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Thermal Instability in a Porous Medium Layer Saturated by a Nanofluid: Brinkman Model

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Abstract The onset of convection in a horizontal layer of a porous medium saturated by a nanofluid is studied analytically. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis. For the porous medium, the Brinkman model is employed. Three cases of free–free, rigid–rigid, and rigid–free boundaries are considered. The analysis reveals that for a typical nanofluid (with large Lewis number), the prime effect of the nanofluids is via a buoyancy effect coupled with the conservation of nanoparticles, whereas the contribution of nanoparticles to the thermal energy equation is a second-order effect. It is found that the critical thermal Rayleigh number can be reduced or increased by a substantial amount, depending on whether the basic nanoparticle distribution is top-heavy or bottom-heavy, by the presence of the nanoparticles. Oscillatory instability is possible in the case of a bottom-heavy nanoparticle distribution.

Keywords Thermal instability · Nanoparticles · Nanofluids · Horton-Rogers-Lapwood problem

List of Symbols

D_B Brownian diffusion coefficient
D_T thermophoretic diffusion coefficient
Da Darcy number, defined by Eq. 15b
g Gravitational acceleration

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$\mathbf{g}$	Gravitational acceleration vector
H	Dimensional layer depth
k_{m}	Effective thermal conductivity of the porous medium
K	Permeability of the porous medium
Le	Lewis number, defined by Eq. 15c
N_{A}	Modified diffusivity ratio, defined by Eq. 19
N_{B}	Modified particle-density increment, defined by Eq. 20
p^*	Pressure
p	Dimensionless pressure, $p^*K/\mu\alpha_{\mathrm{m}}$
Pr	Prandtl number, defined by Eq. 15a
Ra	Thermal Rayleigh–Darcy number, defined by Eq. 16
Rm	Basic-density Rayleigh number, defined by Eq. 17
Rn	Concentration Rayleigh number, defined by Eq. 18
t^*	Time
t	Dimensionless time, $t^*\alpha_{\mathrm{m}}/\sigma H^2$
T^*	Temperature
T	Dimensionless temperature, $\frac{T^*-T_{\mathrm{c}}^*}{T_{\mathrm{h}}^*-T_{\mathrm{c}}^*}$
T_{c}^*	Temperature at the upper wall
T_{h}^*	Temperature at the lower wall
(u, v, w)	Dimensionless Darcy velocity components, $(u^*, v^*, w^*)H/\alpha_{\mathrm{m}}$
$\mathbf{v}$	Dimensionless Darcy velocity, $H\mathbf{v}_{\mathrm{D}}^*/\alpha_{\mathrm{m}}$
$\mathbf{v}_{\mathrm{D}}^*$	Dimensional Darcy velocity, (u^*, v^*, w^*)
(x, y, z)	Dimensionless Cartesian coordinates, $(x^*, y^*, z^*)/H$; z is the vertically upward coordinate
(x^*, y^*, z^*)	Cartesian coordinates

Greek symbols

α_{m}	Thermal diffusivity of the porous medium, $\frac{k_{\mathrm{m}}}{(\rho c)_{\mathrm{f}}}$
β	Volumetric expansion coefficient of the fluid
ε	Porosity
λ_{i}	Parameter that takes value 0 for the case of a rigid boundary and ∞ for a free boundary, $i = 1, 2$
μ	Viscosity of the fluid
$\tilde{\mu}$	Effective viscosity of the porous medium
ρ_{f}	Fluid density
ρ_{p}	Nanoparticle mass density
$(\rho c)_{\mathrm{f}}$	Heat capacity of the fluid
$(\rho c)_{\mathrm{m}}$	Effective heat capacity of the porous medium
$(\rho c)_{\mathrm{p}}$	Effective heat capacity of the nanoparticle material
σ	Heat capacity ratio, defined by Eq. 8
$\phi*$	Nanoparticle volume fraction
ϕ	Relative nanoparticle volume fraction, $\frac{\phi^*-\phi_0^*}{\phi_1^*-\phi_0^*}$

Superscripts

$*$	Dimensional variable
$'$	Perturbation variable

Subscripts

b basic solution

1 Introduction

The term "nanofluid" refers to a liquid containing a suspension of submicronic solid particles (nanoparticles). The term was coined by Choi (1995). The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Masuda et al. (1993). This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems (Buongiorno and Hu 2005).

A comprehensive survey of convective transport in nanofluids was made by Buongiorno (2006) who says that a satisfactory explanation for the abnormal increase of the thermal conductivity and viscosity is yet to be found. He focused on the further heat transfer enhancement observed in convective situations. Buongiorno notes that several authors have suggested that convective heat transfer enhancement could be due to the dispersion of the suspended nanoparticles, but he argues that this effect is too small to explain the observed enhancement. Buongiorno also concludes that turbulence is not affected by the presence of the nanoparticles; therefore, it cannot explain the observed enhancement. Particle rotation has also been proposed as a cause of heat transfer enhancement, but Buongiorno calculates that this effect is too small to explain the effect. With dispersion, turbulence, and particle rotation ruled out as significant agencies for heat transfer enhancement, Buongiorno proposed a new model based on the mechanics of the nanoparticle/base-fluid relative velocity. Buongiorno (2006) noted that the nanoparticle absolute velocity could be viewed as the sum of the base fluid velocity and a relative velocity (that he calls the slip velocity). He considered, in turn, seven slip mechanisms: inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage, and gravity settling. After examining each of these in turn, he concluded that in the absence of turbulent effects, it is the Brownian diffusion and the thermophoresis that will be important. Buongiorno proceeded to write down conservation equations based on these two effects.

