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The Effect of Strong Heterogeneity on the Onset of Convection in a Porous Medium: Periodic and Localized Variation

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Abstract The effect of strong heterogeneity on the onset of convection induced by a vertical density gradient in a saturated porous medium governed by Darcy's law is investigated. The general case, where there is heterogeneity in both the vertical and horizontal directions, and where there is heterogeneity in permeability, thermal conductivity, and applied temperature gradient, is considered. A computer package has been developed to implement an algorithm giving a criterion for instability, and this is now employed to investigate the case where the variation is of a periodic or localized nature.

Keywords Heterogeneity · Stability · Natural convection · Computer package · Periodic variation · Localized variation

1 Introduction

A large number of articles have been published on the topic of the onset of convection in a heterogeneous porous medium. This work has been surveyed by Nield and Bejan (2006). However, the question has been raised as to whether the classical analysis based on the concept of a Rayleigh number is in fact valid for heterogeneous systems. Simmons et al. (2001) and Prasad and Simmons (2003) pointed out that, in many heterogeneous geologic systems, hydraulic properties such as the hydraulic conductivity of the system under consideration can

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vary by many orders of magnitude and sometimes rapidly over small spatial scales. In such systems heterogeneity occurs over many spatial scales and variable density flow phenomena may be triggered, grow, and decay over a very large mix of different spatial and temporal scales. There are inherent difficulties in not only defining an appropriate average Rayleigh number in a heterogeneous system but also in then determining whether any *Ra* number, should it be accurately quantified, implies stable or unstable flows are theoretically expected.

For the case where the heterogeneity is weak, by which we mean that the amount variation of a quantity like the permeability is small in comparison with its mean value (i.e., the fractional change is of order ε where $\varepsilon << 1$), steps to assess the applicability of an average Rayleigh number as a criterion for the onset of convection were made in a series of articles by Nield and Kuznetsov (2007a–e, 2008a, b) and Kuznetsov and Nield (2008). These authors studied the combined effects of vertical heterogeneity and horizontal heterogeneity, for a two-dimensional situation. (The term *vertical heterogeneity* is used to refer to *variation in the vertical direction*. An extreme example is when one has a series of horizontal layers in each of which the permeability is uniform.)

For the case of strong heterogeneity, Nield and Simmons (2007) proposed a rough and ready criterion for the onset of convection in that situation. This criterion is not restricted to the two-dimensional situation. It is based on the results of Beck (1977). These show the variation of the critical Rayleigh number, and the preferred cellular mode, as functions of the aspect ratios $A_x = H/L_x$ and $A_y = H/L_y$ for a three-dimensional box with height H and horizontal dimensions L_x and L_y. Beck's figures, which are reproduced as Figures 6.22 and 6.23 in Nield and Bejan (1992, 1999, 2006), apply to a box with impermeable conducting top and bottom and impermeable insulating sidewalls, and occupied by a porous medium for which the Darcy model is applicable. Beck showed that, the critical Rayleigh number for a homogeneous medium is given by

$$Ra = \pi^2 \min\left(b + \frac{1}{b}\right)^2, \tag{1}$$

where

$$b = \left[(pA_x)^2 + (qA_y)^2\right]^{1/2}, \tag{2}$$

and the minimum is taken over the set of non-negative integer's p and q. Beck's figures show that, in the region $A_x < 1$, $A_y < 1$, the value of *Ra* does not exceed 40.7. Also, in the region $A_x > A_y > 1$, the critical mode is $p = 1, q = 0$, so that

$$Ra = \pi^2 \left(A_x + A_x^{-1}\right)^2. \tag{3}$$

Furthermore, when $A_x > 1$ and $A_y < 1$ the value of *Ra* does not exceed the value given by the expression in Eq. 3.

Similarly, in the region $A_y > A_x > 1$, the critical mode is $p = 0, q = 1$, so that

$$Ra = \pi^2 \left(A_y + A_y^{-1}\right)^2, \tag{4}$$

and when $A_y > 1$ and $A_x < 1$ the value of *Ra* does not exceed the value given by the expression in Eq. 4.

Nield and Simmons (2007) constructed a generalized Rayleigh number for a heterogeneous box in the following way. Their basic idea is that, if at any stage one finds instability in any part of the enclosure at any time, then the whole system can be considered to be unstable. They started with a domain consisting of a box and considered sub-domains. Each sub-domain is taken to be a rectangular box, of arbitrary size, and with arbitrary aspect ratios, bounded by planes $x = x_1, x = x_2, y = y_1, y = y_2, z = z_1, z = z_2$. They called these special rectangular subdomains "sub-boxes." Then for each sub-box they calculated the aspect ratios and a local Rayleigh number Ra_l based on the height of the sub-box, and with other properties given the mean value over the sub-box. In particular, the sub-box mean of the basic temperature gradient at a particular time is employed here. Then they defined a geometrically adjusted Rayleigh number Ra_{g} defined by

$$Ra_{\mathrm{g}} = \begin{cases} Ra_l & \text{if } A_{\mathrm{x}} < 1 \text{ and } A_{\mathrm{y}} < 1, \\ \dfrac{4Ra_l}{\left(\mathrm{A_x} + \mathrm{A_x^{-1}}\right)^2} & \text{if } A_x > 1 \text{ and } A_x > A_y, \\ \dfrac{4Ra_l}{\left(\mathrm{A}_y + \mathrm{A}_y^{-1}\right)^2} & \text{if } A_y > 1 \text{ and } A_y > A_x. \end{cases} \tag{5}$$

Finally, they defined an overall Rayleigh number Ra_{O} by

$$Ra_{\mathrm{O}} = \max Ra_{\mathrm{g}}, \tag{6}$$

where the maximum is taken over all the sub-boxes and all times.

They argued that one would expect that if $Ra_{\mathrm{O}} > 41$ (taken as a round number), then instability will occur. The criterion for instability will be met in at least one sub-box, and hence, the whole system will be unstable. If $Ra_{\mathrm{O}} << 41$, then it is unlikely that instability will occur. If Ra_{O} is only slightly less than 41 then a closer examination of the particular situation is needed to determine whether or not instability will occur. The number 41 is obviously somewhat arbitrary. On literary grounds one can assess that 42 is a better answer to the question of how large Ra_{O} has to be to ensure instability (Rees, D.A.S., Private communication, 2008).

Nield et al. (2009) described a computer package in FORTRAN code, which they called the Stability Exploration Package for Strong Heterogeneity (SEPSH). The main program of SEPSH is designed to produce a *Rayleigh number multiplication factor*, which gives the value of an effective overall Rayleigh number in terms of the Rayleigh number based on a homogeneous domain. The reader is referred to Nield et al. (2009) for more details of the program. The input into the main program consists of a parameter defined in terms of three factors that are allowed to vary as a result of the heterogeneity (permeability K^*, thermal conductivity k^*, and applied vertical temperature gradient β^*) that are then normalized in terms of their mean values. Spatial coordinates are taken relative to the height of the original box, and the subscript s refers to a sub-box. Its height is $z_{\mathrm{B}} - z_{\mathrm{A}}$ and its aspect ratios (height to horizontal dimension) are

$$A_{\mathrm{sx}} = (z_{\mathrm{B}} - z_{\mathrm{A}})/(x_{\mathrm{B}} - x_{\mathrm{A}}) \quad \text{and} \quad A_{\mathrm{sy}} = (z_{\mathrm{B}} - z_{\mathrm{A}})/(y_{\mathrm{B}} - y_{\mathrm{A}}). \tag{7}$$

The normalized Rayleigh number (relative to the Rayleigh number based on mean quantities, with the mean taken over the whole box) for a sub-box, based on the height of the sub-box and with other properties given their mean values over the sub-box, is given by

$$Ra_{\mathrm{Ns}} = \frac{\beta_s K_s (z_{\mathrm{B}} - z_{\mathrm{A}})}{k_s}, \tag{8}$$

where K_{s}, k_{s}, and β_{s} are the means of K_{r}, k_{r}, and β_{r} taken over the sub-box.

