

# Poroelasticity-I: Governing Equations of the Mechanics of Fluid-Saturated Porous Materials

Sergey L. Lopatnikov · John W. Gillespie Jr.

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**Abstract** We present here the approach to the theory of fluid-filled poroelastics based on consideration of poroelastics as a continuum of “macropoints” (representative elementary volumes), which “internal” states can be described by as a set of internal parameters, such as local relative velocity of fluid and solid, density of fluid, internal strain tensor, specific area, and position of the center of mass of porous space. We use the generalized Cauchy–Born hypothesis and suggest that there is a system of (structural) relationships between external parameters, describing the deformation of the continuum and internal parameters, characterizing the state of representative elementary volumes. We show that in nonhomogeneous (and, particularly, nonlinear) poroelastics, an interaction force between solid and fluid appears. Because this force is proportional to the gradient of porosity, absent in homogeneous poroelastics, and one can neglect with dynamics of internal degrees of freedom, this force is equivalent to the interaction force, introduced earlier by Nikolaevskiy from phenomenological reasons. At last, we show that developed theory naturally incorporates three mechanisms of energy absorption: visco-inertial Darcy mechanism, “squirt flow” attenuation, and skeleton attenuation.

**Keywords** Poroelasticity · Dynamics · Waves · Porous continuum

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S. L. Lopatnikov · J. W. Gillespie Jr.  
Center for Composite Materials, University of Delaware, Newark, DE 19716, USA

J. W. Gillespie Jr.  
Department of Materials Science, University of Delaware, Newark, DE 19716, USA

J. W. Gillespie Jr.  
Department of Civil and Environmental Engineering, University of Delaware,  
Newark, DE 19716, USA

S. L. Lopatnikov (✉)  
Center for Composite Materials, 201 Composites Manufacturing Science Laboratory,  
Newark, DE 19711, USA  
e-mail: lopatnik@udel.edu

## 1 Introduction

The purpose of this paper is to reformulate in a more systematic manner and in a somewhat more general context the mechanics of fluid-saturated porous media and to present this in a form that can be consecutively applied for nonhomogeneous and nonlinear materials with arbitrary equations of state.

### 1.1 Motivation and Goals

The field of applications of poroelastic theory is growing fast. Initially, a significant number of works dealt with geophysical and civil engineering problems and marine acoustics. Today, penetrable poroelastics are considered to be an appropriate mechanical image of living tissues: they find applications as absorbers of sound, are considered as possible protective materials able to mitigate the damage caused by blast impact, have impressive potential in the design of new smart materials, and so on. As a result, it has become necessary to expand current poroelastic theory to provide stronger physical consistency and universality. The theory must be enough general to consider constituents with arbitrary equations of state and must be applicable to arbitrary nonhomogeneous and anisotropic materials as well linear as nonlinear.

However, in spite of a tremendous number of publications in this field, the discussion continues about physical background of the poroelastic theory. Even the form of basic governing equations are sufficiently different (as we will discuss later in the article) in frame of different approaches that one can find in literature. It seems that there is no final agreement about consistency of proposed different approaches. Such key questions of theory as interfacial conditions that must be used on the boundaries between materials with different properties or on the moving shock fronts also are under question (de la Cruz and Spanos 1989; Gurevich and Schoenberg 1999). This disturbing incompleteness of the poroelastic theory is troubling. It was correctly stated by Gurevich and Schoenberg (1999), in respect to the problem of consistent interfacial conditions, who made an attempt to address the problem of boundary conditions in fluid-filled poroelastics, based on the Biot's theory (Gurevich): "Their [de la Cruz and Spanos—SL] concern, if justified, could throw into doubt all the theoretical and numerical results based on the interface conditions of Deresiewicz and Skalak ...."<sup>1</sup> It is even more correct in respect to agreement about basic governing equations.

To our view, a physically consistent theory must have solid background in general physical laws and, particularly, pass through several relatively simple mental tests and checks.

First, theory must lead to total mechanical momentum conservation law independently of the type of the material being considered: homogeneous or nonhomogeneous (and, particularly, nonlinear). At the same time, theory must lead to correct momentum conservation law for the fluid.

Second, theory must allow us to solve simple test problems. For example, in frame of elementary physics, one can immediately answer the following question: Consider a ball of volume  $\Omega$  with rigid (or elastic) walls. One fills this ball with mass  $M$  of a fluid with known equation of state. What will be physical state of the fluid and what will be the force acting on the walls of the ball? This problem has an obvious solution: one simply has to calculate the density of fluid and find the state and pressure from a known equation of state. However, can one solve, for example, in frame of the most common "Biot's theory," a similar problem: Let consider rigid ball that is initially filled with a compressible homogeneous "sponge" with

<sup>1</sup> Deresiewicz and Skalak (1963) proposed the boundary conditions that are the most commonly used now in the theory of poroelasticity.

initial porosity  $\varphi_0$ . The questions are: in which state will the fluid of mass  $M$  filling the porous space in equilibrium be? What porosity will the sponge have at equilibrium and what force will act on the wall of the ball? Will this force depend on the fact that a sponge is perfectly bonded to the wall of the ball or only limited in its expansion? It is obvious that consistent theory must answer such questions. However, in the frame of Biot's theory, it is impossible to solve this problem consecutively. The point is that, as correctly stated by Wilmanski (1998): "M. A. Biot proposed an equation for porosity in which the flux was identified with the relative velocity and the source (relaxation) was absent." In the meantime, in application to the formulated problem, the displacements or velocities of constituents cannot be introduced. This is an indicator of the serious physical limitations of Biot's theory.

Third, it is obvious that interfacial conditions even in a linear case must follow from the general boundary conditions, which, in turn, must follow from governing equations of the general theory of poroelasticity. In particular, the ability of the theory to provide correct "linear" boundary conditions is also an important check of physical consistency of the theory.

Unfortunately, to the best of our best knowledge, there are no publications in which the consistency checks of the theory are presented or even discussed. This article is the first in the series that will address these issues.

## 1.2 State of the Art

Many approaches to modeling the mechanical behavior of poroelastics have been proposed. [Berryman \(2005\)](#) separated all of these approaches into four groups: (1) effective medium theory, (2) mixture theory, (3) multiscale homogenization, and (4) volume averaging. Also, it is necessary to recall the theory of the multiple scattering of waves on the material nonhomogeneities that provides upscaling of the material properties, which we consider as a fifth group, and has, however, significant specifics: it is based on one of the theories from first four groups that play the role of a "seed theory."

One can consider as examples the approaches of the first group of papers by [Kastler and Toksoz \(1974\)](#), [Milton \(2002\)](#), and [Berryman \(1992\)](#), [Berryman and Wang \(1995\)](#) as typical examples of mixture theory, the first publication in the field by [Frenkel \(1944\)](#), [Rakhmatulin \(1956\)](#), [Biot \(1956\)](#), [Biot and Willis \(1957\)](#), [Biot \(1961\)](#), [Biot \(1973\)](#), [Bedford and Drumheller \(1979\)](#), [Berryman and Thigpen \(1985\)](#), [de Boer \(2000\)](#), [Wilmanski \(1998\)](#), and examples of multiscale homogenization are given by well-known works of [Auriault and Sanchez-Palencia \(1977\)](#), [Sanchez-Palencia \(1980\)](#), [Jikov et al. \(1994\)](#), and [Gilbert and Lin \(1997\)](#), [Gilbert \(2000\)](#). At last, examples of approaches based on direct volume averaging of microequations are given by works of [Nikolaevskiy et al. \(1970\)](#), [Nikolaevskiy \(1984\)](#), [Nigmatullin \(1990\)](#), and [Milton \(2002\)](#); works based on microaveraging of energy include those of [Lopatnikov \(1995\)](#), [Lopatnikov and Cheng \(1998, 2002, 2004\)](#), and [Lopatnikov et al. \(2003\)](#).

Theory of the multiple scattering of waves on material nonhomogeneities, which provides upscaling of material properties, are presented by works of [Lopatnikov and Gurevich \(1985\)](#), [Gurevich and Lopatnikov \(1995\)](#), [Lopatnikov and Gorbachev \(1987\)](#), [Gorbachev et al. \(1990\)](#), [Berryman \(1992\)](#).

Obviously, we cannot mention all the works devoted to the problem. The body of appropriate literature includes thousands of published papers and dozens of monographs. One can find a more extensive discussion of different approaches, presented in [Lopatnikov and Cheng \(2004\)](#) and cited works of [Berryman \(2005\)](#). Later reviews are presented by [Schanz \(2009\)](#) and [Nikolaevskiy \(2005\)](#).