The Bénard problem (the onset of convection in a horizontal layer uniformly heated from below) for a nanofluid was studied by Tzou (2008a,b) and Nield and Kuznetsov (2009a) on the basis of the transport equations of Buongiorno (2006). The corresponding problem for flow in a porous medium (the Horton–Rogers–Lapwood problem) was studied by Nield and Kuznetsov (2009b) using the Darcy model. In this article, that study is extended to the Brinkman model. This necessitates the introduction of an additional parameter, namely, a Darcy number.

2 Analysis

It is assumed that nanoparticles are suspended in the nanofluid using either surfactant or surface charge technology. This prevents the particles from agglomeration and deposition on the porous matrix. We select a coordinate frame in which the z-axis is aligned vertically upward. We consider a horizontal layer of a porous medium confined between the planes $z^* = 0$ and $z^* = H$. Asterisks are used to denote the dimensional variables. Each boundary wall is assumed to be impermeable and perfectly thermally conducting. The temperatures at the lower and upper wall are taken to be T_h^* and T_c^*, the former being the greater. The

Oberbeck–Boussinesq approximation is employed. The reference temperature is taken to be T_c^*. In the linear theory being applied here, the temperature change in the fluid is assumed to be small in comparison with T_c^*.

Homogeneity and local thermal equilibrium in the porous medium are assumed. We are aware that thermal lagging between the particles and the fluid has been proposed as an explanation of the increased thermal conductivity that has been observed in nanofluids (see, for example, Vadasz 2005, 2006), but this is not our concern here. The extra complication of local thermal non-equilibrium could well be the subject of future research.

We consider a porous medium whose porosity is denoted by ε and permeability by K. The Darcy velocity is denoted by $\mathbf{v}_D$. The following four field equations embody the conservation of total mass, momentum, thermal energy, and nanoparticles, respectively. The field variables are the Darcy velocity $\mathbf{v}_D$, the temperature T^*, and the nanoparticle volume fraction ϕ^*.

$$\nabla^* \cdot \mathbf{v}_D^* = 0, \tag{1}$$

$$\frac{\rho_f}{\varepsilon}\frac{\partial \mathbf{v}_D^*}{\partial t^*} = -\nabla^* p^* + \tilde{\mu}\nabla^{*2}\mathbf{v}_D - \frac{\mu}{K}\mathbf{v}_D^* + \left[\phi^*\rho_p + (1-\phi^*)\left\{\rho_f(1-\beta(T^*-T_c^*))\right\}\right]\mathbf{g}, \tag{2}$$

$$(\rho c)_m \frac{\partial T^*}{\partial t^*} + (\rho c)_f \mathbf{v}_D^* \cdot \nabla^* T^* = k_m \nabla^{*2} T^* + \varepsilon(\rho c)_p \left[D_B \nabla^*\phi^* \cdot \nabla^* T^* + (D_T/T_c^*)\nabla^* T^* \cdot \nabla^* T^*\right], \tag{3}$$

$$\frac{\partial \phi^*}{\partial t^*} + \frac{1}{\varepsilon}\mathbf{v}_D^* \cdot \nabla^*\phi^* = D_B \nabla^{*2}\phi^* + (D_T/T_c^*)\nabla^{*2} T^*. \tag{4}$$

We write $\mathbf{v}_D^* = (u^*, v^*, w^*)$.

Here ρ_f, μ, and β are the density, viscosity, and volumetric volume expansion coefficient of the fluid, while ρ_p is the density of the particles. The gravitational acceleration is denoted by $\mathbf{g}$. We have introduced the effective viscosity $\tilde{\mu}$, the effective heat capacity $(\rho c)_m$, and the effective thermal conductivity k_m of the porous medium. The coefficients that appear in Eqs. 3 and 4 are the Brownian diffusion coefficient D_B and the thermophoretic diffusion coefficient D_T. Details of the derivation of Eqs. 3 and 4 are given in the articles by Buongiorno (2006); Tzou (2008a,b), and Nield and Kuznetsov (2009a,b). The flow is assumed to be slow so that an advective term and a Forchheimer quadratic drag term do not appear in the momentum equation.

We assume that the temperature and the volumetric fraction of the nanoparticles are constant on the boundaries. Thus, the boundary conditions are

$$w^* = 0, \quad \frac{\partial w^*}{\partial z^*} + \lambda_1 H \frac{\partial^2 w^*}{\partial z^{*2}} = 0, \quad T^* = T_h^*, \quad \phi^* = \phi_0^* \quad \text{at } z^* = 0, \tag{5}$$

$$w^* = 0, \quad \frac{\partial w^*}{\partial z^*} - \lambda_2 H \frac{\partial^2 w^*}{\partial z^{*2}} = 0, \quad T^* = T_c^*, \quad \phi^* = \phi_1^* \quad \text{at } z^* = H. \tag{6}$$

The parameters λ_1 and λ_2 each take the value 0 for the case of a rigid boundary and ∞ for a free boundary.