A geometrically adjusted Rayleigh number Ra_{Nsg} is defined by

$$Ra_{\mathrm{Nsg}} = \begin{cases} Ra_{\mathrm{Ns}} \quad \text{if } A_{\mathrm{sx}} \leq 1 \quad \text{and} \quad A_{\mathrm{sy}} \leq 1, \\ \dfrac{4Ra_{\mathrm{Ns}}}{\left(\mathrm{A}_{\mathrm{sx}} + A_{\mathrm{sx}}^{-1}\right)^2} \text{ if } A_{\mathrm{sx}} > 1 \text{ and } A_{\mathrm{sx}} \geq A_{\mathrm{sy}}, \\ \dfrac{4Ra_{\mathrm{Ns}}}{\left(\mathrm{A}_{\mathrm{sy}} + A_{\mathrm{sy}}^{-1}\right)^2} \text{ if } A_{\mathrm{sy}} > 1 \text{ and } A_{\mathrm{sy}} > A_{\mathrm{sx}}. \end{cases} \tag{9}$$

An overall normalized Rayleigh number Ra_{NO} is then defined by

$$Ra_{\mathrm{NO}} = \max Ra_{\mathrm{Nsg}}, \tag{10}$$

where the maximum is taken over all the subboxes, including that subbox which is identical with the whole of the original box. Then Ra_{NO} is the required Rayleigh number multiplication factor. Note is taken of which subbox (or subboxes) gives the maximum. This gives an indication of the favored mode of instability (the one that is most unstable and hence is likely to be observed in practice).

In our previous article (Nield et al. 2009), the SEPSH package was applied to various special cases (piecewise-constant, linear, and quadratic variation) where the variation was non-periodic and extended over the whole of the cube. In the present paper, we continue our investigation of the capabilities of SEPSH by applying it to further special cases, this time where the variation is either periodic across the cube or where the departure from mean values is confined to localized portions of the cube.

The SEPSH code now has been extended to a stretched numerical grid system so that the resolution in one or two directions can be increased at the expense of reduced resolution in the remaining directions.

2 Results and Discussion

In our previous article (Nield et al. 2009), we presented results in terms of the variations of the permeability K and the thermal conductivity k treated as separate quantities, with the assumption that the applied temperature gradient β was homogeneous. In the present paper, we compress the computations by working in terms of what we call the *potential convectivity* defined as

$$\tilde{K} = K\beta/k. \tag{11}$$

For practical purposes, variation of $\tilde{K}$ can be considered as variation of K with no variation of either k or β.

We now consider two types of variation, periodic, and localized.

2.1 Periodic Variation

We consider a distribution of the form

$$\tilde{K} = 1 + A_{\mathrm{x}} \sin 2\alpha_x \pi x + A_{\mathrm{y}} \sin 2\alpha_y \pi y + A_{\mathrm{z}} \sin 2\alpha_z \pi z. \tag{12}$$

This has been chosen to have unit mean. The wavenumbers $\alpha_x, \alpha_y, \alpha_z$ and the amplitudes $A_{\mathrm{x}}, A_{\mathrm{y}}, A_{\mathrm{z}}$ are arbitrary constants.

In the following tables, N is the number of nominal grid-points in each direction. If all the grid-points are in fact used, then $N = 21$ is about the maximum value for which

Table 1 Results for periodic variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_A, y_A, z_A, x_B, y_B, z_B$) for various values of the potential convectivity parameters (A_x, A_y, A_z) for wavenumbers $(\alpha_x, \alpha_y, \alpha_z) = (1, 1, 1)$

Row	A_x, A_y, A_z	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
1	0.2, 0, 0	1.0000	0, 0, 0; 1, 1, 1
2	0.4, 0, 0	1.0039	0, 0, 0; 0.70, 0.70, 1
3	0.6, 0, 0	1.0392	0, 0, 0; 0.70, 0.70, 1
4	0.8, 0, 0	1.0928	0, 0, 0; 0.65, 0.65, 1
5	1.0, 0, 0	1.1572	0, 0, 0; 0.65, 0.65, 1
6	0, 0, 0.2	1.0000	0, 0, 0; 1, 1, 1
7	0, 0, 0.4	1.0000	0, 0, 0; 1, 1, 1
8	0, 0, 0.6	1.0000	0, 0, 0; 1, 1, 1
9	0, 0, 0.8	1.0000	0, 0, 0; 1, 1, 1
10	0, 0, 1.0	1.0000	0, 0, 0; 1, 1, 1
11	0.2, 0.2, 0	1.0039	0, 0, 0; 0.85, 0.85, 1
12	0.4, 0.4, 0	1.0928	0, 0, 0; 0.65, 0.65, 1
13	0.2, 0, 0.2	1.0000	0, 0, 0; 1, 1, 1
14	0.4, 0, 0.4	1.0039	0, 0, 0; 0.85, 0.85, 1
15	0.2, 0.2, 0.2	1.0039	0, 0, 0; 0.85, 0.85, 1

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$

extensive computations are feasible. A simple way of stretching the grid is to increment the x-, y-, and z-indices by integer values M_{gx}, M_{gy}, M_{gz}, respectively. Thus, with $N = 101$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 5, 20)$, for example, the number of subboxes used is $100 \times 20 \times 5$, which is comparable with $20 \times 20 \times 20$, something that is feasible for computations. Tables 1, 2, and 4 are with $N = 21$ and no grid stretching, while Tables 3 and 5 are with $N = 101$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 5, 20)$. Tables 1, 2, 3, 4, 5 are for wavenumber values (α, α, α) where $\alpha = 1, 2, 2, 5, 10$, respectively. In these tables, the amplitudes are incremented so that their sum varies up to the value 1, the maximum value for which $\tilde{K}$ takes non-negative values. This means that within the cube, the value of $\tilde{K}$ varies between 0 and 2.

As one would expect, the values of Ra_{NO} increase as the amplitudes are increased, and the significant flow is confined to smaller and smaller portions of the cube. Here, heterogeneity in a horizontal direction has a substantial effect, but heterogeneity in the vertical direction has much less effect (whether acting alone or in conjunction with horizontal heterogeneity).

Comparison of Table 2 with Table 1 (and with Table 4) shows that, the effect of heterogeneity decreases as the wavenumber increases.

Comparison of Table 3 with Table 2 shows that, the change to the stretched grid produced a small increase in the values of Ra_{NO}. As expected, it virtually removed the variation with respect to z and also largely removed the variation with respect to y.

Calculations for a wavenumber value of 10 with an unstretched grid gave values close to unity, and these are not reported. The change to a stretched grid with a greater number of effective grid-points in the x-direction picked up a small effect of heterogeneity in the x-direction.

