ \Phi_t+U\Phi_x\right)\right\|+\left\|\rho(\nabla\Phi)^2\right\|\right\} \\ +\frac{2}{m}\left\{\eta_x^2\left\|\mu\Phi_{xx}\right\|+\eta_y^2\left\|\mu\Phi_{yy}\right\|+\left(1-\eta_x^2-\eta_y^2\right)\left\|\mu\Phi_{zz}\right\|\right. \\ \left.+2\eta_x\eta_y\left\|\mu\Phi_{xy}\right\|-2\eta_x\left\|\mu\Phi_{xz}\right\|-2\eta_y\left\|\mu\Phi_{yz}\right\|\right\}+\frac{1}{\lambda_1}\left\|\mu\Phi\right\| \\ +\frac{1}{2}\left\|\varepsilon\left(\Psi_z^2-\Psi_x^2-\Psi_y^2\right)\right\|-2\eta_x\left\|\varepsilon\Psi_x\Psi_z\right\|-2\eta_y\left\|\varepsilon\Psi_y\Psi_z\right\| \\ -\left(\eta_x^2+\eta_y^2\right)\left\|\varepsilon E_0^2\right\|+2\eta_x\left\|\varepsilon E_0\Psi_x\right\|+2\eta_y\left\|\varepsilon E_0\Psi_y\right\| \\ +\left(2\eta_x^2+2\eta_y^2-1\right)\left\|\varepsilon E_0\Psi_z\right\|+gz\left\|\rho\right\|+T\left\{\frac{1}{2}\eta_{xx}\left(2-3\eta_x^2-\eta_y^2\right)\right. \\ \left.-2\eta_x\eta_y\eta_{xy}+\frac{1}{2}\eta_{yy}\left(2-3\eta_y^2-\eta_x^2\right)\right\}=0\quad \text{at } z=\eta(x,y,t) \tag{15}$$

3 Multiple Time Scales Method

In order to describe the nonlinear interactions of small but finite amplitude waves, we use the derivative expansion method with multiple scales. Following the usual procedure (Nayfeh 1973), let us first expand η , Φ_j, and Ψ_j $(j=1,2)$ in the following asymptotic series

$$\eta(x,y,t)=\sum_{n=1}^{3}\epsilon^n f_n(x_0,x_1,x_2,y_0,y_1,y_2,t_0,t_1,t_2)+O\left(\epsilon^4\right)$$

$$\begin{pmatrix}\Phi_j\\ \Psi_j\end{pmatrix}(x,y,z,t)=\sum_{n=1}^{3}\epsilon^n\begin{pmatrix}\Phi_{jn}\\ \Psi_{jn}\end{pmatrix}(x_0,x_1,x_2,y_0,y_1,y_2,z,t_0,t_1,t_2)+O\left(\epsilon^4\right) \tag{16}$$

where ϵ is a small parameter indicating the weakness of the nonlinearity. The multiple scales $x_n = \epsilon^n x$, $y_n = \epsilon^n y$, and $t_n = \epsilon^n t$ are assumed to satisfy the following derivative expansions

$$\frac{\partial}{\partial \beta} = \sum_{n=0}^{3} \epsilon^n \frac{\partial}{\partial \beta_n} + O\left(\epsilon^4\right) \tag{17}$$

where β is any of the variables x, y, and t. The short scales x_0 and y_0 and the fast scale t_0 denote, respectively, the wavelength and the frequency of the wave. Here t_1 and t_2 represent the slow temporal scales of the phase and the amplitude, respectively, whereas the long scales x_1, y_1 and x_2, y_2 stand for the spatial modulations of the phase and the amplitude, respectively. Expanding now the boundary conditions (8)–(15) into Taylor series around the undisturbed surface $z = 0$, then substituting Eqs. 16 and 17 into Eqs. 4, 7, and the boundary conditions (8)–(15), and equating the coefficients of the same powers in ϵ, we obtain a sequence of sets of equations for η_n, Φ_n, and Ψ_n. They are not given here because they are very lengthy. In order that the starting solutions of the first-order problem (see the Appendix) should not be trivial, the wavenumber K and the frequency ω must satisfy the following characteristic equation

$$\begin{aligned} S(\omega, K) = {} & \frac{1}{Km^2}\left[\frac{\rho_1 (m\omega - kU_1)^2}{\sigma_1} + \frac{\rho_2 (m\omega - kU_2)^2}{\sigma_2}\right] \\ & + i\left(\frac{2\lambda_1 K^2 + m}{\lambda_1 m K}\right)\left[\frac{\mu_1 (m\omega - kU_1)}{\sigma_1} + \frac{\mu_2 (m\omega - kU_2)}{\sigma_2}\right] \\ & + \left(\frac{2iK\left[\mu_2(kU_2 - m\omega) - \mu_1(kU_1 - m\omega)\right]}{m(\varepsilon_2 E_{02} - \varepsilon_1 E_{01})}\right)\left(\frac{\varepsilon_1 E_{01}}{\sigma_1} + \frac{\varepsilon_2 E_{02}}{\sigma_2}\right) \\ & + K\left(\frac{\varepsilon_1 E_{01}^2}{\sigma_1} + \frac{\varepsilon_2 E_{02}^2}{\sigma_2}\right) + g(\rho_2 - \rho_1) - TK^2 = 0 \end{aligned} \tag{18}$$

where $\sigma_j = \tanh Kh_j$ $(j = 1, 2)$, $\theta = kx_0 + ly_0 - \omega t_0$ is the phase of the carrier wave, $K = \sqrt{k^2 + l^2}$, k and l being, respectively, the wavenumber components along the x- and y-directions, ω is the angular frequency, together with the condition that

$$\frac{1}{\lambda_1}\left(\mu_2 p_{21} - \mu_1 p_{11}\right) = 0 \tag{19}$$

The characteristic Eq. 18 is reduced to the following dispersion relation

$$a_0\omega^2 + (a_1 + ib_1)\omega + (a_2 + ib_2) = 0, \tag{20}$$

where

$$\begin{aligned} a_0 &= \left(\frac{\rho_1}{\sigma_1} + \frac{\rho_2}{\sigma_2}\right) \\ a_1 &= -\frac{2k}{m}\left(\frac{\rho_1 U_1}{\sigma_1} + \frac{\rho_2 U_2}{\sigma_2}\right) \end{aligned}$$

$$b_1 = \frac{1}{\lambda_1}\left(2\lambda_1 K^2 + m\right)\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) - \frac{2K^2(\mu_2 - \mu_1)}{(\varepsilon_2 E_{02} - \varepsilon_1 E_{01})}\left(\frac{\varepsilon_1 E_{01}}{\sigma_1} + \frac{\varepsilon_2 E_{02}}{\sigma_2}\right)$$

$$a_2 = \frac{k^2}{m^2}\left(\frac{\rho_1 U_1^2}{\sigma_1} + \frac{\rho_2 U_2^2}{\sigma_2}\right) + K^2\left(\frac{\varepsilon_1 E_{01}^2}{\sigma_1} + \frac{\varepsilon_2 E_{02}^2}{\sigma_2}\right) + gK(\rho_2 - \rho_1) - TK^3$$

$$b_2 = \frac{2kK^2(\mu_2 U_2 - \mu_1 U_1)}{m(\varepsilon_2 E_{02} - \varepsilon_1 E_{01})}\left(\frac{\varepsilon_1 E_{01}}{\sigma_1} + \frac{\varepsilon_2 E_{02}}{\sigma_2}\right) - \frac{k\left(2\lambda_1 K^2 + m\right)}{\lambda_1 m}\left(\frac{\mu_1 U_1}{\sigma_1} + \frac{\mu_2 U_2}{\sigma_2}\right)$$

Note that, the dispersion relation (20) reduces to the same equation obtained earlier by El-Sayed and Callebaut (1998) in the limiting case of two-dimensional disturbances in absence of velocities and porous medium. Now, applying the Routh–Hurwitz stability criterion (Zahreddine and Elshehawey 1988) to Eq. 20, we obtain the necessary and sufficient conditions for stability as

$$a_1 > 0 \quad \text{and} \quad a_2 a_1^2 + a_1 b_1 b_2 - a_0 b_2^2 > 0 \tag{21}$$

The first stability condition in Eq. 21 is not satisfied, because $a_1 < 0$ usually. The system is unstable even in the presence of electric field and porous medium. Therefore, in the linear theory of stability, we conclude that the system is always unstable for all physical parameters used here, and this instability increases with increasing the dimension l which has also a destabilizing effect.