### 1.3 The Problems and Goal

In this article, we will view the most general form of governing dynamic equations on poroelasticity, consider and discuss appropriate conservation equations, and compare developed theory with other approaches. We will show also that interaction force introduced by [Nikolaevskiy et al. \(1970\)](#) is universal and naturally appears in frame of our theory. However, if the internal dynamics of REV is significant, it has a more complex structure in comparison with Nikolaevskiy's suggestion.

In future publications, we will specifically address the problem of equilibrium of porous material with compressible matrix with fluid, obtain interfacial boundary conditions for shock fronts and boundaries between two porous materials, come to canonical representation for linear waves in homogeneous and nonhomogeneous material, and investigate the relationship between this canonical representation of wave propagation and theory of (effective) materials with memory ([Hanyga and Rok 2000](#)).

### 1.4 Approach

The principal point of the approach presented in this article, which differentiates it from most early works, is the specific view on the modeling of porous materials.

We approach the porous material as *one* continuum of material points, each with an “internal structure.” Deformation of this continuum represents deformation of the smoothed (solid) matrix of the material, identified by the movement of the centers of masses of the “material point” (or REVs), having some internal structure. Thus, one can consider presented theory as generalization of the mechanics of continua with internal degrees of freedom such as Cosserat (micropolar) continuum ([Cosserat and Cosserat 1909](#); [Eringen 1968](#)), continuum with voids ([Dunin and Surkov 1979](#); [Cowin and Nunziato 1983](#)), micromorphic Mindlin's continuum ([Mindlin and Tiersten 1962](#); [Mindlin 1964](#)).

This view leads to a clear distinction between “external parameters” characterizing the deformation of the continuum as a whole and “internal parameters” that characterize the state and internal dynamics of the REVs.

If the number of external parameters is fixed by the fact that the REVs are associated with the points of the macro-continuum, the number and character of internal parameters differ, depending on the “depth of the model” and construction of the continuum.

In terms of a generalization of the Cauchy–Born hypothesis (See for example, [Eringen 1968](#); [Ball 1977](#); [Ericksen 1984](#)), widely used in continuum theory of crystals to frame our approach: we suggest (in frame of local theory) that a full set of parameters fully characterizing the state of “macro-point” (REV) includes only internal parameters plus coefficients of the gradient of deformation in appropriate point of macro-continuum.

In this context, in contrast to our approach, we previously presented ([Lopatnikov and Cheng 1998, 2002, 2004](#)) that the averaging procedure is an important but limited part of the theory. Its specific goal in the frame of the presented theory is to identify the relationship between (variations of) the external and introduced internal parameters.

Thus, the problem of structural modeling of the material consists of these steps.

- Identify the material points (REVs) of the material.
- Identify a set of internal parameters (in accordance with external parameters are universal and follow from representation of the material as the continuum)
- Establish relationships between macro-deformation and changes of the internal parameters

**REV.** The key idea of the presented poroelastic approach (as in the theory of composites) is the concept of REV. This term was first used by Bear (1972, 1993).

The common view on the REV is completely phenomenological: “In order to establish a given porous medium’s properties, we are going to have to measure samples of the porous medium. If the sample is too small, the readings tend to oscillate. As we increase the sample size, the oscillations begin to dampen out. Eventually the sample size will become large enough that we begin to get consistent readings. This sample size is referred to as the representative elementary volume.”

This definition picks up two important ideas: the relationship between the type of measurement and definition of REV size and suggestion that fluctuations of a taken value must decrease with increasing of the size of specimen (which is a strong enough condition, but cannot always be satisfied), this definition is not completely satisfactory for theory. An attempt to provide a more rigorous definition of REV is presented by Evesque (2000). Evesque based his approach on the calculation of correlation function for the parameters, characterizing the structure of the material (porosity, density), and suggested that appropriate correlation functions tend to zero at infinity and, thus, one can define characteristic size on which correlation is negligibly small. As a result, one can consider values defined for volumes separated by such distance as independent and can consider appropriate volumes as REV’s.

**External parameters.** As mentioned previously, external parameters appear as a result of the deformation of the continuum, each point of which represents some REV. If one considers an arbitrary smooth deformation of continuum:  $\xi_S = \xi_S(\mathbf{X}, t)$ , where  $\xi_S$  is a coordinate of the material point (REV)  $\mathbf{X}$  in its deformed state, locally this deformation can be described by the deformation gradient  $\widehat{\Gamma}_{\mathbf{X}}(\bar{\mathbf{X}}) = \left\| \left( \frac{\partial \xi_{Si}}{\partial X_j} \right)_{\mathbf{X}} \right\|$ . In frame of an extended Cauchy–Born hypothesis, gradient of deformation is the only value which a material point can “feel,” if long-range interactions such as electric or gravity fields are absent.

The deformation gradient contains nine coefficients–functions, whose form is restricted by the condition that all of them are appropriate derivatives of only three functions:  $\xi_S = \xi_S(\mathbf{X}, t)$ .

The major statement of the presented theory is that nine coefficients of the deformation gradient (and, probably, their time derivatives) completely represent the information available for a given REV about the behavior of the body as a whole. This includes both information about the deformation of the neighborhood of the given REV and information about changes of its spatial orientation. Thus, external parameters completely describe the *mechanical state of the environment* of a given REV.

However, because the state of REV cannot depend on its orientation in space, the potential part of REV’s energy cannot depend on the part of deformation gradient defining rotation of REV as a solid body. Thus, potential energy of REV can depend only on associated strain tensor  $\widehat{\Sigma}(\mathbf{X})$  having, due to symmetry, only six components.

**Internal parameters.** If external parameters reflect the continual nature of the REV environment, then internal parameters represent the “construction” of the REV itself. Examples of internal degrees of freedom are relative velocity of fluid in the REV, porosity, microstrain of solid matrix, density of fluid, etc. The number of such internal degrees of freedom depends only on the structure of REV’s and “depth” of the modeling and, thus, must be defined from practical reasons.

The set of internal parameters represent the finite-dimensional model of the REV. Dependent on considering physics, one can take in account, for example, one can add the coordinate of the “geometrical center” of porosity in taken REV  $\xi_{\varphi}(\mathbf{X}, t)$  as an independent parameter and characterize its position in respect to the center of REV mass by vector of relative

displacement:  $\Delta \xi(\mathbf{X}, t) = \xi_\varphi(\mathbf{X}, t) - \xi_S(\mathbf{X}, t)$ . Similarly, one can introduce as the internal parameter orientation of major axes of the momentum of inertia of REV. In some problems, such as the destruction of the material, if REV's energy depends on a specific area of internal space, one can consider also the average specific surface within the REV as a parameter of state, etc.

**Constitutive relationships for REVs.** In the scope of this article, we will restrict ourselves to the *local theory*, which means that no long-range fields are significant and thus, only external and internal parameters of an REV take part in the constitutive equations. However, an appropriate generalization for the non-local case is also possible in the frame of the logic presented here.

First, it is necessary to introduce a set of relationships describing how internal and external degrees of freedom of REV are related with each other. We will refer to such relationships as "constitutive relationships."

The formulation of these relationships is a problem of micromodeling the material REVs and a wide range of approaches can be used to accomplish this task.

The most general way to introduce this structural relationship is to define it for variations of external and internal parameters. In such incremental form, the theory can describe reversible (potential) and irreversible (non-potential) deformations of porous materials. Moreover, these structural relationships are defined between variations of parameters, therefore they can always be considered to be linear.

**Procedure.** The physical theory is completed as governing equations are formulated. The most general approach to obtain these governing equations is the Minimal Action Principle and its associated variational procedure. We will use this principle to get an appropriate set of governing equations.

Such a general approach to the theory of fluid-filled poroelastics provides the bridge between the multiscale modeling of complex materials and the theory of poroelasticity and, also, between the theory of poroelasticity and numerical modeling.

## 2 Outline of Lagrangian Approach to the Mechanics of Fluid-filled Poroelastics

In this section, we outline the Lagrangian approach to the mechanics of fluid-filled poroelastics, using described above approach. This procedure is slightly different and more general than used by us earlier (Lopatnikov and Cheng 2002, 2004).