We introduce dimensionless variables as follows. We define

$$(x, y, z) = (x^*, y^*, z^*)/H, \quad t = t^*\alpha_m/\sigma H^2, \quad (u, v, w) = (u^*, v^*, w^*)H/\alpha_m,$$
$$p = p^* K/\mu\alpha_m, \quad \phi = \frac{\phi^* - \phi_0^*}{\phi_1^* - \phi_0^*}, \quad T = \frac{T^* - T_c^*}{T_h^* - T_c^*}, \tag{7}$$

where

$$\alpha_{\mathrm{m}} = \frac{k_{\mathrm{m}}}{(\rho c)_{\mathrm{f}}}, \quad \sigma = \frac{(\rho c)_{\mathrm{m}}}{(\rho c)_{\mathrm{f}}}. \tag{8}$$

Then Eqs. 1–6 take the form:

$$\nabla \cdot \mathbf{v} = 0 \tag{9}$$

$$\frac{Da}{Pr}\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + Da\nabla^2\mathbf{v} - \mathbf{v} - Rm\hat{\mathbf{e}}_z + RaT\hat{\mathbf{e}}_z - Rn\phi\hat{\mathbf{e}}_z \tag{10}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \nabla^2 T + \frac{N_{\mathrm{B}}}{Le}\nabla\phi\cdot\nabla T + \frac{N_{\mathrm{A}}N_{\mathrm{B}}}{Le}\nabla T\cdot\nabla T \tag{11}$$

$$\frac{1}{\sigma}\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla\phi = \frac{1}{Le}\nabla^2\phi + \frac{N_{\mathrm{A}}}{Le}\nabla^2 T \tag{12}$$

$$w = 0, \quad \frac{\partial w}{\partial z} + \lambda_1\frac{\partial^2 w}{\partial z^2} = 0, \quad T = 1, \quad \phi = 0 \quad \text{at } z = 0, \tag{13}$$

$$w = 0, \quad \frac{\partial w}{\partial z} - \lambda_2\frac{\partial^2 w}{\partial z^2} = 0, \quad T = 0, \quad \phi = 1 \quad \text{at } z = 1. \tag{14}$$

Here

$$Pr = \frac{\mu}{\rho_{\mathrm{f}}\alpha_{\mathrm{m}}}, \quad Da = \frac{\tilde{\mu}K}{\mu H^2}, \quad Le = \frac{\alpha_{\mathrm{m}}}{D_{\mathrm{B}}}, \tag{15a,b,c}$$

$$Ra = \frac{\rho g\beta KH(T_{\mathrm{h}}^* - T_{\mathrm{c}}^*)}{\mu\alpha_{\mathrm{m}}}, \tag{16}$$

$$Rm = \frac{[\rho_{\mathrm{p}}\phi_1^* + \rho(1-\phi_1^*)]gKH}{\mu\alpha_{\mathrm{m}}}, \tag{17}$$

$$Rn = \frac{(\rho_{\mathrm{p}} - \rho)(\phi_1^* - \phi_0^*)gKH}{\mu\alpha_{\mathrm{m}}}, \tag{18}$$

$$N_{\mathrm{A}} = \frac{D_T(T_{\mathrm{h}}^* - T_{\mathrm{c}}^*)}{D_{\mathrm{B}}T_{\mathrm{c}}^*(\phi_1^* - \phi_0^*)}, \tag{19}$$

$$N_{\mathrm{B}} = \frac{\varepsilon(\rho c)_{\mathrm{p}}}{(\rho c)_{\mathrm{f}}}(\phi_1^* - \phi_0^*). \tag{20}$$

The parameter Pr is the Prandtl number and Da is the Darcy number modified by the viscosity ratio, while Le is the Lewis number and Ra is the familiar thermal Rayleigh–Darcy number. The new parameters Rm and Rn may be regarded as the basic-density Rayleigh number and the concentration Rayleigh number, respectively. The parameter N_{A} is a modified diffusivity ratio and is somewhat similar to the Soret parameter that arises in cross-diffusion phenomena in solutions, while N_{B} is a modified particle-density increment.
In the spirit of the Oberbeck–Boussinesq approximation, Eq. 10 has been linearized by the neglect of a term proportional to the product of ϕ and T. This assumption is likely to be valid in the case of small temperature gradients in a dilute suspension of nanoparticles.

2.1 Basic Solution

We seek a time-independent quiescent solution of Eqs. 9–14 with temperature and nanoparticle volume fraction varying in the z-direction only, which is a solution of the form

$$\mathbf{v} = 0, \quad T = T_{\mathrm{b}}(z), \quad \phi = \phi_{\mathrm{b}}(z).$$

Equations 11 and 12 reduce to

$$\frac{d^2 T_b}{dz^2} + \frac{N_B}{Le}\frac{d\phi_b}{dz}\frac{dT_b}{dz} + \frac{N_A N_B}{Le}\left(\frac{dT_b}{dz}\right)^2 = 0 \tag{21}$$

$$\frac{d^2\phi_b}{dz^2} + N_A\frac{d^2 T_b}{dz^2} = 0 \tag{22}$$

Using the boundary conditions (13) and (14), Eq. 22 may be integrated to give

$$\phi_b = -N_A T_b + (1 - N_A)z + N_A, \tag{23}$$

and substitution of this into Eq. 21 gives

$$\frac{d^2 T_b}{dz^2} + \frac{(1 - N_A)N_B}{Le}\frac{dT_b}{dz} = 0. \tag{24}$$

The solution of Eq. 24 satisfying Eqs. 13 and 14 is

$$T_b = \frac{1 - e^{-(1-N_A)N_B(1-z)/Le}}{1 - e^{-(1-N_A)N_B/Le}}. \tag{25}$$

The remainder of the basic solution is easily obtained by first substituting in Eq. 23 to obtain ϕ_b and then using integration of Eq. 10 to obtain p_b.