Since, our aim is to study the amplitude modulation for traveling waves, we shall substitute the linear solutions given in the Appendix into the second-order problem. The solutions of the second-order problem are also given in the Appendix. The nonsecularity condition for the second perturbation consists of the following two parts, the first one is

$$\frac{\partial A}{\partial t_1} + v_k \frac{\partial A}{\partial x_1} + v_l \frac{\partial A}{\partial y_1} = 0, \tag{22}$$

together with its c.c., and the second one is

$$\frac{2K(\sigma_1 E_{02} - \sigma_2 E_{01})}{\sigma_1 \sigma_2}\frac{\partial |A|^2}{\partial x_1} - (E_{02} - E_{01})\frac{\partial B_0}{\partial x_1} = 0 \tag{23}$$

where

$$v_k = -(\partial S/\partial k)(\partial S/\partial \omega)^{-1} \quad \text{and} \quad v_l = -(\partial S/\partial l)(\partial S/\partial \omega)^{-1}$$

are the group velocities of the wave train in the x- and y-directions. Equation 22 shows the modulations on the time scale ϵ^{-1} propagate without change of shape with the group velocities (v_k, v_l).

Now, we proceed to the third-order problem. By using the first- and second-order solutions and simplifying the right-hand side of the third-order equations and after some straightforward reductions, we can express the particular solutions for η_3, Φ_{j3}, and Ψ_{j3} $(j = 1, 2)$ but they will not be included here, because they are very lengthy. Finally substituting from the third-order solution into the last boundary condition in the third-order problem, we obtain the nonsecularity condition from the coefficient of $\exp(i\theta)$, in the form

$$i\left\{\frac{\partial S}{\partial \omega}\frac{\partial A}{\partial t_2}-\frac{\partial S}{\partial k}\frac{\partial A}{\partial x_2}-\frac{\partial S}{\partial l}\frac{\partial A}{\partial y_2}\right\}$$
$$+\left\{\frac{\partial^2 S}{\partial \omega \partial k}\frac{\partial^2 A}{\partial t_1 \partial x_1}+\frac{\partial^2 S}{\partial \omega \partial l}\frac{\partial^2 A}{\partial t_1 \partial y_1}-\frac{\partial^2 S}{\partial k \partial l}\frac{\partial^2 A}{\partial x_1 \partial y_1}\right\}$$
$$-\frac{1}{2}\left\{\frac{\partial^2 S}{\partial \omega^2}\frac{\partial^2 A}{\partial t_1^2}+\frac{\partial^2 S}{\partial k^2}\frac{\partial^2 A}{\partial x_1^2}+\frac{\partial^2 S}{\partial l^2}\frac{\partial^2 A}{\partial y_1^2}\right\}+GA^2\bar{A}+RA=0, \quad (24)$$

where G is given in the Appendix, and

$$R=(Q_1+iQ_2)B_0+\left[\frac{2\rho_1(kU_1-m\omega)}{\sigma_1 K m^2}-\frac{2iK\mu_1\varepsilon_2 E_{02}(\sigma_1+\sigma_2)}{m\sigma_1\sigma_2(\varepsilon_2 E_{02}-\varepsilon_1 E_{01})}\right.$$
$$\left.-\frac{i\mu_1}{\sigma_1\lambda_1 K}\right]\left(k\frac{\partial p_{11}}{\partial x_1}+l\frac{\partial p_{11}}{\partial y_1}\right)+\left[\frac{2\rho_2(kU_2-m\omega)}{\sigma_2 K m^2}\right.$$
$$\left.-\frac{2iK\mu_2\varepsilon_1 E_{01}(\sigma_1+\sigma_2)}{m\sigma_1\sigma_2(\varepsilon_2 E_{02}-\varepsilon_1 E_{01})}-\frac{i\mu_2}{\sigma_2\lambda_1 K}\right]\left(k\frac{\partial p_{21}}{\partial x_1}+l\frac{\partial p_{21}}{\partial y_1}\right) \quad (25)$$

in which

$$Q_1=\frac{1}{m^2}\left[\left(\rho_2/\sigma_2^2\right)\left(1-\sigma_2^2\right)(kU_2-m\omega)^2-\left(\rho_1/\sigma_1^2\right)\left(1-\sigma_1^2\right)(kU_1-m\omega)^2\right]$$
$$+K^2\left[\frac{\varepsilon_2 E_{02}^2\left(1-\sigma_2^2\right)}{\sigma_2^2}-\frac{\varepsilon_1 E_{01}^2\left(1-\sigma_1^2\right)}{\sigma_1^2}\right]$$
$$Q_2=\frac{-2K^2\left(\sigma_2^2-\sigma_1^2\right)}{m\sigma_1^2\sigma_2^2(\varepsilon_2 E_{02}-\varepsilon_1 E_{01})}\left[\mu_2(kU_2-m\omega)\varepsilon_1 E_{01}-\mu_1(kU_1-m\omega)\varepsilon_2 E_{02}\right]$$
$$-\frac{1}{\lambda_1}\left[\left(\mu_2/\sigma_2^2\right)(kU_2-m\omega)-\left(\mu_1/\sigma_1^2\right)(kU_1-m\omega)\right] \quad (26)$$

Furthermore, from the nonsecularity condition for η_3, we have

$$m\frac{\partial B_0}{\partial t_1}+U_j\frac{\partial B_0}{\partial x_1}\pm h_j\left(\frac{\partial^2 p_{j1}}{\partial x_1^2}+\frac{\partial^2 p_{j1}}{\partial y_1^2}\right)$$
$$\mp\frac{2\left(kU_j-m\omega\right)}{K\sigma_j}\left(k\frac{\partial |A|^2}{\partial x_1}+l\frac{\partial |A|^2}{\partial y_1}\right)=0 \quad (27)$$

In general, one may consider the general nonlinear evolution equation (24) in one of three regions. The first is the linearly stable region where $\partial S/\partial\omega \neq 0$, $\partial S/\partial k \neq 0$, and $\partial S/\partial l \neq 0$. The second region is on the neutral surface (or stability boundary) $\mathrm{Im}(\omega)=0$, but away from its extremum, so that $\partial S/\partial\omega = 0$, but $\partial S/\partial k \neq 0$, and $\partial S/\partial l \neq 0$. Finally, there is the third region near the extremum of the neutral surface where $\partial S/\partial\omega = \partial S/\partial k = \partial S/\partial l = 0$. Hence, we shall concentrate on the third region in the next section; while we will exclude the other regions because they can be dealt with following the same procedure used by El-Sayed et al. (2009) in the case when the applied electric field acts horizontally.

Now, we shall consider Eqs. 22 and 24 for the general case of nonvanishing of the first derivative of S. In this case, the coefficients of Eq. 22 do not vanish, i.e., $\partial S/\partial\omega \neq 0$, $\partial S/\partial k \neq 0$, and $\partial S/\partial l \neq 0$. By using Eq. 22, derivatives with respect to x_1, y_1, and t_1 can be calculated, then substituting into Eq. 24, and divide the resulting equation by $(\partial S/\partial\omega)$, we obtain

$$i\left[\frac{\partial A}{\partial t_2}+v_k\frac{\partial A}{\partial x_2}+v_l\frac{\partial A}{\partial y_2}\right]+\frac{1}{2}\left[v_{kk}\frac{\partial^2 A}{\partial x_1^2}+2v_{kl}\frac{\partial^2 A}{\partial x_1\partial y_1}+v_{ll}\frac{\partial^2 A}{\partial y_1^2}\right]$$
$$+\frac{GA^2\bar{A}+RA}{\partial S/\partial\omega}=0, \tag{28}$$