### 2.1 Action of the Materials With Internal Structure and Outline of Variational Procedure

This follows the classic formulation of the mechanics and is based on consideration of the action of continuously distributed in space material:

$$A = \int_{t_0}^{t_1} \mathcal{L} dt \quad (1)$$

We will identify the movement of porous body as a continuum with movement of in solid matrix. Considering the body as a set of REVs. In absence of long-range interactions, one can presented Lagrangian  $L$  in terms of the mass density of Lagrangian  $l(\mathbf{x}, t)$ :

$$\mathcal{L}(t) = \iiint_M \mathcal{L}(\mathbf{x}, t) dm \quad (2)$$

In case of two-phase material, “micro-equations” 1, 2 must be replaced with their averaged equivalents:

$$\langle A \rangle = \int_{t_0}^{t_1} \langle L \rangle dt; \tag{3}$$

$$\langle \mathcal{L} \rangle (t) = \sum_{\mathbf{x}_i} \langle \mathcal{L}(\mathbf{x}, t) \rangle_{\mathbf{x}_i} \cdot \Delta M_{\mathbf{x}_i} \equiv \sum_{\mathbf{x}_i} \left( \frac{1}{\Delta m_{\mathbf{x}_i}} \iiint_{\Omega_{\mathbf{x}_i}} \mathcal{L}(\mathbf{x}, t) dm_{\mathbf{x}_i} \right) \cdot \Delta m_{\mathbf{x}_i} \tag{4}$$

Here  $M_{\mathbf{x}_i}$  is the mass of an  $i$ -th REV, over which the averaging of Lagrangian is making an  $\mathbf{X}_i$ —the coordinate of the center of mass of appropriate REV in the referent state. One can see that on this stage, material is presented as a set of discrete “macro-points” and overall Lagrangian is a sum of average “local” Lagrangians of REVs. Each local Lagrangian depends of the state of appropriate REV.

The next step in creation continuum model of the fluid-filled porous body is the transition from integral sum (4) to it continuum equivalent. One has:

$$\langle A \rangle = \int_{t_0}^{t_1} \langle \mathcal{L} \rangle dt = \int_{t_0}^{t_1} \iiint_{M_{\mathbf{X}}} \mathcal{L}(\alpha_i(\mathbf{X}), \mathbf{x}(\mathbf{X}, t), t) dmdt \tag{5}$$

Here we introduced:  $\mathcal{L}(\alpha_i, \mathbf{X}, t) \equiv \langle \mathcal{L}(\mathbf{x}, t) \rangle_{\mathbf{X}} \cdot d\Omega_{\mathbf{X}}$  is the volume element in Lagrangian macrocoordinates (i.e., each vector  $\mathbf{X}$  marks specific REV). Also, we denote here as  $\alpha_i(\mathbf{X})$  the set of parameters characterizing the state of REV  $\mathbf{X}$ .

The first problem is to identify the set of parameters of state of REVs (i.e., parameters, which define the state of taken REV. For propose of this article, to make the results easily comparable with previous publications, we will use the simplest set of such parameters, which include external parameters, strain of solid matrix, mass (or density) of the fluid, and its relative (in respect to REV) velocity and volume available for fluids (Lopatnikov and Cheng 1998, 2002, 2004). However, the procedure of incorporation into other internal degrees of freedom is identical to those used in this article and should not provide any difficulties.

One has to define the movement (deformation) of the material. The major point here is that for penetrable materials, the movement of solid matrix and fluid independently must be considered independently. It means that in referent state one Lagrangian coordinate  $\mathbf{X}_0$  refers to two separate macropoints (REVs): one for fluids, one for solids. Thus, movement for each face must be defined separately. Let us define the movement of solid phase:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t); \mathbf{x}(\mathbf{X}, 0) = \mathbf{X} \tag{6}$$

This means that center of mass of REV  $\mathbf{X}$  is moved in time  $t$  in the point with coordinates  $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ . Similarly, one can define the movement of fluid:

$$\mathbf{x}_f = \mathbf{x}_f(\mathbf{X}, t); \mathbf{x}_f(\mathbf{X}, 0) = \mathbf{X} \tag{7}$$

This means that the liquid part of the REV  $\mathbf{X}$  is moved in time  $t$  in the REV, with its center of mass occupying coordinate  $\xi_f(\mathbf{X}, t)$  and belonging now to the REV  $\mathbf{X}'$  with the following coordinate (Fig. 3):

$$\mathbf{x}(\mathbf{X}', t) = \mathbf{x}_f(\mathbf{X}, t) \tag{8}$$

Relationship (8) explains the meaning of deformation of a two-phase continuum, which includes deformation of a matrix related to the movement of macropoints (REVs) and independent movement of fluid and its distribution between different macropoints (REVs).

Now one has to specify the form of action. Because fluid and solid occupy separate volumes on the micro level, and because a major property of energy is to be (in absence of long-range interaction between different parts of system) an additive function of the system constituents, the Lagrangian of the system can be separated in two parts that represent Lagrangian solid and fluid parts of the system.

This property is still valid for average Lagrangians, too. Using an exact averaging procedure, presented in details in Lopatnikov and Cheng (2004), if only an internal strain of the body, fluid, and solid densities, porosity, and their time derivatives are considered as parameters of REV state, one can present Lagrangian of the system as:

$$A = \int_{t_1}^{t_2} \iiint_{\Omega_X} \left[ \left( \frac{V_S^2}{2} + k_S - e_s(\bar{\epsilon}_{ij}, \varphi) \right) dm_S + \left( \frac{V_f^2}{2} + k_f - e_s(\rho_f) \right) dm_f \right] dt \quad (9)$$

Here  $V_S = \frac{dx}{dt}$ ;  $V_f = \frac{dx_f}{dt}$ . Several remarks must be made about this expression (9).

First, in this expression for clarity, we separated masses of fluid and solid in taken in instances of time to stress the difference between variation procedures for one-phase and two-phase bodies. Also, we omitted brackets, which we used above to notate operation of averaging over REV volumes. Thus, notation  $A$  in (9) now means average action.

Also, we took (9) in account in that potential energy density of fluid must depend only on the fluid density.

One does not have this restriction for solid body energy. Because average energy density of deformation of the body per unit of mass in general case depends on porosity changes, the solid body deformation energy density is a function of internal strain and porosity.

Average kinetic energy density of the constituents is a sum of kinetic energy density related with average movement of the REV and kinetic energy, related with fluctuations of particle velocity on the micro level with each REV. Density of the kinetic energy related to the average movement of REV is equal simply to half of second power of velocity. The density of kinetic energy of fluctuations must be calculated from the micro-model of REVs and can be presented as:

$$k_{S,f} = \frac{1}{\omega_{S,f}} \int_{\Omega_{S,f}} \left( \frac{\tilde{V}_{S,f}^2}{2} + \frac{\tilde{\rho}_{S,f}}{\rho_{S,f}} \mathbf{V}_{S,f} \cdot \tilde{\mathbf{V}}_{S,f} + \frac{\tilde{\rho}_{S,f}}{\rho_{S,f}} \frac{\tilde{V}_{S,f}^2}{2} \right) d\omega \quad (10)$$

Integration here must be made over the volumes  $\omega_{S,f}$ , occupied within REV by appropriate phase;  $\rho_{S,f}$ ,  $\mathbf{V}_{S,f}$  are average density and average of constituent material and  $(\sim)$  denotes fluctuations of associated values. If one can neglect the correlation between fluctuations of density and fluctuation of particle velocity, or if one can consider material as approximately incompressible, expression (10) becomes very simple:

$$k_{S,f} = \frac{\langle \tilde{V}_{S,f}^2 \rangle_{S,f}}{2} \quad (11)$$

Here we denoted  $\langle \dots \rangle_{S,f}$  as operation of averaging over volumes of phases. If we assume that the microscopic fluctuations are caused by macroscopic asynchronous motion of solid and fluid, one can present the kinetic energy of fluctuations in a general form:



$$k_S = k_S (\bar{\varepsilon}_{ij}, \varphi, \dot{\varepsilon}_{ij}, \varphi, \mathbf{V}_S - \mathbf{V}_f, \dots); k_f = k_f (\rho_f, \dot{\rho}_f, \mathbf{V}_S - \mathbf{V}_f) \tag{12}$$

Expression (10) provides the way to calculate and estimate the value of additional mass for particular REV's micro-model. One can see that just this portion of kinetic energy, related to microscale fluctuations of material movement, is responsible for the appearance of additional mass. For the additional mass related to the inclination of fluid particle velocities from an average value, it was explicitly shown by us (Vinogradova and Lopatnikov 1990) in some periodic structures.

Let us consider a variation of expression (9):

$$\begin{aligned} \delta A = & \int_{t_1}^{t_2} \iiint_{\Omega_X} \left[ \left( \frac{\mathbf{V}_S^2}{2} + k_S - e_s (\bar{\varepsilon}_{ij}, \varphi) \right) \delta dm_S + \left( \frac{\mathbf{V}_f^2}{2} + k_f - e_s (\rho_f) \right) \delta m_f dt \right] \\ & + \int_{t_1}^{t_2} \iiint_{\Omega_X} \left[ \delta \left( \frac{\mathbf{V}_S^2}{2} + k_S - e_s (\bar{\varepsilon}_{ij}, \varphi) \right) dm_S (t) + \delta \left( \frac{\mathbf{V}_f^2}{2} + k_f - e_s (\rho_f) \right) dm_f (t) dt \right] \end{aligned} \tag{13}$$

One can see that this variation can be separated in two parts. The first part is related to a variation of values:  $dm_S = (1 - \varphi) \rho_S d\Omega_{X_S} (t)$  and  $dm_f = \varphi \rho_f d\Omega_{X_S} (t)$ . The second part is related to a variation of the densities of Lagrangians of different phases.