According to Buongiorno (2006), for most nanofluids investigated so far Le is large, of the order of 10^5–10^6, while N_A is not greater than about 10. Then, the exponents in Eq. 25 are small, and so to a good approximation, one has

$$T_b = 1 - z, \tag{26}$$

and so,

$$\phi_b = z. \tag{27}$$

2.2 Perturbation Solution

We now superimpose perturbations on the basic solution. We write

$$\mathbf{v} = \mathbf{v}', \quad p = p_b + p', \quad T = T_b + T', \quad \phi = \phi_b + \phi', \tag{28}$$

substitute in Eqs. 9–14, and linearize by neglecting the products of primed quantities. The following equations are obtained when Eqs. 26 and 27 are used.

$$\nabla \cdot \mathbf{v}' = 0, \tag{29}$$

$$\frac{Da}{Pr}\frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + Da\nabla^2\mathbf{v}' - \mathbf{v}' + RaT'\hat{\mathbf{e}}_z - Rn\phi'\hat{\mathbf{e}}_z, \tag{30}$$

$$\frac{\partial T'}{\partial t} - w' = \nabla^2 T' + \frac{N_B}{Le}\left(\frac{\partial T'}{\partial z} - \frac{\partial \phi'}{\partial z}\right) - \frac{2N_A N_B}{Le}\frac{\partial T'}{\partial z}, \tag{31}$$

$$\frac{1}{\sigma}\frac{\partial \phi'}{\partial t} + \frac{1}{\varepsilon}w' = \frac{1}{Le}\nabla^2\phi' + \frac{N_A}{Le}\nabla^2 T', \tag{32}$$

$$w' = 0, \quad \frac{\partial w}{\partial z} + \lambda_1\frac{\partial^2 w}{\partial z^2} = 0, \quad T' = 0, \quad \phi' = 0 \quad \text{at } z = 0. \tag{33}$$

$$w' = 0, \quad \frac{\partial w}{\partial z} - \lambda_2\frac{\partial^2 w}{\partial z^2} = 0, \quad T' = 0, \quad \phi' = 0 \quad \text{at } z = 1. \tag{34}$$

It will be noted that the parameter *Rm* is not involved in these and subsequent equations. It is just a measure of the basic static pressure gradient.

For the case of a regular fluid (not a nanofluid) the parameters *Rn* , N_A, and N_B are zero, the second term in Eq. 32 is absent because $d\phi_b/dz = 0$ rather than 1, and then, Eq. 32 is satisfied trivially. The remaining equations are reduced to the familiar equations for the Brinkman extension of the Horton–Roger–Lapwood problem.

The six unknowns u', v', w', p', T', and, ϕ' can be reduced to three by operating on Eq. 30 with $\hat{\mathbf{e}}_z \cdot curlcurl$ and using Eq. 29. The result is

$$\frac{Da}{Pr}\frac{\partial}{\partial t}\nabla^2 w' - Da\nabla^4 w' + \nabla^2 w' = Ra\nabla_H^2 T' - Rn\nabla_H^2\phi'. \tag{35}$$

Here ∇_H^2 is the two-dimensional Laplacian operator on the horizontal plane.

The differential Eqs. 31, 32, and 35 the boundary conditions (33) and (34) constitute a linear boundary-value problem that can be solved using the method of normal modes.

We write

$$(w', T', \phi') = [W(z), \Theta(z), \Phi(z)]\exp(st + ilx + imy), \tag{36}$$

and substitute into the differential equations to obtain

$$\left[Da(D^2-\alpha^2)^2 - \left(1+\frac{sDa}{Pr}\right)(D^2-\alpha^2)\right]W - Ra\alpha^2\Theta + Rn\alpha^2\Phi = 0, \tag{37}$$

$$W + \left(D^2 + \frac{N_B}{Le}D - \frac{2N_A N_B}{Le}D - \alpha^2 - s\right)\Theta - \frac{N_B}{Le}D\Phi = 0, \tag{38}$$

$$\frac{1}{\varepsilon}W - \frac{N_A}{Le}(D^2-\alpha^2)\Theta - \left(\frac{1}{Le}(D^2-\alpha^2) - \frac{s}{\sigma}\right)\Phi = 0, \tag{39}$$

$$W = 0, \quad DW + \lambda_1 D^2 W = 0, \Theta = 0, \quad \Phi = 0 \quad \text{at } z = 0. \tag{40}$$

$$W = 0, \quad DW - \lambda_2 D^2 W = 0, \Theta = 0, \quad \Phi = 0 \quad \text{at } z = 1, \tag{41}$$

where

$$D \equiv \frac{d}{dz} \quad \text{and} \quad \alpha = (l^2 + m^2)^{1/2}. \tag{42}$$

Thus, α is a dimensionless horizontal wavenumber.