4 Nonlinear Klein–Gordon Equation

Since the nonsecularity condition (22) for the second-order perturbation is satisfied by the transformation $\gamma = x_1 + y_1 - (v_k + v_l)\,t_1$, then substituting into the nonsecularity conditions (23) and (27), using the nonsecularity condition $\mu_2 p_{21} = \mu_1 p_{11}$, and solving the resulting equations, we obtain

$$B_0 = \frac{2K\,(\sigma_1 E_{02} - \sigma_2 E_{01})}{\sigma_1\sigma_2\,(E_{02}-E_{01})}|A|^2 \tag{29}$$

$$\frac{\partial p_{11}}{\partial\gamma} = \left(\frac{\mu_2}{\mu_2 h_1 + \mu_1 h_2}\right)\left[\frac{(k+l)}{K}\left(\frac{(kU_1 - m\omega)}{\sigma_1}+\frac{(kU_2-m\omega)}{\sigma_2}\right)\right.$$
$$\left.-\frac{K\,(\sigma_1 E_{02}-\sigma_2 E_{01})}{\sigma_1\sigma_2\,(E_{02}-E_{01})}\,(U_1-U_2)\right]|A|^2 \tag{30}$$

Also, using the above-mentioned transformation, and the result that $p_{21} = (\mu_1/\mu_2)\,p_{11}$, we get

$$k\frac{\partial p_{j1}}{\partial x_1}+l\frac{\partial p_{j1}}{\partial y_1} = \left(1,\frac{\mu_1}{\mu_2}\right)(k+l)\frac{\partial p_{11}}{\partial\gamma},\quad j=1,2 \tag{31}$$

Substituting from Eq. 31 into Eq. 25, we obtain

$$R = \left\{\frac{2K\,(\sigma_1 E_{02}-\sigma_2 E_{01})}{\sigma_1\sigma_2\,(E_{02}-E_{01})}\,(Q_1+iQ_2)+\frac{(k+l)}{(\mu_2 h_1+\mu_1 h_2)}\right.$$
$$\times\left[\frac{(k+l)}{K}\left(\frac{(kU_1-m\omega)}{\sigma_1}+\frac{(kU_2-m\omega)}{\sigma_2}\right)\right.$$
$$\left.-\frac{K\,(\sigma_1 E_{02}-\sigma_2 E_{01})}{\sigma_1\sigma_2\,(E_{02}-E_{01})}\,(U_1-U_2)\right]$$
$$\times\left[\frac{2}{Km^2}\left(\frac{\rho_1\mu_2\,(kU_1-m\omega)}{\sigma_1}+\frac{\rho_2\mu_1\,(kU_2-m\omega)}{\sigma_2}\right)\right.$$
$$\left.\left.-\frac{2iK\mu_1\mu_2\,(\sigma_1+\sigma_2)\,(\varepsilon_2 E_{02}+\varepsilon_1 E_{01})}{m\sigma_1\sigma_2\,(\varepsilon_2 E_{02}-\varepsilon_1 E_{01})}-\frac{i\mu_1\mu_2}{\lambda_1 K}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)\right]\right\}|A|^2 \tag{32}$$

Using Eq. 32, then the nonlinear evolution equation (24) can be written in the form

$$i\left\{\frac{\partial S}{\partial \omega}\frac{\partial A}{\partial t_2}-\frac{\partial S}{\partial k}\frac{\partial A}{\partial x_2}-\frac{\partial S}{\partial l}\frac{\partial A}{\partial y_2}\right\}$$
$$+\left\{\frac{\partial^2 S}{\partial \omega\partial k}\frac{\partial^2 A}{\partial t_1\partial x_1}+\frac{\partial^2 S}{\partial \omega\partial l}\frac{\partial^2 A}{\partial t_1\partial y_1}-\frac{\partial^2 S}{\partial k\partial l}\frac{\partial^2 A}{\partial x_1\partial y_1}\right\}$$
$$-\frac{1}{2}\left\{\frac{\partial^2 S}{\partial \omega^2}\frac{\partial^2 A}{\partial t_1^2}+\frac{\partial^2 S}{\partial k^2}\frac{\partial^2 A}{\partial x_1^2}+\frac{\partial^2 S}{\partial l^2}\frac{\partial^2 A}{\partial y_1^2}\right\}+\widehat{G}A^2\bar{A}=0 \tag{33}$$

where $\widehat{G}=G+R$. Hence, we shall concentrate on the third region, Eq. 33 reduces to

$$\frac{1}{2}\left\{S_{\omega\omega}\frac{\partial^2 A}{\partial t_1^2}+S_{kk}\frac{\partial^2 A}{\partial x_1^2}+S_{ll}\frac{\partial^2 A}{\partial y_1^2}\right\}-S_{\omega k}\frac{\partial^2 A}{\partial t_1\partial x_1}=\widehat{G}A^2\bar{A} \tag{34}$$

Following Weissman (1979), we may rewrite Eq. 34 in the form

$$\left(\frac{\partial}{\partial t_1}+v_k^+\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial t_1}+v_k^-\frac{\partial}{\partial x_1}\right)A+v_l^+v_l^-\frac{\partial^2 A}{\partial y_1^2}=\tilde{G}A^2\bar{A} \tag{35}$$

where $\tilde{G}=2\widehat{G}/S_{\omega\omega}$. Also $v_k^{\pm}=S_{\omega\omega}^{-1}\left[-S_{\omega k}\pm\sqrt{S_{\omega k}^2-S_{\omega\omega}S_{kk}}\right]$ and $v_l^{\pm}=\pm\sqrt{-S_{ll}/S_{\omega\omega}}$ are the two values of the x-and y-group velocities, which can be expressed, near the extremum of the neutral surface. Another instructive way of writing the amplitude equation (35), using the mean group velocity in the x-direction $c=(-S_{\omega k}/S_{\omega\omega})=\left(v_k^++v_k^-\right)/2$, we may rewrite Eq. 35 in the form

$$\left(\frac{\partial}{\partial t_1}+c\frac{\partial}{\partial x_1}\right)^2 A-\omega_k^2\frac{\partial^2 A}{\partial x_1^2}-\omega_l^2\frac{\partial^2 A}{\partial y_1^2}=\tilde{G}A^2\bar{A} \tag{36}$$

where $\omega_k=\pm v_k^{\pm}\mp c=S_{\omega\omega}^{-1}\sqrt{S_{\omega k}^2-S_{\omega\omega}S_{kk}}$ and $\omega_l=\pm v_l^{\pm}$. Consequently, in a frame of reference moving with this convection velocity ($X_1=x_1-ct_1$, $T_1=t_1$), the amplitude equation (36) takes the form

$$\frac{\partial^2 A}{\partial T_1^2}-(P_1+iP_2)\frac{\partial^2 A}{\partial X_1^2}-(W_1+iW_2)\frac{\partial^2 A}{\partial y_1^2}=(G_1+iG_2)A^2\bar{A} \tag{37}$$

which is the nonlinear Klein–Gordon equation. The nonlinear evolution equation (36) or (37) has various types of solutions (Weissman 1979; Murakami 1986; Parkes 1991): (a) uniform, time-dependent wave trains, (b) steady-state, space-dependent wave trains, (c) nonlinear envelopes, or traveling wave solutions, and (d) modulational stability or linearized stability of uniform solutions. Note that in writing the above equations, we have used the following substitutions for the complex coefficients $\omega_k^2=P_1+iP_2$, $\omega_l^2=W_1+iW_2$, and $\tilde{G}=G_1+iG_2$. Let the solution of Eq. 37 varies with time only, i.e., $A=n\exp(i\sigma T_1)$, where n and σ are constants. Substituting into Eq. 37, then separating real and imaginary parts, we obtain $\sigma^2+G_1n^2=0$ and $G_2n^2=0$, and hence, we conclude that $\sigma^2=-G_1n^2$ and $G_2=0$. Since the stability constraint on the main solution is that σ should be pure imaginary, this solution is bounded when

$$G_1>0 \tag{38}$$

To study the stability criteria for Eq. 37, we perturb the solution according to

$$A = [n + \alpha(X_1, y_1, T_1) + i\beta(X_1, y_1, T_1)] \exp(i\sigma T_1) \tag{39}$$

where $\alpha(X_1, y_1, T_1)$ and $\beta(X_1, y_1, T_1)$ are unknowns to be determined. Substituting Eq. 39 into Eq. 37, and linearizing in α and β, we get

$$\begin{aligned} &-\sigma^2(n + \alpha + i\beta) + 2i\sigma\left(\frac{\partial\alpha}{\partial T_1} + i\frac{\partial\beta}{\partial T_1}\right) + \left(\frac{\partial^2\alpha}{\partial T_1^2} + i\frac{\partial^2\beta}{\partial T_1^2}\right) \\ &-(P_1 + iP_2)\left(\frac{\partial^2\alpha}{\partial X_1^2} + i\frac{\partial^2\beta}{\partial X_1^2}\right) - (W_1 + iW_2)\left(\frac{\partial^2\alpha}{\partial y_1^2} + i\frac{\partial^2\beta}{\partial y_1^2}\right) \\ &= (G_1 + iG_2)(n^3 + 3n^2\alpha + in^2\beta) \end{aligned} \tag{40}$$