However, by definition of solid phase movement, the mass of solid phase for each REV during variation is constant and, thus:

$$\delta (dm_S) = \delta ((1 - \varphi) \rho_S d\Omega_{X_S} (t)) = 0 \tag{14}$$

Similarly, for fluid one has:

$$\delta (dm_f) = \delta (\varphi \rho_f d\Omega_{X_f} (t)) = 0 \tag{15}$$

Thus, the integral in (13) is equal to zero and integration in the second integral can be made without distinction of fluid and solid volumes. Consequently, one can write:

$$\delta A = \int_{t_1}^{t_2} \iiint_{\Omega} \left[ (1 - \varphi) \rho_S \delta \left( \frac{\mathbf{V}_S^2}{2} + k_S - e_s (\bar{\varepsilon}_{ij}, \varphi) \right) + \varphi \rho_f \delta \left( \frac{\mathbf{V}_f^2}{2} + k_f - e_s (\rho_f) \right) d\Omega \right] \tag{16}$$

The total variation of Eq. 16 is thus given by:

$$\begin{aligned} \delta A = & \int_{t_1}^{t_2} \iiint_{\Omega} (1 - \varphi) \rho_S \left[ \mathbf{V}_S \delta \dot{\xi}_S + \frac{\partial k_S}{\partial \mathbf{V}_-} (\delta \dot{\xi}_f - \delta \dot{\xi}_S) + \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} \delta \dot{\varepsilon}_{ij} + \frac{\partial k_S}{\partial \varphi} \delta \varphi \right. \\ & \left. + \frac{\partial k_S - e_s}{\partial \bar{\varepsilon}_{ij}} \delta \bar{\varepsilon}_{ij} + \frac{\partial k_S - e_s}{\partial \varphi} \delta \varphi \right] d\Omega dt \\ & + \int_{t_1}^{t_2} \iiint_{\Omega} \varphi \rho_f \left[ \mathbf{V}_f \delta \dot{\xi}_f + \frac{\partial k_f}{\partial \mathbf{V}_-} (\delta \dot{\xi}_f - \delta \dot{\xi}_S) + \frac{\partial k_f}{\partial \dot{\rho}_f} \delta \dot{\rho}_f + \frac{\partial k_f}{\partial \varphi} \delta \varphi + \frac{\partial k_f}{\partial \rho_f} \delta \rho_f \right. \\ & \left. + \rho_f \frac{\partial k_f - e_f}{\partial \rho_f} \frac{\delta \rho_f}{\rho_f} \right] d\Omega dt \end{aligned} \tag{17}$$

Here we took in account the definition of velocity,  $\delta \mathbf{V}_{S,f} = \delta \dot{\xi}_S$ .

One can see that expression (17) contains variations of only internal variables.

Thus, to complete the construction of the theory, based on macrodeformations of the porous material, it is necessary to establish relationships between variations of internal parameters of REV's and variations of external parameters (i.e., to specify the construction of REV's).

It is useful to recall that one can always consider the variations as infinitely small values and, thus, in each taken state, these relationships are always linear.

### 2.2 Relationships between variations of external and internal parameters of the material and consolidation coefficients

Independently in a specific construction of the REV's, two relationships between external and internal parameters are automatically defied by mass conservation law for solids and fluids (14, 15). Let us consider these relationships in more detail.

One has:

$$\delta_S (dm_S) = -\rho_S d\Omega_{\mathbf{X}_S}(t) \delta_S \varphi + (1 - \varphi) d\Omega_{\mathbf{X}_S}(t) \delta_S \rho_S + (1 - \varphi) \rho_S \delta_S d\Omega_{\mathbf{X}_S}(t) = 0 \tag{18}$$

$$\delta_f (dm_f) = \rho_f d\Omega_{\mathbf{X}_f}(t) \delta_f \varphi + \varphi d\Omega_{\mathbf{X}_f}(t) \delta_f \rho_f + \varphi \rho_f \delta_f d\Omega_{\mathbf{X}_f}(t) = 0 \tag{19}$$

We used index (S,f) to stress that these variations are related with movement of different phases. First, one can rewrite these equations in a more convenient form.

$$-\frac{\delta_S \varphi}{1 - \varphi} + \frac{\delta_S \rho_S}{\rho_S} + \frac{\delta_S d\Omega_{\mathbf{X}_S}}{d\Omega_{\mathbf{X}_S}} = 0 \tag{20}$$

$$\frac{\delta_f \varphi}{\varphi} + \frac{\delta_f \rho_f}{\rho_f} + \frac{\delta_f d\Omega_{\mathbf{X}_f}}{d\Omega_{\mathbf{X}_f}} = 0 \tag{21}$$

Considering this variations, one must remember that porosity and density of solids and volumes  $d\Omega_{\mathbf{X}_S}$  are properties of a solid part of considering elementary macrovolumes. Similarly, the density of fluid and volume  $d\Omega_{\mathbf{X}_f}$  is a property of the fluid part of elementary macrovolumes. Thus, their variations represent simply variations of associated function defined in macropoints. One can omit indexes for this variation. However, porosity is not the property of the liquid part for considering volume. Its variation  $\delta_f \varphi$  consists of two parts. One part represents variation of porosity itself:

$$\bar{\delta}_f \varphi = \delta \varphi \tag{22}$$

The second part is related to the movement of mass of fluid from its initial REV into the other REV. If we suggest that porosity is a smooth function of REV (i.e.,  $\varphi(\mathbf{X})$  is a smooth function), we can present this part of variation as:

$$\bar{\bar{\delta}}_f \varphi = (\delta \xi_f - \delta \xi_S) \cdot \nabla_{\mathbf{X}} \varphi \tag{23}$$

Thus, we can write:

$$\delta_f \varphi = \delta \varphi + (\delta \xi_f - \delta \xi_S) \cdot \nabla_{\mathbf{X}} \varphi \tag{24}$$

As a result, relationship (21) obtain the form:

$$\frac{\delta \varphi}{\varphi} + (\delta \xi_f - \delta \xi_S) \cdot \frac{\nabla \varphi}{\varphi} + \frac{\delta \rho_f}{\rho_f} + \frac{\delta d\Omega_{\mathbf{X}_f}}{d\Omega_{\mathbf{X}_f}} = 0 \tag{25}$$

Presence in (25) of the term *proportional to gradient of porosity* qualitatively differs our proposed variational procedure from the variational procedures used earlier by Bedford and Drumheller (1979), in a later paper by Berryman and Thigpen (1985), and other authors. We will see that just this term is responsible to appearance of the force  $\pm p\nabla\varphi$ , introduced by Nikolaevskiy in momentum conservation equations.

Now one can express variation of densities  $\delta\rho_{S,f}$  and volumes  $\delta d\Omega_{\mathbf{X}_{S,f}}$  in terms of variations of external and internal strain tensors.

Variation of volume  $\delta d\Omega_{\mathbf{X}_{S,f}}$ , by construction of macromodel, is related with external strain as:

$$\frac{\delta d\Omega_{\mathbf{X}_S}}{d\Omega_{\mathbf{X}_S}} = \delta\varepsilon_S \tag{26}$$

Here:

$$\delta\varepsilon_S = Tr\delta\hat{\boldsymbol{\varepsilon}}_S. \tag{27}$$

In the meantime, virtual external strain tensor for (virtually) small deformations representing virtual movement of the material  $\delta\xi_S$ , can be always presented as:

$$\delta\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial\delta\xi_S^j}{\partial X_i} + \frac{\delta\xi_S^i}{\partial X_j} \right) \tag{28}$$

Thus:

$$\delta\varepsilon_S = \partial_i\delta\xi_S^i \tag{29}$$

By definition of internal strain as average strain in solid body, one has:

$$\frac{\delta\rho_S}{\rho_S} = -\delta\bar{\varepsilon}_S \tag{30}$$

where:

$$\delta\bar{\varepsilon}_S = Tr\delta\hat{\boldsymbol{\varepsilon}}_S \tag{31}$$

However, there is not any universal relationship of (27) type for virtual internal strain tensor, because there is no *a priori* defined vector field, generating this tensor. One must consider this value simply as a symmetric tensor field defined on the space of macropoints and containing six independent components.