2.2 Localized Variation

We now consider a situation where the variation of $\tilde{K}$ is localized and its distribution is given by a square-wave function which involves a single hump relative to a mean value of unity. We refer to this as a "localized tower distribution." To be explicit, we consider a distribution, with unit mean, with an elevation for $\xi_x < x < \eta_x, \xi_y < y < \eta_y, \xi_z < z < \eta_z$, given by

Table 2 Results for periodic variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}}, x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}}$) for various values of the potential convectivity parameters ($A_{\mathrm{x}}, A_{\mathrm{y}}, A_{\mathrm{z}}$) and for wavenumbers $(\alpha_{\mathrm{x}}, \alpha_{\mathrm{y}}, \alpha_{\mathrm{z}}) = (2, 2, 2)$

Row	$A_{\mathrm{x}}, A_{\mathrm{y}}, A_{\mathrm{z}}$	Ra_{NO}	$x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}}; x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}}$
1	0.2, 0, 0	1.0007	0, 0, 0; 0.90, 1, 1
2	0.4, 0, 0	1.0202	0, 0, 0; 0.85, 1, 1
3	0.6, 0, 0	1.0512	0, 0, 0; 0.80, 1, 1
4	0.8, 0, 0	1.0843	0, 0, 0; 0.80, 1, 1
5	1.0, 0, 0	1.1174	0, 0, 0; 0.80, 1, 1
6	0, 0, 0.2	1.0000	0, 0, 0; 1, 1, 1
7	0, 0, 0.4	1.0000	0, 0, 0; 1, 1, 1
8	0, 0, 0.6	1.0000	0, 0, 0; 1, 1, 1
9	0, 0, 0.8	1.0000	0, 0, 0; 1, 1, 1
10	0, 0, 1.0	1.0000	0, 0, 0; 1, 1, 1
11	0.2, 0.2, 0	1.0202	0, 0, 0; 0.85, 0.85, 1
12	0.4, 0.4, 0	1.0843	0, 0, 0; 0.80, 0.80, 1
13	0.2, 0, 0.2	1.0007	0, 0, 0; 0.85, 0.85, 1
14	0.4, 0, 0.4	1.0202	0, 0, 0; 0.80, 0.80, 1
15	0.2, 0.2, 0.2	1.0202	0, 0, 0; 0.85, 0.85, 1

$N = 21, (M_{\mathrm{gx}}, M_{\mathrm{gy}}, M_{\mathrm{gz}}) = (1, 1, 1)$

Table 3 Results for periodic variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}}, x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}}$) for various values of the potential convectivity parameters ($A_{\mathrm{x}}, A_{\mathrm{y}}, A_{\mathrm{z}}$) for wavenumbers $(\alpha_{\mathrm{x}}, \alpha_{\mathrm{y}}, \alpha_{\mathrm{z}}) = (2, 2, 2)$

Row	$A_{\mathrm{x}}, A_{\mathrm{y}}, A_{\mathrm{z}}$	Ra_{NO}	$x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}}; x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}}$
1	0.2, 0, 0	1.0011	0, 0, 0; 0.91, 1, 1
2	0.4, 0, 0	1.0229	0, 0, 0; 0.83, 1, 1
3	0.6, 0, 0	1.0544	0, 0, 0; 0.80, 1, 1
4	0.8, 0, 0	1.0893	0, 0, 0; 0.79, 1, 1
5	1.0, 0, 0	1.1257	0, 0, 0; 0.78, 1, 1
6	0, 0, 0.2	1.0000	0, 0, 0; 1, 1, 1
7	0, 0, 0.4	1.0000	0, 0, 0; 1, 1, 1
8	0, 0, 0.6	1.0000	0, 0, 0; 1, 1, 1
9	0, 0, 0.8	1.0000	0, 0, 0; 1, 1, 1
10	0, 0, 1.0	1.0000	0, 0, 0; 1, 1, 1
11	0.2, 0.2, 0	1.0210	0, 0, 0; 0.85, 0.85, 1
12	0.4, 0.4, 0	1.0865	0, 0, 0; 0.80, 0.80, 1
13	0.2, 0, 0.2	1.0011	0, 0, 0; 0.91, 1, 1
14	0.4, 0, 0.4	1.0229	0, 0, 0; 0.83, 1, 1
15	0.2, 0.2, 0.2	1.0210	0, 0, 0; 0.85, 0.85, 1

$N = 101, (M_{\mathrm{gx}}, M_{\mathrm{gy}}, M_{\mathrm{gz}}) = (1, 5, 20)$

$$\begin{aligned} \tilde{K} = 1 &+ \Lambda_{\mathrm{x}} \left\{ \frac{1}{\eta_x - \xi_x} [H(x - \xi_x) - H(x - \eta_x)] - 1 \right\} \\ &+ \Lambda_{\mathrm{y}} \left\{ \frac{1}{\eta_y - \xi_y} \left[H(y - \xi_y) - H(y - \eta_y)\right] - 1 \right\} \\ &+ \Lambda_{\mathrm{z}} \left\{ \frac{1}{\eta_z - \xi_z} \left[H(z - \xi_z) - H(z - \eta_z)\right] - 1 \right\}, \end{aligned} \tag{13}$$

Table 4 Results for periodic variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_A, y_A, z_A, x_B, y_B, z_B$) for various values of the potential convectivity parameters (A_x, A_y, A_z) for wavenumbers $(\alpha_x, \alpha_y, \alpha_z) = (5, 5, 5)$

Row	A_x, A_y, A_z	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
1	0.2, 0, 0	1.0026	0, 0, 0; 0.95, 1, 1
2	0.4, 0, 0	1.0110	0, 0, 0; 0.90, 1, 1
3	0.6, 0, 0	1.0219	0, 0, 0; 0.90, 1, 1
4	0.8, 0, 0	1.0329	0, 0, 0; 0.90, 1, 1
5	1.0, 0, 0	1.0439	0, 0, 0; 0.90, 1, 1
6	0, 0, 0.2	1.0000	0, 0, 0; 1, 1, 1
7	0, 0, 0.4	1.0000	0, 0, 0; 1, 1, 1
8	0, 0, 0.6	1.0000	0, 0, 0; 1, 1, 1
9	0, 0, 0.8	1.0000	0, 0, 0; 1, 1, 1
10	0, 0, 1.0	1.0000	0, 0, 0; 1, 1, 1
11	0.2, 0.2, 0	1.0110	0, 0, 0; 0.90, 0.90, 1
12	0.4, 0.4, 0	1.0329	0, 0, 0; 0.90, 0.90, 1
13	0.2, 0, 0.2	1.0026	0, 0, 0; 0.95, 1, 1
14	0.4, 0, 0.4	1.0110	0, 0, 0; 0.90, 1, 1
15	0.2, 0.2, 0.2	1.0110	0, 0, 0; 0.90, 0.90, 1

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$

Table 5 Results for periodic variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_A, y_A, z_A, x_B, y_B, z_B$) for various values of the potential convectivity parameters (A_x, A_y, A_z) for wavenumbers $(\alpha_x, \alpha_y, \alpha_z) = (10, 10, 10)$