Using the result $\sigma^2 = -G_1 n^2$ and $G_2 = 0$, and separating Eq. 40 into real and imaginary parts, we get

$$\begin{aligned} &\frac{\partial^2\alpha}{\partial T_1^2} - 2\sigma\frac{\partial\beta}{\partial T_1} - \left(P_1\frac{\partial^2\alpha}{\partial X_1^2} + W_1\frac{\partial^2\alpha}{\partial y_1^2}\right) + \left(P_2\frac{\partial^2\beta}{\partial X_1^2} + W_2\frac{\partial^2\beta}{\partial y_1^2}\right) - 2n^2G_1\alpha = 0 \\ &\frac{\partial^2\beta}{\partial T_1^2} + 2\sigma\frac{\partial\alpha}{\partial T_1} - \left(P_1\frac{\partial^2\beta}{\partial X_1^2} + W_1\frac{\partial^2\beta}{\partial y_1^2}\right) - \left(P_2\frac{\partial^2\alpha}{\partial X_1^2} + W_2\frac{\partial^2\alpha}{\partial y_1^2}\right) = 0 \end{aligned} \tag{41}$$

Suppose that the above system has the following solutions

$$(\alpha, \beta) = (a, b) \exp i(qX_1 + ry_1 - \Omega T_1) + c.c. \tag{42}$$

Substituting Eq. 42 into the system (41), we get

$$\begin{aligned} &a\left[-\Omega^2 + P_1q^2 + W_1r^2 - 2n^2G_1\right] + b\left[2i\sigma\Omega - P_2q^2 - W_2r^2\right] = 0 \\ &a\left[-2i\sigma\Omega + P_2q^2 + W_2r^2\right] + b\left[-\Omega^2 + P_1q^2 + W_1r^2\right] = 0 \end{aligned} \tag{43}$$

For nontrivial solution of Eqs. 43, the determinant of the coefficients of a and b should vanish. Hence, we obtain the following dispersion relation

$$\begin{aligned} &\Omega^4 - 2\Omega^2\left(P_1q^2 + W_1r^2 - 3G_1n^2\right) - 4i\sigma\Omega\left(P_2q^2 + W_2r^2\right) \\ &+\left(P_1q^2 + W_1r^2\right)\left(P_1q^2 + W_1r^2 - 2n^2G_1\right) + \left(P_2q^2 + W_2r^2\right)^2 = 0 \end{aligned} \tag{44}$$

The marginal state arises when the disturbance wave numbers q and r satisfied the following relation $P_2q^2 + W_2r^2 = 0$. Substituting this equation and the expression $P_1q^2 + W_1r^2 = q^2[P_1 - (W_1/W_2)P_2]$ into the dispersion relation (44), we obtain

$$\begin{aligned} &\Omega^4 - 2\Omega^2\left[q^2\left(P_1 - \frac{W_1}{W_2}P_2\right) - 3G_1n^2\right] \\ &+\left[q^2\left(P_1 - \frac{W_1}{W_2}P_2\right)\right]^2 - 2G_1n^2\left[q^2\left(P_1 - \frac{W_1}{W_2}P_2\right)\right] = 0 \end{aligned} \tag{45}$$

Now, applying the Routh–Hurwitz stability criterion (Zahreddine and Elshehawey 1988) which states that the system $Z' = BZ$ where B is 4×4 real matrix whose characteristic polynomial $f(\Omega) = \Omega^4 + c_1\Omega^2 + c_2$ having no zero repeated root is stable if and only if

$c_1 > 0$, $c_2 > 0$, and $c_1^2 - 4c_2 \geq 0$, where asymptotic stability occurs only when all inequalities are strict. Then, from Eq. 45, we found that the three conditions for stability are satisfied if the following condition occurs

$$q^2 \left(P_1 - \frac{W_1}{W_2} P_2 \right) < \frac{9}{4} G_1 n^2 \tag{46}$$

and since the right-hand side is positive, then this condition holds if

$$W_2 \left(W_2 P_1 - W_1 P_2 \right) > 0 \tag{47}$$

Therefore, the stability conditions are given by Eqs. 38 and 47, and these conditions must hold simultaneously for the considered system to be stable; otherwise, it is unstable if at least one of these conditions does not occur. Now, we shall discuss numerically the stability of the system under consideration by drawing the transition curves G_1 and $W_2\left(W_2 P_1 - W_1 P_2\right)$ versus the wavenumbers
k and l for different parameters including the analysis for this limiting case near the minimum of the neutral curve (at which $G_1 = 0$ and $W_2\left(W_2 P_1 - W_1 P_2\right) = 0$). Figure 1a and b shows the variation of these transition curves versus the wavenumbers k for various electric field values, and it indicates that except that at this minimum point the system is stable for small electric field values corresponding to small wavenumber range. By increasing the electric field values we found that the stability wavenumber range increases. Therefore, the electric fields have stabilizing effects in this case. The critical wavenumber value (corresponding to the minimum value of the neutral curve) is found to increase by increasing the electric field values. Figure 2a and b is drawn for different values of the medium permeability λ_1, and it is shown that the medium permeability has a stabilizing effect in the two wavenumber regions separated by the critical wavenumber and that the critical points obtained by increasing $\lambda_1$values will coincide in this case. Similar effect is found for the fluid depths h_1 and h_2 on the stability of the considered system as shown in Fig. 2a and b, but the corresponding figure is not given here. While opposite effect is found for the fluid viscosities μ_1 and μ_2 on the stability of the system to that given in Fig. 2a and b, the corresponding figure is also removed. Figure 3a and b shows the variations of the two stability conditions with the wavenumbers k for various values of fluid velocities. It is found that the system is neutrally stable, i.e., it is stable in a wavenumber range separated by a critical wavenumber value. In these two regions, it is found also that the fluid velocities have stabilizing effect in the first region as well as destabilizing effect in the second region, and the corresponding critical wavenumber values increase by increasing the fluid velocities values. Figure 4a and b shows the stability conditions versus wavenumber k for various values of surface tension coefficient T, and it indicates that at any constant value of T, the system is stable in the two regions separated by the critical wavenumber. By increasing the surface tension values, we found that the surface tension T has a destabilizing effect (since the wavenumber range for stability decreases), and the corresponding critical wavenumbers decrease by increasing T values. Similarly, for various values of the porosity of porous medium m, it shows that for small values of m, the system is always unstable and by increasing the porosity values, the system becomes stable in a wavenumber range $2 \leq k \leq 15.7$ (figure is not given here). Therefore, we conclude that the porosity of porous medium has a stabilizing effect.