Similarly, one can relate the variation  $d\Omega_{\mathbf{X}_f}$  with virtual macromovement of fluid. One has:

$$\frac{\delta d\Omega_{\mathbf{X}_f}}{d\Omega_{\mathbf{X}_f}} = \delta\varepsilon_f \tag{32}$$

Here:

$$\delta\varepsilon_f = \partial_i\delta\xi_f^i \tag{33}$$

One can also define formally virtual internal strain tensor for liquid:

$$\frac{\delta\rho_f}{\rho_f} = -\delta\bar{\varepsilon}_f \tag{34}$$

With these additions, one can present (20, 21) as:

$$-\frac{\delta\varphi}{1-\varphi} - \delta\bar{\varepsilon}_S + \partial_i\delta\xi_S^i = 0 \tag{35}$$

$$\frac{\delta\varphi}{\varphi} + \frac{1}{\varphi} (\delta\xi_f^i - \delta\xi_S^i) \cdot \frac{\partial\varphi}{\partial X_i} - \delta\bar{\varepsilon}_f + \partial_i\delta\xi_f^i = 0 \tag{36}$$

Thus, two universal relations are given by Eqs. 35, 36, representing mass conservation laws for solid and fluid constituents of REV's. However, the total number of static internal parameters equals eight (six components of internal strain tensor, density of fluid, and porosity). Thus, in general, a maximum of six additional relationships between these variations can be defined. It is convenient to introduce these additional relationships as the relationships between external and internal strain tensors as:

$$\delta\bar{\varepsilon}_{Sij} = \hat{L}_{klij} (\bar{\varepsilon}_{ij}, \varphi, \rho_f, t) \delta\varepsilon_{Sij} \tag{37}$$

We will refer relationships of (37) type as “material structural equations,” or simply as “structural equations” and introduced coefficients  $L_{klij}$  as “structural” or “consolidation” coefficients, because they define whether the external strain causes the strain of matrix or not. Absence of such a relationship means that material is not statically consolidated.

Here we explicitly marked that these coefficients could be functions of parameters of the REV state and, in a general case, they also could be some operators of time, dependent on the specific “construction” of material REV's.

When coefficients  $\hat{L}_{klij}$  connect two tensors of second rank, by well-know property of tensors, they form a tensor of the fourth rank. Also, because virtual strain tensors are symmetric, and, thus, contain six independent coefficients, fourth-rank tensor  $L_{klij}$  contains a maximum of 21 independent structural coefficients.

However, it seems too rigid a suggestion. If external deformation uniquely defines the deformation of solid matrix, it follows from (35) that it also uniquely defines the porosity. Consequently, porosity cannot be changed without external deformation of solid matrix, and means that if  $\delta\varepsilon_{Sij} = 0$ , the variation  $\delta\varphi$  is also equal to zero. In this case, from (36), it follows that if there is no flow of fluid into or out of the considering volume and, thus,  $\partial_i\delta\xi_f^i = 0$ , the density of fluid can not be changed:  $\delta\bar{\varepsilon}_f = 0$ . In other words, if relationship (37) provides a one to-one map, one comes to conclusion that compressible components of porous material encapsulated in rigid ball *cannot change their densities and volumes*, which looks physically incorrect. The suggestion that (37) provides a one-to-one map makes the system “too rigid.”<sup>2</sup> Thus, for description of porous material having maximal consolidation, one must define not six, but only five relationships between five components of external and internal strains tensors. It is easy to calculate that consolidation tensor  $\hat{L}$  can contain now only contain 15 different coefficients.

If dilatation is absent, the most convenient suggestion is to define these relationships as a linear relationship between traceless parts of variation of external and internal strain tensor. In this case, one can write:

$$\{\delta\bar{\varepsilon}_{ij}\} = L_{ijkl} \{\delta\varepsilon_{kl}\} \tag{38}$$

<sup>2</sup> This point is important, because, to the best of our knowledge, it has never been discussed in literature and refers to one of the key questions of porous material modeling: can one change the porosity of porous material without changing its external shape, for example, by application homogeneous pressure to “internal surface”? It seems that a positive or negative answer on this question is an additional “axiom.” From a physical standpoint, authors prefer to give a positive answer, which matches, particularly, the logic of Frenkel, presented in his famous paper (Frenkel 1944).

as it was proposed in (Lopatnikov and Cheng 2002). This suggestion means that external strain does not affect the density (and, consequently, porosity) of the solid material. Here we used notation:

$$\{a_{ij}\} = a_{ij} - \frac{1}{3}a_{ii}\delta_{ij} \tag{39}$$

One can see that this choice does not change the relationship (35), because shear components of variation of internal strain, which can be expressed in terms of variation of external strain, simply do not take part in this equation. All other possible choices (which depend on structure of the materials) will affect this relationship. It can happen, for example, for materials with dilatation.

Mechanics of porous materials are not limited to materials, which have a maximal level of consolidation. For (partially) nonconsolidated materials, the number of consolidation equations is smaller than five. In the case of non-fully consolidated materials, only some (<5) internal parameters can be uniquely related with external deformations. All other parameters one must consider as statically nondefined. As we will see, free parameters lead to the appearance of additional dynamic equations.

Considering the most consolidated porous material, one can present variation (17) in the form:

$$\begin{aligned} \delta A = & \int_{t_1}^{t_2} \iiint_{\Omega} (1 - \varphi) \rho_S \left[ \mathbf{V}_S \delta \dot{\xi}_S + \frac{\partial k_S}{\partial \mathbf{V}_-} (\delta \dot{\xi}_f - \delta \dot{\xi}_S) + \frac{1}{3} \frac{\partial k_S}{\partial \dot{\bar{\varepsilon}}_{ij}} \delta \dot{\bar{\varepsilon}} + \frac{\partial k_S}{\partial \dot{\varphi}} \delta \dot{\varphi} \right. \\ & \left. + \frac{\partial k_S}{\partial \dot{\bar{\varepsilon}}_{ij}} \{\delta \dot{\bar{\varepsilon}}_{ij}\} + \frac{1}{3} \frac{\partial k_S - e_S}{\partial \bar{\varepsilon}_{ij}} \delta \bar{\varepsilon} + \frac{\partial k_S - e_S}{\partial \varphi} \delta \varphi + \frac{\partial k_S - e_S}{\partial \bar{\varepsilon}_{ij}} \{\delta \bar{\varepsilon}_{ij}\} \right] d\Omega dt \\ & + \int_{t_1}^{t_2} \iiint_{\Omega} \varphi \rho_f \left[ \mathbf{V}_f \delta \dot{\xi}_f + \frac{\partial k_f}{\partial \mathbf{V}_-} (\delta \dot{\xi}_f - \delta \dot{\xi}_S) + \frac{\partial k_f}{\partial \dot{\rho}_f} \delta \dot{\rho}_f + \frac{\partial k_f}{\partial \dot{\varphi}} \delta \dot{\varphi} + \frac{\partial k_f}{\partial \varphi} \delta \varphi \right. \\ & \left. + \frac{\partial k_f - e_f}{\partial \rho_f} \delta \rho_f \right] d\Omega dt \end{aligned} \tag{40}$$

One has from (36, 37):

$$\delta \bar{\varepsilon}_S = \partial_i \delta \xi_S^i - \frac{\delta \varphi}{1 - \varphi} \tag{42}$$

$$\delta \rho_f = -\rho_f \left( \partial_i \delta \xi_f^i + \frac{\delta \varphi}{\varphi} + \frac{1}{\varphi} \frac{\partial \varphi}{\partial X_i} \cdot (\delta \xi_f^i - \delta \xi_S^i) \right) \tag{43}$$

For variations of time derivatives of these values, one has:

$$\delta \dot{\bar{\varepsilon}}_S = \partial_i \delta \dot{\xi}_S^i - \frac{\delta \dot{\varphi}}{1 - \varphi} - \frac{\dot{\varphi}}{(1 - \varphi)^2} \delta \varphi \tag{44}$$

$$\begin{aligned} \delta \dot{\rho}_f = & -\rho_f \partial_i \delta \dot{\xi}_f^i - \dot{\rho}_f \partial_i \delta \xi_f^i - \frac{\rho_f}{\varphi} \delta \dot{\varphi} - \frac{\rho_f}{\varphi} \left( \frac{\dot{\rho}_f}{\rho_f} - \frac{\dot{\varphi}}{\varphi} \right) \delta \varphi - \frac{\rho_f}{\varphi} \frac{\partial \varphi}{\partial X_i} \cdot (\delta \dot{\xi}_f^i - \delta \dot{\xi}_S^i) \\ & - \frac{\rho_f}{\varphi} \left[ \frac{\partial \dot{\varphi}}{\partial X_i} + \left( \frac{\dot{\rho}_f}{\rho_f} - \frac{\dot{\varphi}}{\varphi} \right) \frac{\partial \varphi}{\partial X_i} \right] (\delta \xi_f^i - \delta \xi_S^i) \end{aligned} \tag{45}$$

and, using suggestion (39):

$$\{\delta \dot{\bar{\varepsilon}}_{ij}\} = L_{klij} \{\delta \dot{\varepsilon}_{kl}\} + \dot{L}_{klij} \{\delta \varepsilon_{kl}\} \tag{46}$$