For neutral stability, the real part of s is zero. Hence, we now write $s = i\omega$, where ω is real and is a dimensionless frequency.

We now employ a Galerkin-type weighted residuals method to obtain an approximate solution to the system of Eqs. 37–41. We choose as trial functions (satisfying the boundary conditions) W_p, Θ_p, Φ_p; $p = 1, 2, 3, \ldots$, and write

$$W = \sum_{p=1}^{N} A_p W_p, \quad \Theta = \sum_{p=1}^{N} B_p \Theta_p, \quad \Phi = \sum_{p=1}^{N} C_p \Phi_p, \tag{43}$$

substitute into Eqs. 37–39, and make the expressions on the left-hand sides of those equations (the residuals) orthogonal to the trial functions, thereby obtaining a system of $3N$ linear algebraic equations in the $3N$ unknowns A_p, B_p, C_p; $p = 1, 2, \ldots, N$. The vanishing of the determinant of coefficients produces the eigenvalue equation for the system. One can regard *Ra* as the eigenvalue. Thus, *Ra* is found in terms of the other parameters.

3 Results and Discussion

3.1 Free–Free Boundaries

For this case, the boundary conditions are

$$W=0,\quad D^2W=0,\quad \Theta=0,\quad \Phi=0 \quad \text{at } z=0 \quad \text{and at } z=1, \tag{44}$$

and the trial functions can be chosen as

$$W_p=\Theta_p=\Phi_p=\sin p\pi z;\quad p=1,2,3,\ldots \tag{45}$$

3.1.1 Non-Oscillatory Convection

First, we consider the case of non-oscillatory instability, when $\omega=0$.
For a first approximation, we take $N=1$. This produces the result

$$Ra=\frac{Da\left(\pi^2+\alpha^2\right)^3+\left(\pi^2+\alpha^2\right)^2}{\alpha^2}-\left(\frac{Le}{\varepsilon}+N_{\mathrm{A}}\right)Rn. \tag{46}$$

For the case when $Da=0$, the minimum is attained with $\alpha=\pi$, and the minimum value is

$$Ra=4\pi^2-\left(\frac{Le}{\varepsilon}+N_{\mathrm{A}}\right)Rn. \tag{47}$$

On the other hand, in the case where Da is large compared with unity, the minimum being attained at $\alpha=\pi/\sqrt{2}$, and the minimum value is

$$Ra=\frac{27\pi^4}{4}Da-\left(\frac{Le}{\varepsilon}+N_{\mathrm{A}}\right)Rn. \tag{48}$$

One recognizes that in the absence of nanoparticles, one recovers the well-known results that the critical Rayleigh–Darcy number is equal to $4\pi^2$ when $Da=0$, and that the critical value of the fluid Rayleigh number is $27\pi^4/4=657.5$ in the case where Da tends to infinity. Usually when one employs a single-term Galerkin approximation in this context, one gets an overestimate; however, in this case, the approximation happens to give the exact result.

As noted above, for a typical nanofluid, Le is of the order of 10^5–10^6, and N_{A} is not much >10. Hence, the coefficient of Rn in Eq. 46 is large and negative. Thus, under the approximations we have made so far, we have the result that the presence of nanoparticles lowers the value of the critical Rayleigh number, usually by a substantial amount, in the case when Rn is positive, that is, when the basic nanoparticle distribution is a top-heavy one.

It will be noted that in Eq. 46, the parameter N_{B} does not appear. The instability is almost purely a phenomenon due to buoyancy coupled with the conservation of nanoparticles. It is independent of the contributions of Brownian motion and thermophoresis to the thermal energy equation. Rather, the Brownian motion and thermophoresis enter to produce their effects directly into the equation expressing the conservation of nanoparticles, so that the temperature and the particle density are coupled in a particular way, and that results in the thermal and concentration buoyancy effects being coupled in the same way. It is useful to emphasize this by rewriting Eq. 46 in the form

$$Ra+\left(\frac{Le}{\varepsilon}+N_{\mathrm{A}}\right)Rn=\frac{Da\left(\pi^2+\alpha^2\right)^3+\left(\pi^2+\alpha^2\right)^2}{\alpha^2}, \tag{49}$$

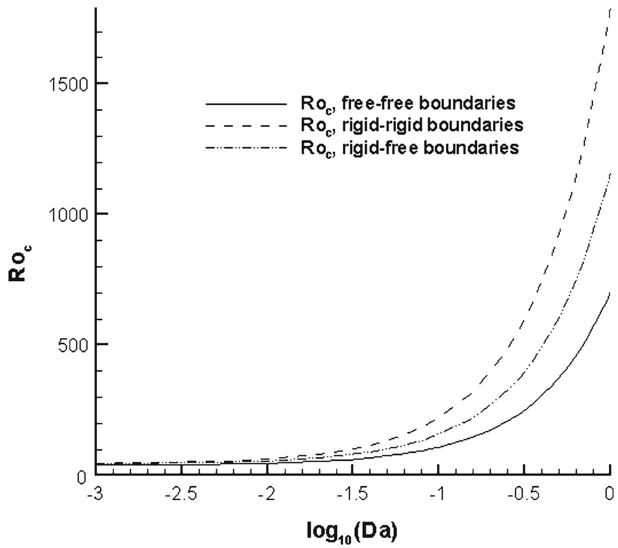

Fig. 1 Plots of Ro_c, the minimum of the right-hand sides of Eqs. 49, 67, 71, respectively, as functions of the Darcy number *Da*