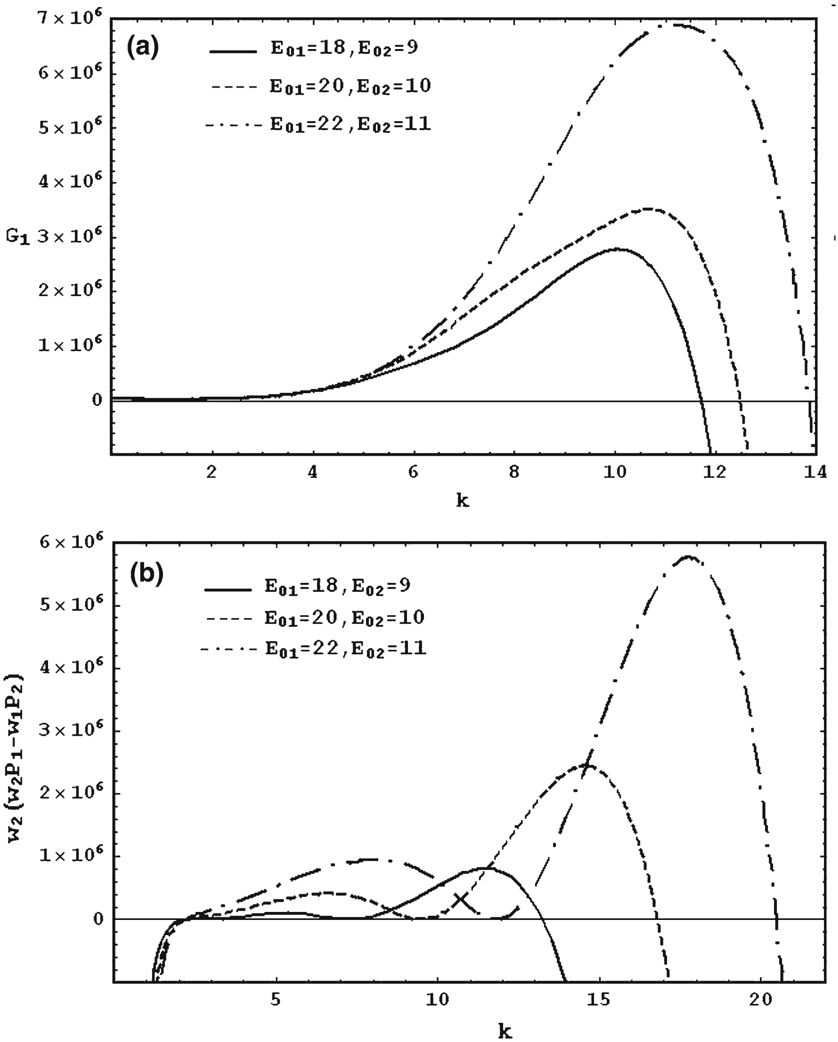

Fig. 1 Variation of **a** G_1 and **b** $W_2\,(W_2P_1 - W_1P_2)$ given by Eqs. 38 and 47 with k for various values of the electric field values $E_{01}$and E_{02}, for the system having $\rho_1 = 0.9856\,\text{g/cm}^3$, $\rho_2 = 0.0012\,\text{g/cm}^3$, $\varepsilon_1 = 1.7$ farad/cm, $\varepsilon_2 = 1.007$ farad/cm, $\mu_1 = 0.8\,\text{cm}^2/\text{s}$, $\mu_2 = 0.07\,\text{cm}^2/\text{s}$, $U_1 = 20\,\text{cm/s}$, $U_2 = 10\,\text{cm/s}$, $h_1 = 0.9\,\text{cm}$, $h_2 = 0.5\,\text{cm}$, $l = 0$, $\beta = 1$, $m = 0.5\,\text{s/cm}$, $\lambda_1 = 0.005\,\text{cm}^2$, $T = 76\,\text{dyn/cm}$ and $g = 980\,\text{cm/s}^2$, when $E_{01} = 18\,\text{V/cm}$, $E_{02} = 9\,\text{V/cm}$ (*solid*), $E_{01} = 20\,\text{V/cm}$, $E_{02} = 10\,\text{V/cm}$ (*dashed*) and $E_{01} = 22\,\text{V/cm}$, $E_{02} = 11\,\text{V/cm}$ (*dashed dotted*)

5 Conclusions

The nonlinear stability analysis of two superposed bounded dielectric fluids streaming through porous media under the effect of uniform vertical electric fields in the presence of interfacial surface charges is investigated in (2 + 1)-dimensions. Using the method of multiple scales, we obtain a general dispersion relation for the linear problem. The results in this case show that the system is always unstable for all physical parameters, and this instability increases with increasing the dimension, even in the presence of electric fields

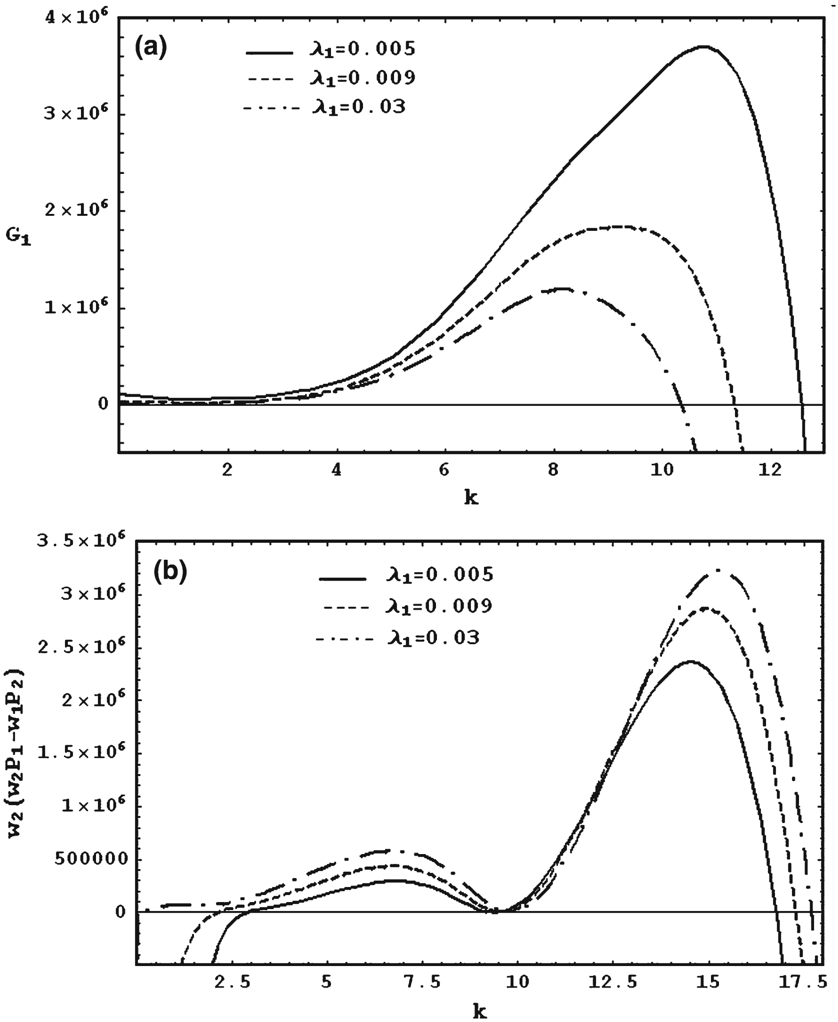

Fig. 2 Variation of **a** G_1 and **b** $W_2\,(W_2P_1 - W_1P_2)$ given by Eqs. 38 and 47 with k for various values of the permeability of the medium λ_1, for the system having $\rho_1 = 0.9856\,\text{g/cm}^3$, $\rho_2 = 0.0012\,\text{g/cm}^3$, $\varepsilon_1 = 1.7$ farad/cm, $\varepsilon_2 = 1.007$ farad/cm, $\mu_1 = 0.8\,\text{cm}^2/\text{s}$, $\mu_2 = 0.07\,\text{cm}^2/\text{s}$, $U_1 = 20\,\text{cm/s}$, $U_2 = 10\,\text{cm/s}$, $E_{01} = 20\,\text{V/cm}$, $E_{02} = 10\,\text{V/cm}$, $h_1 = 0.9\,\text{cm}$, $h_2 = 0.5\,\text{cm}$, $l = 0$, $\beta = 1$, $m = 0.5\,\text{s/cm}$, $T = 76\,\text{dyn/cm}$ and $g = 980\,\text{cm/s}^2$, when $\lambda_1 = 0.005\,\text{cm}^2$ (*solid*), $\lambda_1 = 0.09\,\text{cm}^2$ (*dashed*) and $\lambda_1 = 0.03\,\text{cm}^2$ (*dashed dotted*)

and porous medium. In the nonlinear analysis, we have obtained a nonlinear Klein–Gordon equation with complex coefficients, describing the behavior of the perturbed system at the critical point of the neutral curve. The obtained results show that:

(1) The electric fields have stabilizing effects, and the critical wavenumber value increases by increasing the electric fields.
(2) The medium permeability and fluid depths have stabilizing effects, while fluid viscosities have destabilizing effects and the critical points obtained by increasing their values coincide.