Using standard variational procedure, one comes in laboratory frame of coordinates, to the next system of three momentum conservation equations for solid and fluid movement and for changes in porosity:

$$(1 - \varphi) \rho_S \left( \frac{\partial \pi_{Si}}{\partial t} + V_{Sj} \frac{\partial \pi_{Si}}{\partial X_j} \right) - \frac{\partial \Sigma_{Sij}}{\partial x_j} + F_{\text{inti}} = F_{S(\text{dis})i} + f_{S(\text{ext})i} \tag{47}$$

$$\varphi \rho_f \left( \frac{\partial \pi_{fi}}{\partial t} + V_{fj} \frac{\partial \pi_{fi}}{\partial X_j} \right) + \frac{\partial P_f}{\partial X_i} - F_{\text{inti}} = F_{f(\text{dis})i} + f_{f(\text{ext})i} \tag{48}$$

$$\frac{\partial \Pi_{\varphi i}}{\partial t} + V_{Sj} \frac{\partial \Pi_{\varphi i}}{\partial X_j} - F_{\varphi} = F_{\varphi(\text{dis})} \tag{49}$$

A significant singularity of the system (47, 48) is that, in opposition to poromechanics theories proposed earlier, these governing equations define not the only equations of external deformation of solid matrix and movement of fluid, but also of a complete set of internal parameters of the material.

Let us enumerate complete set of parameters of unknowns. These unknowns are: three components of external deformation of solid body  $\xi_{ij}^i$ ; three components of the fluid velocity  $V_f^i$ , density of fluid  $\rho_f$ , density of solid  $\rho_S$ , porosity  $\varphi$ , and, at last, six components of internal strain tensor  $\bar{\varepsilon}_{ij}$ . Thus, the total number of unknowns in (47–49) is equal to 15.

In the mean time, the system (50, 51) contains only seven equations. What are the final relationships?—Two additional equations come from mass conservation laws and represent continuity equations for solid and fluid phases:

$$\frac{\partial (1 - \varphi) \rho_S}{\partial t} + \frac{\partial (1 - \varphi) \rho_S V_{Si}}{\partial X_i} = 0 \tag{50}$$

$$\frac{\partial \varphi \rho_f}{\partial t} + \frac{\partial \varphi \rho_f V_{fi}}{\partial X_j} = 0 \tag{51}$$

The relationship between the variation of solid density  $\rho_S$  and the bulk part of the solid internal strain tensor:

$$d\bar{\varepsilon} = - \frac{d\rho_S}{\rho_S} \tag{52}$$

provides the sixth equation. The last five equations relate, as discussed previously, small variations of internal and external shear modulus, which can be presented in differential or time-dependent form:

$$\{d\bar{\varepsilon}_{ij}\} = L_{ijkl} \{d\varepsilon_{kl}\} \rightarrow \left\{ \frac{d\bar{\varepsilon}_{ij}}{dt} \right\} = L_{ijkl} \left\{ \frac{d\varepsilon_{kl}}{dt} \right\} \tag{53}$$

Thus, the system of governing equations of porodynamics (47–53) is now complete and can be investigated. Details of introduction in the theory of dissipative and external forces acting on constituents of the material one can find in (Lopatnikov and Cheng 2004).

It is important to mention that general differential relationships are not necessary integrable (potential) and, thus, expression (53) describes as reversible (potential) as irreversible deformations of porous matrix.

### 2.3 Degree of Consolidation and Structural Degradation of Porous Materials

As mention previously, structural coefficients (38) in general can be operators of time. A material can be dynamically defined, but a matrix of coefficients in the stationary state can be degenerated.

A simple example is an isotropic material with viscous type of load transmission from external to internal strain:

$$\{d\bar{\varepsilon}_{ij}\} = \tau_v \frac{d}{dt} \{d\varepsilon_{ij}\} \tag{54}$$

For any dynamic movement, this type of relationship is well-defined. However, in a static case, the matrix of structural coefficients will be completely degenerated. Thus, defining the degree of consolidations, we refer to the stationary state of a porous material. Definitely, some destructive processes can lead to the change in degree of material consolidation.

This observation provides a clear way for classification of porous material degradation. Two factors must be taken in account: the loss of integrity (which can be described by the difference between initial and final degree of consolidation) and symmetry properties of the initial and final matrices of structural coefficients. Because the loss of integrity in all events makes the system “more symmetric,” both factors must be taken into account simultaneously.

### 2.4 A simplified Form of Governing Equations

To establish a physical meaning for the obtained governing equations, one can completely neglect the parts of kinetic energy related to fluctuations in comparison with average kinetic energy. Later we will consider conditions when it possible and which parts one can neglect. In this case, expressions for momentum of constituents, phase pressure of the fluid, stress tensor of matrix, and interaction force are drastically simplifying. One has:

$$\Pi_{fi} = \varphi \rho_f V_{fi} \tag{55}$$

$$P_f = \varphi p_f \tag{56}$$

$$\Pi_{Si} = (1 - \varphi) \rho_S V_{Si} \tag{57}$$

$$\Sigma_{kl} = (1 - \varphi) \rho_S \left( \left( \frac{1}{3} \frac{\partial e_S}{\partial \bar{\varepsilon}_{nm}} \delta_{nm} - \frac{1}{3} \frac{\partial e_S}{\partial \bar{\varepsilon}_{ij}} L_{nmij} \delta_{mn} \right) \delta_{kl} + \frac{\partial e_S}{\partial \bar{\varepsilon}_{ij}} L_{kl ij} \right) \tag{58}$$

$$F_{inti} = p_f \frac{\partial \varphi}{\partial X_i} \tag{59}$$

One can not completely neglect the fluctuational term in expression for  $\Pi_{\varphi i}$ , because appropriate expression does not contain any values with which it can be compared. However, one can neglect in this expressions the terms, containing the derivatives of fluctuational parts of kinetic energy of constituents over  $\dot{\rho}_f$  and  $\dot{\bar{\varepsilon}}_{ij}$  in comparison with derivatives in respect to  $\dot{\varphi}$ . Moreover, if to hold only the major term in respect to  $\dot{\varphi}$  and, thus, to suggest that:

$$k_f \approx \frac{1}{2} l_{f\varphi}^2 \dot{\varphi}^2 \tag{60}$$

and:

$$k_S \approx \frac{1}{2} l_{S\varphi}^2 \dot{\varphi}^2 \tag{61}$$

where we introduced coefficients  $l_{f\varphi}^2$  and  $l_{S\varphi}^2$ , which must have the dimension of a second power of length to provide a proper dimension for the kinetic energy density of fluctuations, one has, for the generalized momentum, related with porosity dynamics:

$$\Pi_\varphi = \rho_f l_{f\varphi}^2 \varphi \dot{\varphi} + \rho_S l_{S\varphi}^2 (1 - \varphi) \dot{\varphi} \tag{62}$$

At last, generalized force  $F_\varphi$  in this approximation is equal to:

$$F_\varphi = \left( - (1 - \varphi) \rho_S \frac{\partial e_S}{\partial \varphi} + \rho_f^2 \frac{\partial e_f}{\partial \rho_f} + \frac{1}{3} \rho_S \frac{\partial e_S}{\partial \bar{\varepsilon}_{ij}} \delta_{ij} \right) \tag{63}$$

or, using definitions of solid and liquid pressures:

$$F_\varphi = \left( - (1 - \varphi) \rho_S \frac{\partial e_S}{\partial \varphi} + p_f - p_S \right) \tag{64}$$

In this approximation, equations of porodynamics in absence of external fields obtain transparent form:

$$(1 - \varphi) \rho_S \left( \frac{\partial V_{Si}}{\partial t} + V_{Sj} \frac{\partial V_{Si}}{\partial X_j} \right) - \frac{\partial \Sigma_{Sij}}{\partial x_j} + p_f \frac{\partial \varphi}{\partial X_i} = F_{S(\text{dis})i} \tag{65}$$

$$\varphi \rho_f \left( \frac{\partial V_{fi}}{\partial t} + V_{fj} \frac{\partial V_{fi}}{\partial X_j} \right) + \frac{\partial \varphi p_f}{\partial X_i} - p_f \frac{\partial \varphi}{\partial X_i} = F_{f(\text{dis})i} \tag{66}$$

$$\rho_S l_{S\varphi}^2 \left[ (1 + \lambda_\varphi \varphi) \frac{d^2 \varphi}{dt^2} + \lambda_\varphi \left( \frac{d\varphi}{dt} \right)^2 \right] = p_f - p_S - (1 - \varphi) \rho_S \frac{\partial e_S}{\partial \varphi} + F_{\text{dis}\varphi} \tag{67}$$

Here we introduce parameter:

$$\lambda_\varphi = \frac{(\rho_f l_{f\varphi}^2 - \rho_S l_{S\varphi}^2)}{\rho_S l_{S\varphi}^2} \tag{68}$$

It is important to mention that one must understand the derivative in (67) as a full derivative over time, reflecting transport of the porosity. Thus, in the Eulerian frame, one has:

$$\frac{d}{dt_\varphi} = \frac{\partial}{\partial t} + V_{\varphi i} \frac{\partial}{\partial x_i} \tag{69}$$

In (69), the velocity of porosity transport is presented. As mentioned previously, we suggested that, because porosity is a property of solid matrix, in some approximation one can use the suggestion that

$$V_{\varphi i} = V_{Si} \tag{70}$$

However, this suggestion is an additional statement that is, from wider point of view, not necessary.