and noting that the left-hand side is a linear combination of the thermal Rayleigh number *Ra* and the concentration Rayleigh number *Rn*. The problem is analogous to the familiar double diffusive problem (Nield and Bejan 2006). It is also analogous to a bioconvection problem discussed in Kuznetsov and Avramenko (2004). We have defined *Rn* in a way so that it is positive when the applied particle density increases upward (the destabilizing situation). We note that *Ra* takes a negative value when *Rn* is sufficiently large. In this case, the destabilizing effect of concentration is so great that the bottom of the fluid layer must be cooled relative to the top to produce a state of neutral stability.

Let us denote by *Ro* the expression on the right-hand side of Eq. 49. Its minimum value (Ro_c) and the value of α that gives the minimum (α_c) are plotted in Figs. 1 and 2, respectively, as functions of *Da*. We observe a smooth transition from the values $(4\pi^2, \pi)$ for $Da=0$ to the values $([27\pi^4/4]Da, \pi/\sqrt{2})$ as $Da \to \infty$.

It is emphasized that the simple expression in Eq. 46 arises because the Lewis number has been assumed to be large. In order to estimate the contribution of the terms involving N_B, the 2-term Galerkin result has been investigated. The expression in the eigenvalue equation is now complicated, and it is difficult to make a statement that is simultaneously precise, simple, and general. However, it is clear that the functions of N_B are of second degree. We conclude that for practical purposes, Eq. 46 is a good approximation.

3.1.2 Oscillatory Convection

We now consider the case $\omega \neq 0$. We confine ourselves to the one-term Galerkin approximation. The eigenvalue equation now takes the form

$$Ra\alpha^2\left(\frac{J}{Le}+\frac{i\omega}{\sigma}\right)+Rn\alpha^2\left(\frac{N_A J}{Le}+\frac{J+i\omega}{\varepsilon}\right)$$
$$=J\left(DaJ+1+\frac{i\omega Da}{Pr}\right)(J+i\omega)\left(\frac{J}{Le}+\frac{i\omega}{\sigma}\right), \tag{50}$$

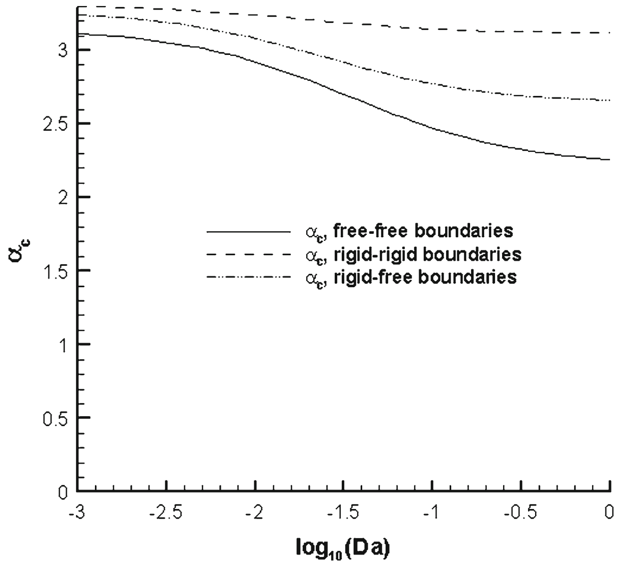

Fig. 2 Plots of α_c, the critical dimensionless wavenumber, for various sets of boundary conditions, as functions of the Darcy number *Da*

where for shorthand, we have written

$$J = \pi^2 + \alpha^2. \tag{51}$$

The real and imaginary parts of Eq. 50 yield

$$Ra + \left(\frac{Le}{\varepsilon} + N_{\mathrm{A}}\right) Rn = \frac{J^2(DaJ + 1)}{\alpha^2} - \frac{\omega^2}{\alpha^2}\left[(DaJ + 1)\frac{Le}{\sigma} + \frac{DaJ}{Pr} + \frac{DaLeJ}{Pr\sigma}\right], \tag{52}$$

$$\frac{Le}{\sigma}Ra + \frac{Le}{\varepsilon}Rn = \frac{J^2}{\alpha^2}\left[(DaJ + 1)\left(1 + \frac{Le}{\sigma}\right) + \frac{DaJ}{Pr}\right] - \frac{\omega^2 DaLeJ}{Pr\sigma\alpha^2} = 0. \tag{53}$$