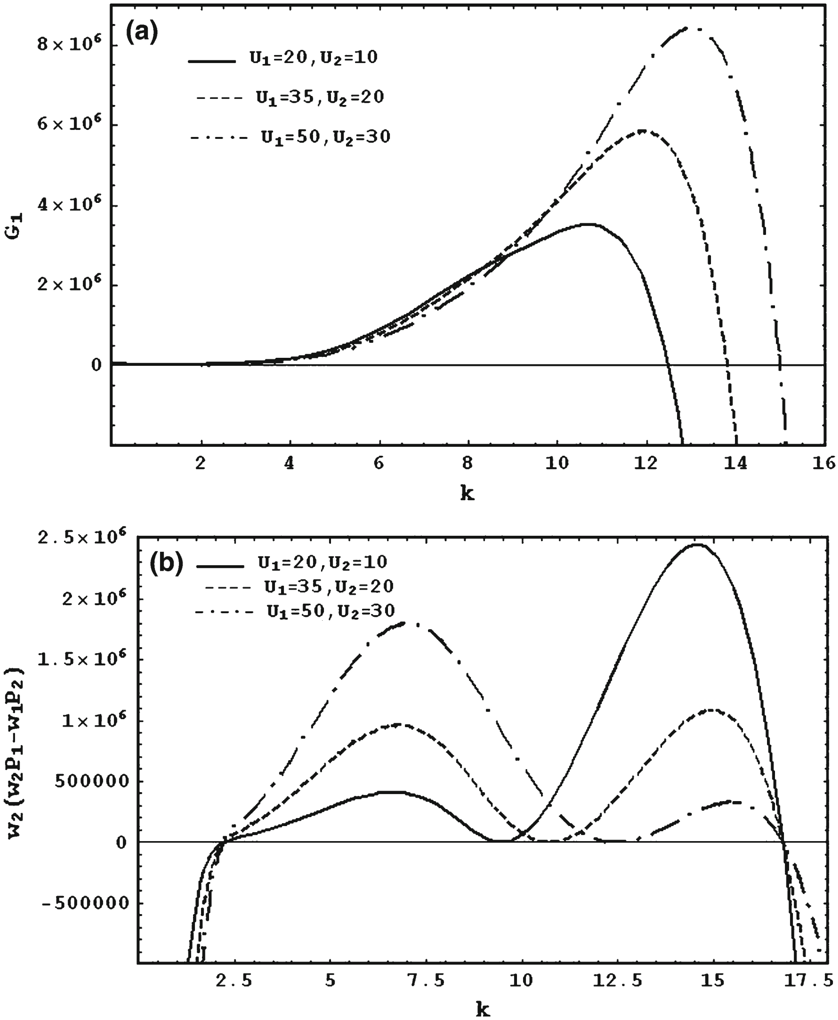

Fig. 3 Variation of **a** G_1 and **b** W_2 $(W_2P_1 - W_1P_2)$ given by Eqs. 38 and 47 with k for various values of the fluid velocities $U_1$and U_2, for the system having $\rho_1 = 0.9856\,\text{g/cm}^3$, $\rho_2 = 0.0012\,\text{g/cm}^3$, $\varepsilon_1 = 1.7$ farad/cm, $\varepsilon_2 = 1.007$ farad/cm, $\mu_1 = 0.8\,\text{cm}^2/\text{s}$, $\mu_2 = 0.07\,\text{cm}^2$ /s, $h_1 = 0.9$ cm, $h_2 = 0.5$ cm, $E_{01} = 20$ V/cm, $E_{02} = 10$ V/cm, $l = 0$, $\beta = 1$, $m = 0.5$ s/cm, $\lambda_1 = 0.005\,\text{cm}^2$ and $g = 980\,\text{cm/s}^2$, $T = 0.09$ dyn/cm when $U_1 = 20$ cm/s, $U_2 = 10$ cm/s (*solid*), $U_1 = 35$ cm/s, $U_2 = 20$ cm/s (*dashed*) and $U_1 = 50$ cm/s, $U_2 = 30$ cm/s (*dashed dotted*)

(3) The fluid velocities have stabilizing as well as destabilizing effects on the considered system, and the system has been found to be neutrally stable. The corresponding critical wavenumber values increase by increasing the fluid velocities.

(4) At any constant value of surface tension, the system has been found to be stable, and the surface tension has been found to have a destabilizing effect and the critical wavenumbers decrease by increasing surface tension values.

(5) For small values of the porosity of porous medium, the system is always unstable, and the porosity of porous medium has been found to have a stabilizing effect.

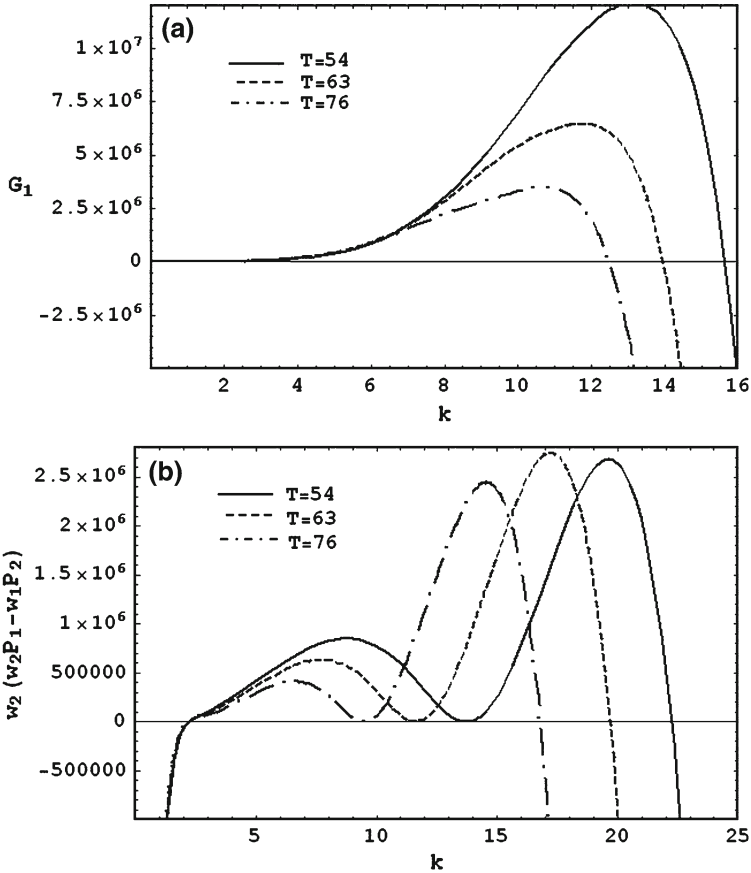

Fig. 4 Variation of **a** G_1 and **b** W_2 $(W_2 P_1 - W_1 P_2)$ given by Eqs. 38 and 47 with k for various values of the surface tension T, for the system having $\rho_1 = 0.9856\,\text{g/cm}^3$, $\rho_2 = 0.0012\,\text{g/cm}^3$, $\varepsilon_1 = 1.7$ farad/cm, $\varepsilon_2 = 1.007$ farad/cm, $\mu_1 = 0.8\,\text{cm}^2/\text{s}$, $\mu_2 = 0.07\,\text{cm}^2/\text{s}$, $U_1 = 20$ cm/s, $U_2 = 10$ cm/s, $h_1 = 0.9$ cm, $h_2 = 0.5$ cm, $E_{01} = 20$ V/cm, $E_{02} = 10$ V/cm, $l = 0$, $\beta = 1$, $m = 0.5$ s/cm, $\lambda_1 = 0.005\,\text{cm}^2$ and $g = 980\,\text{cm/s}^2$, when $T = 54$ dyn/cm (*solid*), $T = 63$ dyn/cm (*dashed*) and $T = 76$ dyn/cm (*dashed dotted*)