After consideration of dissipative terms, it is possible to show (Lopatnikov and Cheng 2004) that

$$F_{f(\text{dis})i} = -F_{S(\text{dis})i} \tag{71}$$

In quadratic approximation for dissipation function, one has:

$$F_{f(\text{dis})i} = \varphi (1 - \varphi) \hat{C}_{ji} (t) (V_{fj} - V_{Sj}); F_{\text{dis}\varphi} = -\hat{v}_\varphi (t) \dot{\varphi} \tag{72}$$

In (71, 72), coefficients could be time-dependent operators. In frequency domain this operator can be presented as a function of frequency as suggested by Biot, or in later works (See, for example Johnson et al. 1987)

It is possible to show that if we consider “liquid” matrix and porosity formed by a low concentration of gas-filled voids (“bubbles”), the dynamic equation for porosity (69) becomes



practically equivalent to the most common Rayleigh-Plesset equation of “bubble dynamics” (Rayleigh 1917; Plesset 1949) with a specific dissipation force of  $F_{\text{dis}\varphi}$  (72).

It is important to mention that if this dissipation force  $F_{\text{f}(\text{dis})i}$  represents visco-inertial dissipation related with “global flow” of fluid, investigated in many papers, starting with Frenkel (Frenkel 1944), dissipation force  $F_{\text{dis}\varphi}$  represents relaxation of the porosity within each REV and, thus, it is quite a different source of dissipation that does not present in the theories of Biot type and mixture theories. However, the last type of attenuation is well known in the poroelastic theory as the “squirt flow” mechanism of attenuation (Dvorkin et al. 1993; Jakobsen 2009).

There is also the third source of dissipation of mechanical energy in the presented theory. This source is related to the form of structural relationships (53) describing the “construction” of REVs and the relationship between external and internal parameters. These relationships, as mentioned previously, are not necessarily potential, or, even if they are potential are not necessarily conservative. This dissipation is related to “skeleton” attenuation, considered, for example, in early papers of Stall and Bryan (1970) and Stall (1974).

### 3 Conservation Laws

As discussed in the introduction of this article, conservation laws are a critically important element of the theory. It is obvious that presented theory contains separate mass conservation laws for each phase.

Situations with momentum conservation are more complicated. It is easy to check that universal momentum conservation law, which theory supports, is the total momentum conservation law, which has (if one can neglect the dynamics of internal degrees of freedom and use a simplified form of governing equations) the form:

$$\left( \frac{\partial \varphi \rho_f V_{fi} + (1 - \varphi) \rho_s V_{si}}{\partial t} + \frac{\partial \varphi \rho_f V_{fi} V_{fj} + (1 - \varphi) \rho_s V_{si} V_{sj}}{\partial X_j} \right) \tag{73}$$

$$- \frac{\partial (1 - \varphi) \sigma_{sij} - \varphi p_f}{\partial X_j} = 0 \tag{74}$$

Here we introduce solid phase stress  $\sigma_{sij}$  as:

$$\sigma_{sij} = \rho_s \left( \left( \frac{1}{3} \frac{\partial e_s}{\partial \bar{\epsilon}_{nm}} \delta_{nm} - \frac{1}{3} \frac{\partial e_s}{\partial \bar{\epsilon}_{ij}} L_{nmij} \delta_{mn} \right) \delta_{kl} + \frac{\partial e_s}{\partial \bar{\epsilon}_{ij}} L_{kl ij} \right) \tag{75}$$

In the meantime, using continuity equation for fluid, after simple manipulations one can present equation for fluid momentum as:

$$\rho_f \left( \frac{\partial V_{fi}}{\partial t} + V_{fj} \frac{\partial V_{fi}}{\partial X_j} \right) + \frac{\partial p_f}{\partial X_i} = \frac{1}{\varphi} F_{\text{f}(\text{dis})i} \tag{76}$$

One can see that interaction force  $p_f \frac{\partial \varphi}{\partial X_i}$  annihilates the part of “phase force”  $-\frac{\partial \varphi p_f}{\partial X_i}$  and force, acting on the fluid to become equal to  $-\frac{\partial p_f}{\partial X_i}$  independently whether porosity is constant in space or not. In stationary case, Eq. 76 predicts correct behavior of fluid pressure even in case of non-homogeneous porosity: it becomes constant in space.

In the meantime, one can present Eq. 65 for momentum of solid as:

$$(1 - \varphi) \rho_s \left( \frac{\partial V_{si}}{\partial t} + V_{sj} \frac{\partial V_{si}}{\partial X_j} \right) - \frac{\partial (1 - \varphi) \sigma_{sij} + \varphi p_f}{\partial X_j} - \varphi \frac{\partial p_f}{\partial X_i} = F_{\text{S}(\text{dis})i} \tag{77}$$

If the value:

$$\Sigma_{ij} = (1 - \varphi) \sigma_{Sij} - \varphi p_f \tag{78}$$

in total momentum conservation obviously represents the effective stress acting on porous material as a whole, the value:

$$\Sigma_T = (1 - \varphi) \sigma_{Sij} + \varphi p_f \tag{79}$$

replaces Terzaghi’s stress.

If Eq. 66 is equivalent to a regular equation of the fluid motion (77), in a non-homogeneous (and, in nonlinear materials too) equation for the momentum of a solid phase is significantly different from equations considered in literature earlier, except for the phenomenological consideration of Nikolaevskiy et al. (1970).

As to energy conservation laws, we will devote to the special publication, because they must take in account energy exchange between phases, and this process needs extended and careful physical consideration.

### Comparison with earlier theories

Gathering Eqs. 47–53, one comes to the most general formulation of poroelastic theory, in a form convenient for comparison with earlier approaches:

$$\frac{\partial (1 - \varphi) \rho_S}{\partial t} + \frac{\partial (1 - \varphi) \rho_S V_{Si}}{\partial X_i} = 0 \tag{80}$$

$$\frac{\partial \varphi \rho_f}{\partial t} + \frac{\partial \varphi \rho_f V_{fi}}{\partial X_j} = 0 \tag{81}$$

$$\left( \frac{\partial \varphi \rho_f V_{fi} + (1 - \varphi) \rho_S V_{Si}}{\partial t} + \frac{\partial \varphi \rho_f V_{fi} V_{fj} + (1 - \varphi) \rho_S V_{Si} V_{Sj}}{\partial X_j} \right) - \frac{\partial \Sigma_{ij}^{(M)}(\varphi, \rho_f, \bar{\varepsilon}_{ij})}{\partial X_j} = 0 \tag{82}$$

$$\rho_f \left( \frac{\partial V_{fi}}{\partial t} + V_{fj} \frac{\partial V_{fi}}{\partial X_j} \right) + \frac{\partial p_f}{\partial X_i} = \frac{1}{\varphi} F_{f(\text{dis})i} \tag{83}$$

$$\rho_S l_{S\varphi}^2 \left[ (1 + \lambda_\varphi \varphi) \frac{d^2 \varphi}{dt^2} + \lambda_\varphi \left( \frac{d\varphi}{dt} \right)^2 \right] = p_f - p_S - (1 - \varphi) \rho_S \frac{\partial \varepsilon_S}{\partial \varphi} + F_{\text{dis}\varphi} \tag{84}$$

$$\frac{d\rho_S}{dt} = -\rho_S \frac{d\bar{\varepsilon}}{dt} \tag{85}$$

$$\left\{ \frac{d\bar{\varepsilon}_{ij}}{dt} \right\} = L_{ijkl} \left\{ \frac{d\varepsilon_{kl}}{dt} \right\} \tag{86}$$

It is easy to check, that if one considers propagation of linear waves in homogeneous materials, can neglect with porosity dynamics and the material is not degenerated (consolidated material), the theory developed is equivalent to Biot’s governing equations. It is important to mention that the present theory is equivalent to Biot theory even in case of non-homogeneous materials, if one can neglect with porosity dynamics and consider the material as completely consolidated. Particularly, it provides rigorous proof that one can apply Biot theory for wave propagation in non-homogeneous materials without any other restrictions.