Row	A_x, A_y, A_z	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
1	0.2, 0, 0	1.0041	0, 0, 0; 0.96, 1, 1
2	0.4, 0, 0	1.0103	0, 0, 0; 0.95, 1, 1
3	0.6, 0, 0	1.0168	0, 0, 0; 0.95, 1, 1
4	0.8, 0, 0	1.0232	0, 0, 0; 0.95, 1, 1
5	1.0, 0, 0	1.0297	0, 0, 0; 0.95, 1, 1
6	0, 0, 0.2	1.0000	0, 0, 0; 1, 1, 1
7	0, 0, 0.4	1.0000	0, 0, 0; 1, 1, 1
8	0, 0, 0.6	1.0000	0, 0, 0; 1, 1, 1
9	0, 0, 0.8	1.0000	0, 0, 0; 1, 1, 1
10	0, 0, 1.0	1.0000	0, 0, 0; 1, 1, 1
11	0.2, 0.2, 0	1.0041	0, 0, 0; 0.96, 1, 1
12	0.4, 0.4, 0	1.0103	0, 0, 0; 0.95, 0.95, 1
13	0.2, 0, 0.2	1.0041	0, 0, 0; 0.96, 1, 1
14	0.4, 0, 0.4	1.0103	0, 0, 0; 0.95, 0.95, 1
15	0.2, 0.2, 0.2	1.0041	0, 0, 0; 0.96, 1, 1

$N = 101$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 5, 20)$

where $H(x)$ is the Heaviside step function defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} . \tag{14}$$

Here, the amount of variation in $\tilde{K}$ depends on the values of $\eta_x - \xi_x$, $\eta_y - \xi_y$, $\eta_z - \xi_z$, and in practice these in turn are limited in smallness by the grid size, $1/(N-1)$.

Table 6 Results for localized tower variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_A, y_A, z_A, x_B, y_B, z_B$) for various values of the parameters ($\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$)

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
1	0.00, 0.05; 0, 0; 0, 0	1.0230	0, 0, 0; 0.55, 1, 0.95
2	0.05, 0.10; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
3	0.10, 0.15; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
4	0.15, 0.20; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
5	0.20, 0.25; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
6	0.25, 0.30; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
7	0.30, 0.35; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
8	0.35, 0.40; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
9	0.40, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
10	0.45, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
11	0.00, 0.10; 0, 0; 0, 0	1.1935	0, 0, 0; 0.55, 1, 0.95
12	0.05, 0.15; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
13	0.10, 0.20; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
14	0.15, 0.25; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
15	0.20, 0.30; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
16	0.25, 0.35; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
17	0.30, 0.40; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
18	0.35, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
19	0.40, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
20	0.00, 0.15; 0, 0; 0, 0	1.2503	0, 0, 0; 0.55, 1, 0.95
21	0.05, 0.20; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
22	0.10, 0.25; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
23	0.15, 0.30; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
24	0.20, 0.35; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
25	0.25, 0.40; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
26	0.30, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
27	0.35, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
28	0.00, 0.20; 0, 0; 0, 0	1.2787	0, 0, 0; 0.55, 1, 0.95
29	0.05, 0.25; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
30	0.10, 0.30; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
31	0.15, 0.35; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
32	0.20, 0.40; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
33	0.25, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
34	0.30, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
35	0.00, 0.25; 0, 0; 0, 0	1.2958	0, 0, 0; 0.55, 1, 0.95
36	0.05, 0.30; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
37	0.10, 0.35; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
38	0.15, 0.40; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
39	0.20, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
40	0.25, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
41	0.00, 0.30; 0, 0; 0, 0	1.3020	0, 0, 0; 0.55, 1, 0.95
42	0.05, 0.35; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
43	0.10, 0.40; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95

Table 6 Continued

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$. Case $(\Lambda_x, \Lambda_y, \Lambda_z) = (1, 0, 0)$

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
44	0.15, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
45	0.20, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
46	0.00, 0.35; 0, 0; 0, 0	1.3115	0, 0, 0; 0.55, 1, 0.95
47	0.05, 0.40; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
48	0.10, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
49	0.15, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
50	0.00, 0.40; 0, 0; 0, 0	1.3185	0, 0, 0; 0.55, 1, 0.95
51	0.05, 0.45; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
52	0.10, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
53	0.00, 0.45; 0, 0; 0, 0	1.3261	0, 0, 0; 0.55, 1, 0.95
54	0.05, 0.50; 0, 0; 0, 0	1.3640	0, 0, 0; 0.55, 1, 0.95
55	0.00, 0.50; 0, 0; 0, 0	1.3299	0, 0, 0; 0.55, 1, 0.95

There are nine parameters involved. The following tables are restricted to the case of Λ values of 0 or 1, yielding the maximum contrast while keeping $\tilde{K}$ non-negative, and to the case where $\xi_x = \xi_y = \xi_z = \xi$ and $\eta_x = \eta_y = \eta_y = \eta$. The reported calculations have all been done with an unstretched grid and with $N = 21$, and with the coordinates ξ and η taking integer multiples of $\Delta = 1/(N-1) = 0.05$.

In order that $\tilde{K}$ is does not take negative values anywhere within the cube, we need $\Lambda_x + \Lambda_y + \Lambda_z \leq 1$. For our first investigation, we considered some extreme cases for which $\Lambda_x + \Lambda_y + \Lambda_z = 1$, so that $\tilde{K}$ takes a zero value on a portion of the cube. Tables 6, 7, 8, 9, and 10 are for the values $(\Lambda_x, \Lambda_y, \Lambda_z) = (1, 0, 0), (0, 0, 1), (1/2, 1/2, 0), (1/2, 0, 1/2), (1/3, 1/3, 1/3)$, respectively. In most cases there are multiple cells yielding the optimum Rayleigh number, and the cell coordinates given in the tables are representative only once.

Table 6 shows that the effect of heterogeneity in the x-direction. Essentially, because of round-off error introduced by the fine grid, the last two digits in the reported values of Ra_{NO} are not significant. The value of Ra_{NO} increases with the lateral extent of the hump, when the tower is adjacent to the cube boundary, but it takes a value independent of the tower position and width.

Table 7 shows that the effect of heterogeneity in the z-direction varies with tower position and width in a similar manner to that of heterogeneity in the x-direction, but the magnitude of the effect is considerably less.

A comparison of Table 8 with 6 shows that the effects of variation in the two horizontal directions are additive.

A comparison of Table 9 with 6 shows that when the amplitudes of the vertical and horizontal heterogeneity are shared, then there is a reduction of the destabilizing effect. What is important is the amplitude of the horizontal heterogeneity.

The results in Table 10 show trends similar to those seen in the previous tables but also some reinforcement between the effects of horizontal and vertical variation. We proceeded to investigate further this reinforcement.