Appendix

The solutions of the first-order problem are

$$
\begin{aligned}
\eta_1 &= A\exp(i\theta) + c.c. \\
\Phi_{j1} &= \pm\frac{i(kU_j - m\omega)}{K\sinh Kh_j}\cosh K(z \pm h_j)A\exp(i\theta) + c.c. + p_{j1} \\
\Psi_{j1} &= \pm\left[1 + \frac{2i\left[\mu_2(kU_2 - m\omega) - \mu_1(kU_1 - m\omega)\right]}{mE_{0j}(\varepsilon_2 E_{02} - \varepsilon_1 E_{01})}\right] \\
&\quad\times\frac{E_{0j}\sinh K(z \pm h_j)}{\sinh Kh_j}A\exp(i\theta) + c.c.
\end{aligned}
$$

The solutions of the second-order problem are

$$
\begin{aligned}
\eta_2 &= \Lambda A^2 \exp(2i\theta) + c.c. + B_0 \\
\Phi_{j2} &= \left\{ \frac{\cosh K(z \pm h_j)}{K^2 \sigma_j} \left[\pm \frac{K}{\cosh K h_j} \left(m \frac{\partial A}{\partial t_1} + U_j \frac{\partial A}{\partial x_1} \right) \right.\right. \\
&\quad \left. + i(K h_j + \sigma_j) \left(k \frac{\partial C_j}{\partial x_1} + l \frac{\partial C_j}{\partial y_1} \right) \right] - \frac{i \sinh K(z \pm h_j)}{K} (z \pm h_j) \\
&\quad \times \left. \left(k \frac{\partial C_j}{\partial x_1} + l \frac{\partial C_j}{\partial y_1} \right) \right\} \exp(i\theta) \mp \frac{(1 - \sigma_j^2) \cosh 2K(z \pm h_1)}{2K\sigma_1} \\
&\quad \times \left\{ K^2 C_j \cosh K h_j + i \Lambda A (m\omega - kU_j) \right\} A \exp(2i\theta) \\
&\quad + c.c. + p_{j2}, \quad j = 1, 2 \\
\Psi_{j2} &= \left\{ \pm \frac{2 \sinh K(z \pm h_j)}{m(\varepsilon_2 E_{02} - \varepsilon_1 E_{01}) \sinh K h_j} \left[(\mu_2 U_2 - \mu_1 U_1) \frac{\partial A}{\partial x_1} \right.\right. \\
&\quad \left. + m(\mu_2 - \mu_1) \frac{\partial A}{\partial t_1} \right] - \frac{i}{K} \left(k \frac{\partial S_j}{\partial x_1} + l \frac{\partial S_j}{\partial y_1} \right) \left[(z \pm h_j) \cosh K(z \pm h_j) \right. \\
&\quad \left.\left.\left. - h_j \coth K h_j \sinh K(z \pm h_j) \right]\right]\right\} \exp(i\theta) \\
&\quad \mp \frac{\sinh 2K(z \pm h_j)}{4(\varepsilon_2 E_{02} - \varepsilon_1 E_{01}) \sinh 2K h_j} \left\{ K \left[\varepsilon_1 S_1^2 \sinh 2K h_1 \right.\right. \\
&\quad \left. + \varepsilon_2 S_2^2 \sinh 2K h_2 \right] \pm 2AK [S_{3-j} \varepsilon_{3-j} E_{03-j} \cosh K h_{3-j} \\
&\quad + S_j (2\varepsilon_{3-j} E_{03-j} - 3\varepsilon_j E_{0j}) \cosh K h_j] \\
&\quad - \frac{4iA^2}{m} \left[\frac{\mu_1 (kU_1 - m\omega)(K - 4\Lambda\sigma_1)}{\sigma_1} + \frac{\mu_2 (kU_2 - m\omega)(K + 4\Lambda\sigma_2)}{\sigma_2} \right] \\
&\quad \left. - 4A^2 \Lambda E_{0j} (\varepsilon_2 E_{02} - \varepsilon_1 E_{01}) \right\} \exp(2i\theta) + c.c.
\end{aligned}
\tag{48}
$$

where the quantity B_0 represents the induced mean motion or the zero frequency correction to slow modulation of the fundamental mode, $C_j = \pm i A(kU_j - m\omega)/[iA(kU_j - m\omega)]$ and S_j $(j = 1, 2)$, Λ are given by

$$
\begin{aligned}
S_j &= \pm \frac{E_{0j}}{\sinh K h_j} \left[1 + \frac{2i \left[k(\mu_2 U_2 - \mu_1 U_1) - m\omega (\mu_2 - \mu_1) \right]}{m E_{0j} (\varepsilon_2 E_{02} - \varepsilon_1 E_{01})} \right] \\
\Lambda &= \frac{1}{S(2\omega, 2K)} \left\{ \frac{i \left(8\lambda_1 K^2 + m \right)}{2\lambda_1 m} \left[\frac{\mu_2 \left(1 + \sigma_2^2 \right)(kU_2 - m\omega)}{\sigma_2^2} \right.\right. \\
&\quad \left. - \frac{\mu_1 \left(1 + \sigma_1^2 \right)(kU_1 - m\omega)}{\sigma_1^2} \right] + \frac{1}{2m^2} \left[\frac{\rho_2 \left(\sigma_2^2 - 3 \right)(kU_2 - m\omega)^2}{\sigma_2^2} \right. \\
&\quad \left. - \frac{\rho_1 \left(\sigma_1^2 - 3 \right)(kU_1 - m\omega)^2}{\sigma_1^2} \right] - \frac{iK^2}{m(\varepsilon_2 E_{02} - \varepsilon_1 E_{01})} \left[\frac{\varepsilon_1 E_{01} \left(1 + \sigma_1^2 \right)}{\sigma_1} \right. \\
&\quad \left. + \frac{\varepsilon_2 E_{02} \left(1 + \sigma_2^2 \right)}{\sigma_2} \right] \left[\frac{\mu_1 (kU_1 - m\omega)}{\sigma_1} + \frac{\mu_2 (kU_2 - m\omega)}{\sigma_2} \right]
\end{aligned}
$$

$$-\frac{4iK^2\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]}{m(\varepsilon_2E_{02}-\varepsilon_1E_{01})}\left[\frac{\varepsilon_2E_{02}\left(1+\sigma_2^2\right)}{\sigma_2^2}\right.$$

$$\left.-\frac{\varepsilon_1E_{01}\left(1+\sigma_1^2\right)}{\sigma_1^2}\right]+\frac{iK^2\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]}{m(\varepsilon_2E_{02}-\varepsilon_1E_{01})^2}$$

$$\times\left[\frac{\varepsilon_1E_{01}}{\sigma_1}+\frac{\varepsilon_2E_{02}}{\sigma_2}\right]\left[\frac{\varepsilon_1E_{01}\left(1+\sigma_1^2\right)}{\sigma_1}+\frac{\varepsilon_2E_{02}\left(1+\sigma_2^2\right)}{\sigma_2}\right]$$

$$+\frac{2K^2\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]^2}{m^2(\varepsilon_2E_{02}-\varepsilon_1E_{01})^2}\left[\frac{\varepsilon_2\left(1+\sigma_2^2\right)}{\sigma_2^2}\right.$$

$$\left.-\frac{\varepsilon_1\left(1+\sigma_1^2\right)}{\sigma_1^2}\right]-\frac{2K^2\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]^2}{m^2(\varepsilon_2E_{02}-\varepsilon_1E_{01})^3}$$

$$\times\left[\frac{\varepsilon_1}{\sigma_1}+\frac{\varepsilon_2}{\sigma_2}\right]\left[\frac{\varepsilon_1E_{01}\left(1+\sigma_1^2\right)}{\sigma_1}+\frac{\varepsilon_2E_{02}\left(1+\sigma_2^2\right)}{\sigma_2}\right]$$

$$\left.+\frac{1}{2}K^2\left[\frac{\varepsilon_2E_{02}^2\left(\sigma_2^2-3\right)}{\sigma_2^2}-\frac{\varepsilon_1E_{01}^2\left(\sigma_1^2-3\right)}{\sigma_1^2}\right]\right\}.$$