In general, if one considers the case of non-degenerated (consolidated) material and can neglect with porosity dynamics, present theory is also equivalent to the theory, developed by V. Nikolaevskiy. In the mean time, the difference is that interaction force proportional to the

porosity gradient introduced by Nikolaevskiy from phenomenological reasons, automatically appears in the frame of our approach.

As to mentioned theories; Bedford and Drumheller, Berryman and Thigpen, Wilmanski and some other works; these theories does not provide correct equations for fluid motion in case of non-homogeneous materials and, thus, are valid only in the case of propagation of linear waves in homogeneous materials, if one neglects the difference between  $\nabla\varphi p_f$  and  $\varphi\nabla p_f$ , which is possible only if characteristic wavelength of a process is significantly smaller than the characteristic length of porosity changes. In many cases, it is too restrictive a requirement.

#### 4 Discussion and Conclusions

One can see that, in opposition to the majority of approaches, the presented theory clearly separates the deformation of the porous continuum itself from the deformation of a solid matrix, which is identified with an average deformation of a solid matrix and the effect of the external deformation on the internal state of REV's and back. The relationship between macrodynamics and microdynamics is a question to micromodeling. The number of internal degrees of freedom is limited only by practical needs. Relationships between external and internal degrees of freedom can be formulated as the set of linear equations for differentials of appropriate variables. For example, continuity equations and structural equations of the (86) type provide the relationship between external deformation of the continuum as a whole and state of "macropoints" (REV's) of the continuum in approximation used in this article.

One can consider relationships of (86) as the discrete description of the internal "mechanism" of considering macrocontinuum. Such a view on the problem opens the opportunity to use developed theory as a background for new field of application of poroelastic theory" the theory of "conformable robots." Under "conformable robots," the body, which REV's can be an active machine able to provide work and, thus, can include both "passive" as active degrees of freedom as in classic theory of machines and mechanisms (Kolovsky et al. 2000). We will consider this new application of the theory in a special publication.

Second, developed theory automatically leads to the appearance of the interaction force between phases, which has the same physical meaning as the force, introduced by Nikolaevskiy. However, this interaction force is equal to the force, introduced by Nikolaevskiy, only if one can neglect the dynamics of internal degrees of freedom (except porosity) and represent its generalization. Nikolaevskiy's interaction force must be incorporated properly in the governing equations. One can see that if to introduce Terzaghi stress, the "rest force" is equal to  $\varphi \frac{\partial p_f}{\partial X_i}$ ; this term has the same order as the major "elastic force" term  $\frac{\partial(1-\varphi)\sigma_{sij} + \varphi p_f}{\partial X_j}$  and cannot be neglected *ad hoc*. Another consequence is that the appearance of Nikolaevskiy's force questions the interfacial conditions, because it is definitely not small on the interfaces, where porosity is sharply changing. Thus, interfacial conditions boundary which must be reinvestigated. This investigation will be presented in the next part of this work.

The third advantage of the presented theory is that it naturally incorporates all three mechanisms of attenuation discussed in the literature: "global flow" visco-inertial attenuation, which is as usual describing by generalized Darcy force; "squirt flow" attenuation, related with local movement of fluid that appears as an attenuation term in equation for porosity dynamics; and, the "skeleton" attenuation, which can be related to processes such as dry friction, dilatation, and "lubricant-like" interaction between grains of the material, which appears as a part of a "structural equation."

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**Appendix**

Definitions of used values

$$\Pi_{Si} = (1 - \varphi) \rho_S V_{Si} - (1 - \varphi) \rho_S \frac{\partial k_S}{\partial \mathbf{V}_{-i}} - \varphi \rho_f \frac{\partial k_f}{\partial \mathbf{V}_{-i}} + \rho_f^2 \frac{\partial k_f}{\partial \dot{\rho}_f} \frac{\partial \varphi}{\partial X_i} - \frac{\partial \Psi_{ij}}{\partial X_j} \tag{A.1}$$

$$\Psi_{kl} = (1 - \varphi) \rho_S \left[ \frac{1}{3} \left( \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} \delta_{ij} - \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} L_{nmij} \delta_{nm} \right) \delta_{kl} + \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} L_{kl ij} \right] \tag{A.2}$$

$$\Pi_{fi} = \varphi \rho_f V_{fi} + (1 - \varphi) \rho_S \frac{\partial k_S}{\partial \mathbf{V}_{-i}} + \varphi \rho_f \frac{\partial k_f}{\partial \mathbf{V}_{-i}} - \rho_f^2 \frac{\partial k_f}{\partial \dot{\rho}_f} \frac{\partial \varphi}{\partial X_i} - \frac{\partial}{\partial X_i} \left( \varphi \rho_f^2 \frac{\partial k_f}{\partial \dot{\rho}_f} \right) \tag{A.3}$$

$$\pi_{Si} = \frac{1}{(1 - \varphi) \rho_S} \Pi_{Si}; \pi_{fi} = \frac{1}{\varphi \rho_f} \Pi_{fi} \tag{A.4}$$

$$\Pi_\varphi = \left( \varphi \rho_f \frac{\partial k_f}{\partial \dot{\varphi}} + (1 - \varphi) \rho_S \frac{\partial k_S}{\partial \dot{\varphi}} - \rho_f^2 \frac{\partial k_f}{\partial \dot{\rho}_f} - \frac{1}{3} \rho_S \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} \delta_{ij} \right) \tag{A.5}$$

$$\Sigma_{kl} = -\frac{1}{3} \left( (1 - \varphi) \rho_S \frac{\partial k_S - e_S}{\partial \bar{\varepsilon}_{nm}} \delta_{nm} - (1 - \varphi) \rho_S \frac{\partial k_S - e_S}{\partial \bar{\varepsilon}_{ij}} L_{nmij} \delta_{mn} - (1 - \varphi) \rho_S \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} \dot{L}_{nmij} \delta_{mn} \right) \delta_{kl} - \left( (1 - \varphi) \rho_S \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} \dot{L}_{kl ij} + (1 - \varphi) \rho_S \frac{\partial k_S - e_S}{\partial \bar{\varepsilon}_{ij}} L_{kl ij} \right) \tag{A.6}$$

$$P_f = \varphi \rho_f^2 \frac{\partial e_f}{\partial \rho_f} - \varphi \rho_f^2 \frac{\partial k_f}{\partial \rho_f} - \varphi \rho_f \dot{\rho}_f \frac{\partial k_f}{\partial \dot{\rho}_f} = \varphi p_f - \varphi \rho_f^2 \frac{\partial k_f}{\partial \rho_f} - \varphi \rho_f \dot{\rho}_f \frac{\partial k_f}{\partial \dot{\rho}_f} \tag{A.7}$$

$$p_f = \rho_f^2 \frac{\partial e_f}{\partial \rho_f} \tag{A.8}$$

$$f_{Si} = -F_{inti} = -p_f \frac{\partial \varphi}{\partial X_i} + \rho_f^2 \left[ \left( \frac{\partial k_f}{\partial \rho_f} + \left( \frac{\dot{\rho}_f}{\rho_f} - \dot{\varphi} \right) \frac{\partial k_f}{\partial \dot{\rho}_f} \right) \frac{\partial \varphi}{\partial X_i} + \frac{\partial k_f}{\partial \dot{\rho}_f} \frac{\partial \dot{\varphi}}{\partial X_i} \right] \tag{A.9}$$

$$f_{fi} = F_{inti} = p_f \frac{\partial \varphi}{\partial X_i} - \rho_f^2 \left[ \left( \frac{\partial k_f}{\partial \rho_f} + \left( \frac{\dot{\rho}_f}{\rho_f} - \dot{\varphi} \right) \frac{\partial k_f}{\partial \dot{\rho}_f} \right) \frac{\partial \varphi}{\partial X_i} + \frac{\partial k_f}{\partial \dot{\rho}_f} \frac{\partial \dot{\varphi}}{\partial X_i} \right] \tag{A.10}$$

$$F_\varphi = \left( (1 - \varphi) \rho_S \frac{\partial k_S - e_S}{\partial \varphi} - \rho_f^2 \frac{\partial k_f - e_f}{\partial \rho_f} - \frac{1}{3} \rho_S \frac{\partial k_S - e_S}{\partial \dot{\varepsilon}_{ij}} \delta_{ij} - \left( -\frac{1}{3} \frac{\dot{\varphi}}{(1 - \varphi)} \rho_S \frac{\partial k_S}{\partial \dot{\varepsilon}_{ij}} \delta_{ij} - \rho_f^2 \left( \frac{\dot{\rho}_f}{\rho_f} - \dot{\varphi} \right) \frac{\partial k_f}{\partial \dot{\rho}_f} + \varphi \rho_f \frac{\partial k_f}{\partial \varphi} \right) \right) \tag{A.11}$$

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