Elimination of ω^2 between the last two equations gives

$$\begin{aligned} &\left\{\frac{LePr}{\sigma}(DaJ + 1) + \frac{Le}{\sigma}DaJ\right\} Ra + \left\{\frac{LePr}{\varepsilon}(DaJ + 1) + \left(\frac{\sigma}{\varepsilon} - N_{\mathrm{A}}\right) DaJ\right\} Rn \\ &= \frac{J^2}{\alpha^2}\left\{\left[(DaJ + 1)\left(1 + \frac{Le}{\sigma}\right) + \frac{DaJ}{Pr}\right]\left[Pr\,(DaJ + 1) + \left(1 + \frac{\sigma}{Le}\right) DaJ\right]\right. \\ &\qquad \left. - DaJ\,(DaJ + 1)\right\}. \end{aligned} \tag{54}$$

One observes from Eq. 52 that in order for ω to be real, it is necessary that

$$Ra + \left(\frac{Le}{\varepsilon} + N_{\mathrm{A}}\right) Rn \le \frac{J^2(DaJ + 1)}{\alpha^2}. \tag{55}$$

Hence, Eq. 54 gives the oscillatory stability boundary when Eq. 55 holds, and the angular frequency ω of the oscillation is then given by

$$\omega^2 = \frac{J^2(DaJ+1) - Ra\alpha^2 - \left(\frac{Le}{\varepsilon} + N_A\right) Rn\alpha^2}{(DaJ+1)\frac{Le}{\sigma} + \frac{DaJ}{Pr} + \frac{DaLeJ}{Pr\sigma}}. \tag{56}$$

3.2 Rigid–Rigid Boundaries

We confine our analysis to the one-term Galerkin approximation. Appropriate trial functions satisfying the boundary conditions, which are now

$$W = 0, \quad DW = 0, \quad \Theta = 0, \quad \Phi = 0 \quad \text{at } z = 0 \quad \text{and at } z = 1, \tag{57}$$

are

$$W_1 = z^2(1-z)^2, \quad \Theta_1 = z(1-z), \quad \Phi_1 = z(1-z). \tag{58}$$

With this choice of trial functions, the eigenvalue equation takes the form

$$\begin{aligned} &\left[\tilde{\mu} + i\omega\frac{Le}{\sigma}\right] Ra + \left[\tilde{\mu}\left(\frac{Le}{\varepsilon} + N_A\right) + i\omega\frac{Le}{\varepsilon}\right] Rn \\ &= \frac{1}{\tilde{\rho}}\left[\tilde{\lambda} Da + \tilde{\nu}\left(1 + i\omega\frac{Da}{Pr}\right)\right](\tilde{\mu} + i\omega)\left(\tilde{\mu} + i\omega\frac{Le}{\sigma}\right), \end{aligned} \tag{59}$$

where

$$\tilde{\lambda} = 504 + 24\alpha^2 + \alpha^4, \quad \tilde{\mu} = 10 + \alpha^2, \quad \tilde{\nu} = 12 + \alpha^2, \quad \tilde{\rho} = \frac{27\alpha^2}{28}. \tag{60}$$

The real and imaginary parts of Eq. 59 yield

$$Ra + \left(\frac{Le}{\varepsilon} + N_A\right) Rn = \frac{(\tilde{\lambda} Da + \tilde{\nu})\tilde{\mu}}{\tilde{\rho}} - \frac{\omega^2}{\tilde{\rho}}\left[\frac{(\tilde{\lambda} Da + \tilde{\nu})Le}{\tilde{\mu}\sigma} + \tilde{\nu}\left(\frac{DaLe}{\sigma Pr} + \frac{Da}{Pr}\right)\right], \tag{61}$$

$$\omega\left\{\frac{Le}{\sigma}Ra + \frac{Le}{\varepsilon}Rn - \frac{1}{\tilde{\rho}}\left[\frac{\tilde{\mu}^2\tilde{\nu} Da}{Pr} + (\tilde{\lambda} Da + \tilde{\nu})\tilde{\mu}\left(1 + \frac{Le}{\sigma}\right) - \omega^2\tilde{\nu}\frac{DaLe}{\sigma Pr}\right]\right\} = 0. \tag{62}$$

Hence, either $\omega = 0$ and

$$Ra + \left(\frac{Le}{\varepsilon} + N_A\right) Rn = \frac{(\tilde{\lambda} Da + \tilde{\nu})\tilde{\mu}}{\tilde{\rho}}, \tag{63}$$

or one has the pair of equations

$$\omega^2\left[\frac{(\tilde{\lambda} Da + \tilde{\nu})Le}{\tilde{\mu}\sigma} + \tilde{\nu}\left(\frac{DaLe}{\sigma Pr} + \frac{Da}{Pr}\right)\right] = (\tilde{\lambda} Da + \tilde{\nu})\tilde{\mu} - \tilde{\rho}\left[Ra + \left(\frac{Le}{\varepsilon} + N_A\right) Rn\right], \tag{64}$$

$$\omega^2\tilde{\nu}\frac{DaLe}{\sigma Pr} = \frac{\tilde{\mu}^2\tilde{\nu} Da}{Pr} + (\tilde{\lambda} Da + \tilde{\nu})\tilde{\mu}\left(1 + \frac{Le}{\sigma}\right) - \tilde{\rho}\left(\frac{Le}{\sigma}Ra + \frac{Le}{\varepsilon}Rn\right). \tag{65}$$