In our subsequent investigations, we investigated intermediate values of Λ_x, Λ_y, Λ_z. Having learnt that the Rayleigh multiplier was independent of ξ and η except when the tower was adjacent to the walls, we settled on the values $\xi = 0.2$, $\eta = 0.3$, and then varied Λ_x, Λ_y, Λ_z

Table 7 Results for localized tower variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_A, y_A, z_A, x_B, y_B, z_B$) for various values of the parameters ($\Lambda_x, \Lambda_x, \Lambda_x; \xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$)

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
1	0, 0; 0, 0; 0.00, 0.05	0.7875	0, 0, 0; 0.55, 0.55, 0.10
2	0, 0; 0, 0; 0.05, 0.10	1.0500	0, 0, 0; 0.15, 0.15, 0.15
3	0, 0; 0, 0; 0.10, 0.15	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
4	0, 0; 0, 0; 0.15, 0.20	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
5	0, 0; 0, 0; 0.20, 0.25	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
6	0, 0; 0, 0; 0.25, 0.30	1.0500	0, 0, 0; 0.35, 0.35, 0.35
7	0, 0; 0, 0; 0.30, 0.35	1.0500	0, 0, 0; 0.60, 0.60, 0.60
8	0, 0; 0, 0; 0.35, 0.40	1.0500	0, 0, 0; 0.60, 0.60, 0.60
9	0, 0; 0, 0; 0.40, 0.45	1.0500	0, 0, 0; 0.60, 0.60, 0.60
10	0, 0; 0, 0; 0.45, 0.50	1.0500	0, 0, 0; 0.60, 0.60, 0.60
11	0, 0; 0, 0; 0.00, 0.10	0.9188	0, 0, 0; 0.15, 0.15, 0.15
12	0, 0; 0, 0; 0.05, 0.15	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
13	0, 0; 0, 0; 0.10, 0.20	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
14	0, 0; 0, 0; 0.15, 0.25	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
15	0, 0; 0, 0; 0.20, 0.30	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
16	0, 0; 0, 0; 0.25, 0.35	1.0500	0, 0, 0; 0.55, 0.75, 0.55
17	0, 0; 0, 0; 0.30, 0.40	1.0500	0, 0, 0; 0.60, 0.60, 0.60
18	0, 0; 0, 0; 0.35, 0.45	1.0500	0, 0, 0; 0.60, 0.60, 0.60
19	0, 0; 0, 0; 0.40, 0.50	1.0500	0, 0, 0; 0.60, 0.60, 0.60
20	0, 0; 0, 0; 0.00, 0.15	0.9625	0, 0, 0; 0.30, 0.3 0, 0.30
21	0, 0; 0, 0; 0.05, 0.20	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
22	0, 0; 0, 0; 0.10, 0.25	1.0500	0, 0, 0; 0.30, 0.45, 0.30
23	0, 0; 0, 0; 0.15, 0.30	1.0500	0, 0, 0; 0.60, 0.60, 0.60
24	0, 0; 0, 0; 0.20, 0.35	1.0500	0, 0, 0; 0.60, 0.85, 0.60
25	0, 0; 0, 0; 0.25, 0.40	1.0500	0, 0, 0; 0.60, 0.60, 0.60
26	0, 0; 0, 0; 0.30, 0.45	1.0500	0, 0, 0; 0.25, 0.25, 0.50
27	0, 0; 0, 0; 0.35, 0.50	1.0500	0, 0, 0; 0.60, 0.85, 0.60
28	0, 0; 0, 0; 0.00, 0.20	0.9844	0, 0, 0; 0.85, 0.85, 0.85
29	0, 0; 0, 0; 0.05, 0.25	1.0500	0, 0, 0; 0.30, 0.3 0, 0.30
30	0, 0; 0, 0; 0.10, 0.30	1.0500	0, 0, 0; 0.60, 0.60, 0.60
31	0, 0; 0, 0; 0.15, 0.35	1.0500	0, 0, 0; 0.60, 0.60, 0.60
32	0, 0; 0, 0; 0.20, 0.40	1.0500	0, 0, 0; 0.60, 0.60, 0.60
33	0, 0; 0, 0; 0.25, 0.45	1.0500	0, 0, 0; 1.00, 0.60, 0.60
34	0, 0; 0, 0; 0.30, 0.50	1.0500	0, 0, 0; 0.60, 0.60, 0.60
35	0, 0; 0, 0; 0.00, 0.25	0.9975	0, 0, 0; 0.35, 0.35, 0.30
36	0, 0; 0, 0; 0.05, 0.30	1.0500	0, 0, 0; 0.35, 0.35, 0.35
37	0, 0; 0, 0; 0.10, 0.35	1.0500	0, 0, 0; 0.35, 0.35, 0.40
38	0, 0; 0, 0; 0.15, 0.40	1.0500	0, 0, 0; 0.35, 0.35, 0.45
39	0, 0; 0, 0; 0.20, 0.45	1.0500	0, 0, 0; 0.35, 0.35, 0.50
40	0, 0; 0, 0; 0.25, 0.50	1.0500	0, 0, 0; 0.35, 0.35, 0.55
41	0, 0; 0, 0; 0.00, 0.30	1.0023	0, 0, 0; 0.560, 0.90, 0.30
42	0, 0; 0, 0; 0.05, 0.35	1.0500	0, 0, 0; 0.80, 0.60, 0.60

Table 7 continued

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$. Case $(\Lambda_x, \Lambda_y, \Lambda_z) = (0, 0, 1)$

Row	$\xi_x, \eta_x; , \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
43	0, 0; 0, 0; 0.10, 0.40	1.0500	0, 0, 0; 0.40, 0.40, 0.45
44	0, 0; 0, 0; 0.15, 0.45	1.0500	0, 0, 0; 0.40, 0.40, 0.50
45	0, 0; 0, 0; 0.20, 0.50	1.0500	0, 0, 0; 0.40, 0.40, 0.55
46	0, 0; 0, 0; 0.00, 0.35	1.0096	0, 0, 0; 1, 1, 0.35
47	0, 0; 0, 0; 0.05, 0.40	1.0500	0, 0, 0; 1, 1, 0.95, 0.60
48	0, 0; 0, 0; 0.10, 0.45	1.0500	0, 0, 0; 1, 1, 0.95, 0.60
49	0, 0; 0, 0; 0.15, 0.50	1.0500	0, 0, 0; 1, 1, 0.95, 0.60
50	0, 0; 0, 0; 0.00, 0.40	1.0150	0, 0, 0; 1, 1, 0.55
51	0, 0; 0, 0; 0.05, 0.45	1.0500	0, 0, 0; 0.65, 0.75, 0.55
52	0, 0; 0, 0; 0.10, 0.50	1.0500	0, 0, 0; 0.60, 0.60, 0.60
53	0, 0; 0, 0; 0.00, 0.45	1.0208	0, 0, 0; 0.55, 1, 0.55
54	0, 0; 0, 0; 0.05, 0.50	1.0500	0, 0, 0; 1, 0.95, 0.55
55	0, 0; 0, 0; 0.00, 0.50	1.0237	0, 0, 0; 1, 1, 0.60

Table 8 Results for localized tower variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_A, y_A, z_A, x_B, y_B, z_B$) for various values of the parameters ($\Lambda_x, \Lambda_x, \Lambda_x; \xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$)