The coefficient G appearing in Eq. 24 is defined as

$$G=(3/2)K^4T+2K^3\left[\left(\varepsilon_1E_{01}^2/\sigma_1^3\right)\left(1-2\sigma_1^2\right)+\left(\varepsilon_2E_{02}^2/\sigma_2^3\right)\left(1-2\sigma_2^2\right)\right]$$

$$+\frac{2K}{m^2}\left[\left(\rho_1/\sigma_1^3\right)\left(1-2\sigma_1^2\right)\left(kU_1-m\omega\right)^2+\left(\rho_2/\sigma_2^3\right)\left(1-2\sigma_2^2\right)\left(kU_2-m\omega\right)^2\right]$$

$$-\frac{iK}{2\lambda_1}\left[\left(\mu_1/\sigma_1^3\right)\left(kU_1-m\omega\right)\left(2-\sigma_1^2\right)+\left(\mu_2/\sigma_2^3\right)\left(kU_2-m\omega\right)\left(2-\sigma_2^2\right)\right]$$

$$-\frac{2iK^3\left[\mu_2\left(kU_2-m\omega\right)\varepsilon_1E_{01}-\mu_1\left(kU_1-m\omega\right)\varepsilon_2E_{02}\right]\left(\sigma_1+\sigma_2\right)}{m(\varepsilon_2E_{02}-\varepsilon_1E_{01})^3\sigma_1^3\sigma_2^3}$$

$$\times\left\{\left[4\sigma_1^2\sigma_2^2(\varepsilon_2E_{02}-\varepsilon_1E_{01})^2-(\varepsilon_1\sigma_1E_{01}+\varepsilon_2\sigma_2E_{02})^2\right]\right\}$$

$$+\frac{8K^3\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]^2}{m^2(\varepsilon_2E_{02}-\varepsilon_1E_{01})^5}\left[\varepsilon_2^4\left(E_{02}/\sigma_2\right)^3\left(1-\sigma_2^2\right)\right.$$

$$\left.-\varepsilon_1^4\left(E_{01}/\sigma_1\right)^3\left(1-\sigma_1^2\right)\right]+\frac{4iK^3\varepsilon_1\varepsilon_2\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]^3}{m^3(\varepsilon_2E_{02}-\varepsilon_1E_{01})^5}$$

$$\times\frac{(\sigma_1+\sigma_2)(\varepsilon_2\sigma_1+\varepsilon_1\sigma_2)}{\sigma_1^3\sigma_2^3}\left[E_{02}\sigma_2\left(1-\sigma_1^2\right)+E_{01}\sigma_1(1-\sigma_2^2)\right]$$

$$+\frac{2K^3\varepsilon_1\varepsilon_2\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]}{m^2(\varepsilon_2E_{02}-\varepsilon_1E_{01})^5\sigma_1^3\sigma_2^3}\left\{\mu_2\Big(kU_2-m\omega\Big)\right.$$

$$\times\left[2\varepsilon_2^2\sigma_2^2E_{02}^3\left(\sigma_2-\sigma_1\right)\left(1-\sigma_1^2\right)-\sigma_1\varepsilon_1^2E_{01}^3\left(3\sigma_1^2+\sigma_2^2\right)\left(1-\sigma_2^2\right)\right]$$

$$-\mu_1\left(kU_1-m\omega\right)\left[2\varepsilon_1^2\sigma_1^2E_{01}^3\left(\sigma_2-\sigma_1\right)\left(1-\sigma_2^2\right)-\sigma_2\varepsilon_2^2E_{02}^3\left(3\sigma_2^2+\sigma_1^2\right)\left(1-\sigma_1^2\right)\right.$$

$$+E_{01}E_{02}\mu_2\left(kU_2-m\omega\right)\langle\varepsilon_1E_{01}\left\{\varepsilon_1\sigma_2[\sigma_1^3\left(\sigma_1+4\sigma_2\right)+\sigma_1^2\left(1-9\sigma_2^2\right)+11\sigma_2^2\right]$$

$$+\varepsilon_2\sigma_1\left[\sigma_1^2\left(11-9\sigma_2^2\right)+2\sigma_1\sigma_2\left(1+\sigma_2^2\right)+\left(3-\sigma_2^2\right)\sigma_2^2\right]\}$$

$$+\varepsilon_2 E_{02}\left\{\varepsilon_1\sigma_2\left[\left(7\sigma_1^2-9\right)\sigma_2^2+2\sigma_1\sigma_2\left(1-3\sigma_1^2\right)-\left(1+\sigma_1^2\right)\sigma_1^2\right]\right.$$
$$\left.+2\varepsilon_2\sigma_1\left[\sigma_1^2\left(5\sigma_2^2-6\right)-\sigma_1\sigma_2\left(1+\sigma_2^2\right)-\sigma_2^2\right]\right\}\rangle+E_{01}E_{02}\mu_1\left(kU_1-m\omega\right)$$
$$\times\langle\varepsilon_1 E_{01}\left\{2\varepsilon_1\sigma_2[\sigma_2^2\left(5\sigma_1^2-6\right)-\sigma_1\sigma_2\left(1+\sigma_1^2\right)-\sigma_1^2]+\varepsilon_2\sigma_1[\left(7\sigma_2^2-9\right)\sigma_1^2\right.$$
$$\left.+2\sigma_1\sigma_2\left(1-3\sigma_2^2\right)-\left(1+\sigma_2^2\right)\sigma_2^2]\right\}+\varepsilon_2 E_{02}\left\{\varepsilon_1\sigma_2[\sigma_2^2\left(11-9\sigma_1^2\right)+2\sigma_1\sigma_2\left(1+\sigma_1^2\right)\right.$$
$$\left.\left.+\left(3-\sigma_1^2\right)\sigma_1^2]+\varepsilon_2\sigma_1\left[\sigma_2^3\left(\sigma_2+4\sigma_1\right)+\sigma_2^2\left(1-9\sigma_1^2\right)+11\sigma_1^2\right]\right\}\rangle\right\}$$
$$+\Lambda\left\{K^2\left[\varepsilon_2\left(E_{02}/\sigma_2\right)^2\left(3-\sigma_2^2\right)-\varepsilon_1\left(E_{01}/\sigma_1\right)^2\left(3-\sigma_1^2\right)\right]\right.$$
$$+\frac{1}{m^2}\left[\left(\rho_2/\sigma_2^2\right)\left(3-\sigma_2^2\right)\left(kU_2-m\omega\right)^2-\left(\rho_1/\sigma_1^2\right)\left(3-\sigma_1^2\right)\left(kU_1-m\omega\right)^2\right]$$
$$-\frac{i}{\lambda_1}\left[\left(\mu_2/\sigma_2^2\right)\left(kU_2-m\omega\right)\left(2+\sigma_2^2\right)-\left(\mu_1/\sigma_1^2\right)\left(kU_1-m\omega\right)\left(2+\sigma_1^2\right)\right]$$
$$-\frac{2iK^2\left[\mu_2\left(kU_2-m\omega\right)\varepsilon_1 E_{01}-\mu_1\left(kU_1-m\omega\right)\varepsilon_2 E_{02}\right]\left[\sigma_1+\sigma_2\right]}{m(\varepsilon_2 E_{02}-\varepsilon_1 E_{01})^2\sigma_1^2\sigma_2^2}$$
$$\times\left\{\varepsilon_1 E_{01}\left[\sigma_1\left(2+\sigma_2^2\right)+3\sigma_2\right]+\varepsilon_2 E_{02}\left[3\sigma_1+\left(2+\sigma_1^2\right)\sigma_2\right]\right\}$$
$$-\frac{8K^2\varepsilon_1\varepsilon_2\left[\mu_2\left(kU_2-m\omega\right)-\mu_1\left(kU_1-m\omega\right)\right]^2\left[\sigma_1+\sigma_2\right]}{m^2(\varepsilon_2 E_{02}-\varepsilon_1 E_{01})^3\sigma_1^2\sigma_2^2}$$
$$\left.\times\left\{\sigma_2 E_{02}\left(1-\sigma_1^2\right)+\sigma_1 E_{01}\left(1-\sigma_2^2\right)\right\}\right\}.$$

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