Elimination of ω^2 then gives

$$\left(\frac{\tilde{\lambda}Da+\tilde{\nu}}{\tilde{\mu}\tilde{\nu}\sigma^2}+\frac{Da}{\sigma^2 Pr}\right)Ra+\left(\frac{\tilde{\lambda}Da+\tilde{\nu}}{\tilde{\mu}\tilde{\nu}\varepsilon\sigma}+\frac{Da(\sigma-\varepsilon N_A)}{\varepsilon\sigma Le Pr}\right)Rn$$
$$=\frac{1}{\tilde{\rho}}\left\{\left[(\tilde{\lambda}Da+\tilde{\nu})\tilde{\mu}\left(\frac{1}{\sigma}+\frac{1}{Le}\right)+\frac{\tilde{\mu}^2\tilde{\nu}Da}{Le Pr}\right]\left[\frac{\tilde{\lambda}Da+\tilde{\nu}}{\tilde{\mu}\tilde{\nu}\sigma}+\frac{Da}{\sigma Pr}+\frac{Da}{Le Pr}\right]\right.$$
$$\left.-\frac{(\tilde{\lambda}Da+\tilde{\nu})\tilde{\mu}Da}{\sigma Le Pr}\right\}. \quad (66)$$

The boundary for non-oscillatory instability is given by Eq. 63, namely,

$$Ra+\left(\frac{Le}{\varepsilon}+N_A\right)Rn=\frac{28}{27\alpha^2}[(504+24\alpha^2+\alpha^4)Da+12+\alpha^2](10+\alpha^2). \quad (67)$$

When Da is very large compared with unity the right-hand side of this equation takes a minimum when $\alpha = 3.12$, and its minimum value is 1,750. We recognize that the value 1,750 obtained using the one-term Galerkin approximation is about 3% greater than the well-known exact value 1707.762 for the critical Rayleigh number for the classical Raleigh–Bénard problem.

When $Da = 0$, the right-hand side of Eq. 67 takes a minimum when $\alpha = 3.31$ and the minimum value is 43.91, something that is 11% greater than the exact value $4\pi^2$ for the classical Horton–Rogers–Lapwood problem. For comparison, it may be noted that Platten and Legros (1984) conclude that in a determination of the critical Rayleigh number using the Schmidt–Milverton method, it is difficult to reduce the experimental error to a value 7%. A plot of Ro_c, the minimum of the right-hand side of Eq. 67 is shown in Fig. 1, and a plot of α_c, the minimizing value of α is shown in Fig. 2.

The oscillatory instability boundary is given by Eq. 66 when the right-hand side of Eq. 64 is positive, so that the equation yields a real value for ω. When Le is large, this requires that Rn is negative, so that the basic nanoparticle distribution is stabilizing.

3.3 Rigid–Free Boundaries

The analysis for this case is the same as that for the rigid–rigid case, except that now the boundary conditions on W become

$$\begin{array}{ll} W=0, \quad DW=0 & \text{at } z=0, \\ W=0, \quad D^2W=0 & \text{at } z=1, \end{array} \quad (68)$$

so that a suitable trial function is now,

$$W_1 = z^2(1-z)(3-2z). \quad (69)$$

Instead of Eq. 60, one now has

$$\tilde{\lambda}=4536+432\alpha^2+19\alpha^4,\quad \tilde{\mu}=10+\alpha^2,\quad \tilde{\nu}=216+19\alpha^2,\quad \tilde{\rho}=\frac{507\alpha^2}{28}. \quad (70)$$

Hence, instead of Eq. 67 one now has

$$Ra+\left(\frac{Le}{\varepsilon}+N_A\right)Rn=\frac{28}{507\alpha^2}\left[(4536+432\alpha^2+19\alpha^4)Da+216+19\alpha^2\right](10+\alpha^2). \quad (71)$$

When Da is very large compared with unity, the right-hand side of this equation takes a minimum when $\alpha = 2.67$, and its minimum value is 1,139. We recognize that the value 1139 obtained using the one-term Galerkin approximation is about 3.5% greater than the well-known exact value 1100.65 for the critical Rayleigh number for the classical Raleigh–Bénard problem.

When $Da = 0$, the right-hand side of Eq. 71 takes a minimum when $\alpha = 3.27$, and the minimum value is 48.01, something that is 22% greater than the exact value $4\pi^2$ for the classical Horton–Rogers–Lapwood problem. The relatively poor accuracy in this case is expected since the asymmetric (about the horizontal midline) trial function W_1 significantly differs in shape from the symmetric exact function $W = \sin \pi z$. A plot of Ro_c, the minimum of the right-hand side of Eq. 71 is shown in Fig. 1, and a plot of α_c, the minimizing value of α, is shown in Fig. 2. As one would expect, the values for the rigid–free case are about half way between the corresponding values for the free–free and the rigid–rigid cases, except when Da is very small.

4 Conclusions

We have studied analytically using linear instability theory the onset of convection in a horizontal layer of a porous medium saturated by a nanofluid, employing a model used for the nanofluid that incorporates the effects of Brownian motion and thermophoresis and employing the Brinkman momentum equation. We found that for a typical nanofluid (for which the Lewis number is large), the primary contribution of the nanoparticles is via a buoyancy effect coupled with the conservation of nanoparticles, with the contribution of nanoparticles to the thermal energy equation being a second-order effect. The analysis predicts that oscillatory instability is possible in the case of a bottom-heavy nanoparticle distribution.

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