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
1	0.00, 0.05; 0.00, 0.05; 0, 0	1.0230	0, 0, 0; 0.55, 0.55, 0.95
2	0.05, 0.10; 0.05, 0.10; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
3	0.10, 0.15; 0.10, 0.15; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
4	0.15, 0.20; 0.15, 0.20; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
5	0.20, 0.25; 0.20, 0.25; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
6	0.25, 0.30; 0.25, 0.30; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
7	0.30, 0.35; 0.30, 0.35; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
8	0.35, 0.40; 0.35, 0.40; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
9	0.40, 0.45; 0.40, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
10	0.45, 0.50;0.45, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
11	0.00, 0.10;0.00, 0.10; 0, 0	1.1935	0, 0, 0; 0.55, 0.55, 0.95
12	0.05, 0.15;0.05, 0.15; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
13	0.10, 0.20;0.10, 0.20; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
14	0.15, 0.25;0.15, 0.25; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
15	0.20, 0.30; 0.20, 0.30; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
16	0.25, 0.25; 0.25, 0.25; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
17	0.30, 0.40; 0.30, 0.40; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
18	0.35, 0.45; 0.35, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
19	0.40, 0.50; 0.40, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
20	0.00, 0.15; 0.00, 0.15; 0, 0	1.2503	0, 0, 0; 0.55, 0.55, 0.95
21	0.05, 0.20; 0.05, 0.20; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
22	0.10, 0.25; 0.10, 0.25; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
23	0.15, 0.30; 0.15, 0.30; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
24	0.20, 0.35; 0.20, 0.35; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
25	0.25, 0.40; 0.25, 0.40; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95

Table 8 continued

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
26	0.30, 0.45; 0.30, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
27	0.35, 0.50;0.35, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
28	0.00, 0.20; 0.00, 0.20; 0, 0	1.2787	0, 0, 0; 0.55, 0.55, 0.95
29	0.05, 0.25; 0.05, 0.25; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
30	0.10, 0.30; 0.10, 0.30; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
31	0.15, 0.35; 0.15, 0.35; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
32	0.20, 0.40; 0.20, 0.40; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
33	0.25, 0.45; 0.25, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
34	0.30, 0.50; 0.30, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
35	0.00, 0.25; 0.00, 0.25; 0, 0	1.2958	0, 0, 0; 0.55, 0.55, 0.95
36	0.05, 0.30;0.05, 0.30; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
37	0.10, 0.35; 0.10, 0.35; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
38	0.15, 0.40; 0.15, 0.40; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
39	0.20, 0.45; 0.20, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
40	0.25, 0.50;0.25, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
41	0.00, 0.30; 0.00, 0.30; 0, 0	1.3020	0, 0, 0; 0.55, 0.55, 0.95
42	0.05, 0.35;0.05, 0.35; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
43	0.10, 0.40; 0.10, 0.40; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
44	0.15, 0.45; 0.15, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
45	0.20, 0.50; 0.20, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
46	0.00, 0.35; 0.00, 0.35; 0, 0	1.3115	0, 0, 0; 0.55, 0.55, 0.95
47	0.05, 0.40; 0.05, 0.40; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
48	0.10, 0.45; 0.10, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
49	0.15, 0.50; 0.15, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
50	0.00, 0.40;0.00, 0.40; 0, 0	1.3185	0, 0, 0; 0.55, 0.55, 0.95
51	0.05, 0.45;0.05, 0.45; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
52	0.10, 0.50; 0.10, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
53	0.00, 0.45; 0.00, 0.45; 0, 0	1.3261	0, 0, 0; 0.55, 0.55, 0.95
54	0.05, 0.50; 0.05, 0.50; 0, 0	1.3640	0, 0, 0; 0.55, 0.55, 0.95
55	0.00, 0.50; 0.00, 0.50; 0, 0	1.3299	0, 0, 0; 0.55, 0.55, 0.95

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$. Case $(\Lambda_x, \Lambda_y, \Lambda_z) = (1/2, 1/2, 0)$

in turn through the values 0, 0.1, 0.2, 0.3. We quickly found that, the effects of varying Λ_x, Λ_y were approximately additive, so that, without loss of generality, we only needed to vary the value of $\Lambda_x, +\Lambda_y = \Lambda_H$ rather than Λ_x, Λ_y independently. The results that, we obtained are reported in Table 11.

For small values of Λ_H and Λ_z, the results in Table 11 can be fitted approximately by the formula

$$Ra_{NO} = 1 + 0.030\Lambda_H + 0.036\Lambda_z + 0.20\Lambda_H^2 + 0.15\Lambda_H\Lambda_z + 0.01\Lambda_z^2. \quad (15)$$

It will be observed that, the coefficient of the last term is relatively small. This means that Ra_{NO} is approximately linear in Λ_z. For larger values of Λ_H and Λ_z, Ra_{NO} increases somewhat more rapidly with increasing Λ_H and Λ_z.

Table 9 Results for localized tower variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box ($x_A, y_A, z_A, x_B, y_B, z_B$) for various values of the parameters ($\Lambda_x, \Lambda_x, \Lambda_x; \xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$)

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_{A,} z_A; x_B, y_B, z_B$
1	0.00, 0.05; 0, 0; 0.00, 0.05	0.8467	0, 0, 0; 0.15, 0.15, 0.20
2	0.05, 0.10; 0, 0; 0.05, 0.10	1.1290	0, 0, 0; 0.15, 0.15, 0.20
3	0.10, 0.15; 0, 0; 0.10, 0.15	1.1290	0, 0, 0; 0.30, 0.20, 0.40
4	0.15, 0.20; 0, 0; 0.15, 0.20	1.1290	0, 0, 0; 0.30, 0.30, 0.40
5	0.20, 0.25; 0, 0; 0.20, 0.25	1.1290	0, 0, 0; 0.30, 0.30, 0.40
6	0.25, 0.30; 0, 0; 0.25, 0.30	1.1290	0.10, 0, 0; 0.55, 0.60, 0.40
7	0.30, 0.35; 0, 0; 0.30, 0.35	1.1290	0, 0, 0; 0.60, 0.60, 0.80
8	0.35, 0.40; 0, 0; 0.35, 0.40	1.1290	0, 0, 0; 0.60, 0.60, 0.80
9	0.40, 0.45; 0, 0; 0.40, 0.45	1.1290	0, 0, 0; 0.60, 0.60, 0.80
10	0.45, 0.50; 0, 0; 0.45, 0.50	1.1290	0, 0, 0; 0.60, 0.60, 0.80
11	0.00, 0.10; 0, 0; 0.00, 0.10	0.9878	0, 0, 0; 0.15, 0.15, 0.20
12	0.05, 0.15; 0, 0; 0.05, 0.15	1.1290	0, 0, 0; 0.30, 0.30, 0.40
13	0.10, 0.20; 0, 0; 0.10, 0.20	1.1290	0, 0, 0; 0.30, 0.30, 0.40
14	0.15, 0.25; 0, 0; 0.15, 0.25	1.1290	0, 0, 0; 0.30, 0.30, 0.40
15	0.20, 0.30; 0, 0; 0.20, 0.30	1.1290	0, 0, 0; 0.30, 0.30, 0.40
16	0.25, 0.35; 0, 0; 0.25, 0.35	1.1290	0.10, 0, 0; 0.55, 0.55, 0.60
17	0.30, 0.40; 0, 0; 0.30, 0.40	1.1290	0, 0, 0; 0.60, 0.60, 0.80
18	0.35, 0.45; 0, 0; 0.35, 0.45	1.1290	0, 0, 0; 0.60, 0.60, 0.80
19	0.40, 0.50; 0, 0; 0.40, 0.50	1.1290	0, 0, 0; 0.60, 0.60, 0.80
20	0.00, 0.15; 0, 0; 0.00, 0.15	1.0349	0, 0, 0; 0.75, 1, 1
21	0.05, 0.20; 0, 0; 0.05, 0.20	1.1290	0, 0, 0; 0.70, 0.90, 1
22	0.10, 0.25; 0, 0; 0.10, 0.25	1.1290	0.05, 0, 0; 0.80, 0.90, 1
23	0.15, 0.30; 0, 0; 0.15, 0.30	1.1290	0.10, 0, 0; 0.85, 0.90, 1
24	0.20, 0.35; 0, 0; 0.20, 0.35	1.1290	0.05, 0, 0; 0.35, 0.35, 0.40
25	0.25, 0.40; 0, 0; 0.25, 0.40	1.1290	0, 0, 0; 0.75, 0.90, 1
26	0.30, 0.45; 0, 0; 0.30, 0.45	1.1290	0.10, 0, 0; 0.55, 0.70, 0.60
27	0.35, 0.50; 0, 0; 0.35, 0.50	1.1290	0.10, 0, 0; 0.55, 0.80, 0.60
28	0.00, 0.20; 0, 0; 0.00, 0.20	1.0584	0, 0, 0; 0.30, 0.30, 0.40
29	0.05, 0.25; 0, 0; 0.05, 0.25	1.1290	0, 0, 0; 0.30, 0.30, 0.40
30	0.10, 0.30; 0, 0; 0.10, 0.30	1.1290	0, 0, 0; 0.60, 0.60, 0.80
31	0.15, 0.35; 0, 0; 0.15, 0.35	1.1290	0, 0, 0; 0.60, 0.60, 0.80
32	0.20, 0.40; 0, 0; 0.20, 0.40	1.1290	0, 0, 0; 0.60, 0.60, 0.80
33	0.25, 0.45; 0, 0; 0.25, 0.45	1.1290	0.20, 0, 0.20; 0.80, 0.70, 1
34	0.30, 0.50; 0, 0; 0.30, 0.50	1.1290	0, 0, 0; 0.60, 0.60, 0.80
35	0.00, 0.25; 0, 0; 0.00, 0.25	1.0725	0, 0, 0; 0.75, 0.80, 1
36	0.05, 0.30; 0, 0; 0.05, 0.30	1.1290	0, 0, 0; 0.75, 1, 1
37	0.10, 0.35; 0, 0; 0.10, 0.35	1.1290	0.05, 0, 0; 0.80, 1, 1
38	0.15, 0.40; 0, 0; 0.15, 0.40	1.1290	0.10, 0, 0; 0.85, 1, 1
39	0.20, 0.45; 0, 0; 0.20, 0.45	1.1290	0.15, 0, 0; 0.90, 1, 1
40	0.25, 0.50; 0, 0; 0.25, 0.50	1.1290	0.20, 0, 0; 0.95, 1, 1
41	0.00, 0.30; 0, 0; 0.00, 0.30	1.0776	0, 0, 0; 0.75, 1, 1

Table 9 continued

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$. Case $(\Lambda_x, \Lambda_y, \Lambda_z) = (1/2, 0, 1/2)$

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
42	0.05, 0.35; 0, 0; 0.05, 0.35	1.1290	0, 0, 0; 0.75, 1, 1
43	0.10, 0.40; 0, 0; 0.10, 0.40	1.1290	0, 0, 0; 0.60, 0.75, 0.80
44	0.15, 0.45; 0, 0; 0.15, 0.45	1.1290	0.10, 0, 0; 0.55, 0.55, 0.60
45	0.20, 0.50; 0, 0; 0.20, 0.50	1.1290	0.10, 0, 0; 0.55, 0.60, 0.60
46	0.00, 0.35; 0, 0; 0.00, 0.35	1.0855	0, 0, 0; 0.75, 1, 1
47	0.05, 0.40; 0, 0; 0.05, 0.40	1.1290	0, 0, 0; 0.45, 0.45, 0.60
48	0.10, 0.45; 0, 0; 0.10, 0.45	1.1290	0.05, 0, 0.05; 0.50, 0.45, 0.65
49	0.15, 0.50; 0, 0; 0.15, 0.50	1.1290	0.10, 0, 0.05; 0.55, 0.45, 0.65
50	0.00, 0.40; 0, 0; 0.00, 0.40	1.0913	0, 0, 0; 0.60, 0.70, 0.80
51	0.05, 0.45; 0, 0; 0.05, 0.45	1.1290	0, 0, 0; 0.60, 0.70, 0.80
52	0.10, 0.50; 0, 0; 0.10, 0.50	1.1290	0, 0, 0; 0.60, 0.60, 0.80
53	0.00, 0.45; 0, 0; 0.00, 0.45	1.0976	0, 0, 0; 0.75, 1, 1
54	0.05, 0.50; 0, 0; 0.05, 0.50	1.1290	0, 0, 0; 0.75, 0.90, 1
55	0.00, 0.50; 0, 0; 0.00, 0.50	1.1007	0, 0, 0; 0.75, 1, 1

3 Conclusions

The computer package SEPSH that we have developed to investigate the onset of convection in a strongly heterogeneous porous medium has been tested for additional types of variation of the potential convectivity (defined by Eq. 11) within a cube, namely, periodic and localized variation. The results confirm the conclusions of Nield et al. (2009) that the heterogeneity generally gives rise to an increase in the value of $Ra_{NO,}$ the Rayleigh number multiplier defined in Eq. 10. However, the largest value of Ra_{NO} computed in this study is 1.364, which is not dramatically different from 1. This indicates that the value of a Rayleigh number based on the mean value of the potential convectivity is likely to be a useful qualitative criterion for the onset of convection in a variety of situations. In both the cases of periodic and localized variation, heterogeneity in the horizontal direction has a substantial effect, but heterogeneity in the vertical direction has much less effect.

We emphasize that, we are using “strong heterogeneity “ to refer to a situation were the potential convectivity varies strongly about its mean value, and we are not using it in the sense that the equivalent Rayleigh number varies strongly. It turns out that, our results involving strong variation to two dimensions (the vertical and one horizontal dimension) have not led to strong variation in the effective Rayleigh number.

Our package is still in the process of development. Currently, we are investigating distributions of potential convectivity, which more closely resemble those encountered in geophysical situations. This will involve not only testing more realistic geologic structures but also an exploration of the effects of much stronger heterogeneity than has been investigated here. Our work has explored heterogeneity that is clearly stronger than earlier work on weak heterogeneity (i.e., in this study, we no longer required the amount variation in the potential convectivity parameter to be small in comparison with its mean value). In real geologic settings much greater variation in the potential convectivity parameter $\tilde{K}$ than that generally employed here (over at least several orders of magnitude) may be expected. In this connection, it should be noted that what is important here is not the variation of $\tilde{K}$ per se but rather

Table 10 Results for localized tower variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) and the coordinates of opposite corners of the favored sub-box (x_A, y_A, z_A, x_B, y_B, z_B) for various values of the parameters (Λ_x, Λ_x, Λ_x; ξ_x, η_x; ξ_y, η_y; ξ_z, η_z)

Row	$\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_A, z_A; x_B, y_B, z_B$
1	0.00, 0.05; 0.00, 0.05; 0.00, 0.05	0.8947	0, 0, 0; 0.30, 0.30, 0.45
2	0.05, 0.10; 0.05, 0.10; 0.05, 0.10	1.1929	0, 0, 0; 0.30, 0.30, 0.45
3	0.10, 0.15; 0.10, 0.15; 0.10, 0.15	1.1929	0.05, 0.05, 0; 0.35, 0.35, 0.45
4	0.15, 0.20; 0.15, 0.20; 0.15, 0.20	1.1929	0.10, 0.10, 0.10; 0.40, 0.40, 0.55
5	0.20, 0.25; 0.20, 0.25; 0.20, 0.25	1.1929	0.15, 0.10, 0.00; 0.45, 0.40, 0.45
6	0.25, 0.30; 0.25, 0.30; 0.25, 0.30	1.1929	0.20, 0.15, 0.00; 0.50, 0.45, 0.45
7	0.30, 0.35; 0.30, 0.35; 0.30, 0.35	1.1929	0.25, 025, 0.25; 0.55, 0.55, 0.70
8	0.35, 0.40; 0.35, 0.40; 0.35, 0.40	1.1929	0.25, 0.25, 0.15; 0.45, 0.45, 0.45
9	0.40, 0.45; 0.40, 0.45; 0.40, 0.45	1.1929	0.30, 0.30, 0.25; 0.50, 0.50, 0.55
10	0.45, 0.50; 0.45, 0.50; 0.45, 0.50	1.1929	0.35, 0.35, 0.25; 0.55, 0.55, 0.55
11	0.00, 0.10; 0.00, 0.10; 0.00, 0.10	1.0438	0, 0, 0; 0.60, 0.60, 0.90
12	0.05, 0.15; 0.05, 0.15; 0.05, 0.15	1.1929	0, 0, 0; 0.60, 0.60, 0.90
13	0.10, 0.20; 0.10, 0.20; 0.10, 0.20	1.1929	0.05, 0.05, 0; 0.65, 0.65, 0.90
14	0.15, 0.25; 0.15, 0.25; 0.15, 0.25	1.1929	0.10, 0.10, 0.00; 0.70, 0.70, 0.90
15	0.20, 0.30; 0.20, 0.30; 0.20, 0.30	1.1929	0.15, 0.15, 0.10; 0.75, 0.75, 1.00
16	0.25, 0.35; 0.25, 0.35; 0.25, 0.35	1.1929	0.20, 0.20, 0.00; 0.80, 0.80, 0.90
17	0.30, 0.40; 0.30, 0.40; 0.30, 0.40	1.1929	0, 0, 0; 0.60, 0.60, 0.90
18	0.35, 0.45; 0.30, 0.40; 0.30, 0.40	1.1929	0, 0, 0; 0.60, 0.60, 0.90
19	0.40, 0.50; 0.30, 0.40; 0.30, 0.40	1.1929	0, 0, 0; 0.60, 0.60, 0.90
20	0.00, 0.15; 0.00, 0.15; 0.00, 0.15	1.0935	0, 0, 0; 0.20, 0.20, 0.30
21	0.05, 0.20; 0.05, 0.20; 0.05, 0.20	1.1929	0, 0, 0; 0.30, 0.30, 0.45
22	0.10, 0.25; 0.10, 0.25; 0.10, 0.25	1.1929	0.05, 0.00, 0.05; 0.35, 0.30, 0.50
23	0.15, 0.30; 0.15, 0.30; 0.15, 0.30	1.1929	0.10, 0.10, 0.10; 0.40, 0.40, 0.55
24	0.20, 0.35; 0.20, 0.35; 0.20, 0.35	1.1929	0.15, 0.15, 0.15; 0.35, 0.35, 0.45
25	0.25, 0.40; 0.25, 0.40; 0.25, 0.40	1.1929	0.20, 0.20, 0.20; 0.50, 0.50, 0.65
26	0.30, 0.45; 0.30, 0.45; 0.30, 0.45	1.1929	0.25, 0.25, 0.00; 0.85, 0.85, 0.90
27	0.35, 0.50; 0.35, 0.50; 0.35, 0.50	1.1929	0.05, 0.05, 0.00; 0.55, 0.55, 0.75
28	0.00, 0.20; 0.00, 0.20; 0.00, 0.20	1.1183	0, 0, 0; 0.30, 0.30, 0.45
29	0.05, 0.25; 0.05, 0.25; 0.05, 0.25	1.1929	0, 0, 0; 0.30, 0.30, 0.45
30	0.10, 0.30; 0.10, 0.30; 0.10, 0.30	1.1929	0.05, 0.05, 0; 0.35, 0.35, 0.45
31	0.15, 0.35; 0.15, 0.35; 0.15, 0.35	1.1929	0.10, 0.00, 0; 0.70, 0.60, 0.90
32	0.20, 0.40; 0.20, 0.40; 0.20, 0.40	1.1929	0.15, 0.15, 0.00; 0.45, 0.45, 0.45
33	0.25, 0.45; 0.25, 0.45; 0.25, 0.45	1.1929	0.15, 0.20, 0.00; 0.75, 0.80, 0.90
34	0.30, 0.50; 0.30, 0.50; 0.30, 0.50	1.1929	0.00, 0.00, 0.00; 0.60, 0.60, 0.90
35	0.00, 0.25; 0.00, 0.25; 0.00, 0.25	1.1333	0, 0, 0; 0.50, 0.50, 0.75
36	0.05, 0.30; 0.05, 0.30; 0.05, 0.30	1.1929	0, 0, 0; 0.40, 0.40, 0.60
37	0.10, 0.35; 0.10, 0.35; 0.10, 0.35	1.1929	0.05, 0.00, 0.00; 0.45, 0.40, 0.60
38	0.15, 0.40; 0.15, 0.40; 0.15, 0.40	1.1929	0.00, 0.10, 0.05; 0.6 0, 0.70, 0.95
39	0.20, 0.45; 0.20, 0.45; 0.20, 0.45	1.1929	0.10, 0.00, 0.00; 0.60, 0.50, 0.75

Table 10 continued

Row	$\xi_x, \eta_x; \xi_y, s\eta_y; \xi_z, \eta_z$	Ra_{NO}	$x_A, y_{A,} z_A; x_B, y_B, z_B$
40	0.25, 0.50; 0.25, 0.50; 0.25, 0.50	1.1929	0.15, 0.05, 0.00; 0.65, 0.55, 0.75
41	0.00, 0.30; 0.00, 0.30; 0.00, 0.30	1.1387	0, 0, 0; 0.40, 0.40, 0.60
42	0.05, 0.35; 0.05, 0.35; 0.05, 0.35	1.1929	0, 0, 0; 0.50, 0.50, 0.75
43	0.10, 0.40; 0.10, 0.40; 0.10, 0.40	1.1929	0.00, 0.00, 0.05; 0.55, 0.50, 0.75
44	0.15, 0.45; 0.15, 0.45; 0.15, 0.45	1.1929	0.10, 0.00, 0.00; 0.60, 0.50, 0.75
45	0.15, 0.45; 0.15, 0.45; 0.15, 0.45	1.1929	0.15, 0.05, 0.00; 0.65, 0.55, 0.75
46	0.00, 0.35; 0.00, 0.35; 0.00, 0.35	1.1470	0, 0, 0; 0.60, 0.60, 0.90
47	0.05, 0.40; 0.05, 0.40; 0.05, 0.40	1.1929	0, 0, 0; 0.50, 0.50, 0.75
48	0.10, 0.45; 0.10, 0.45; 0.10, 0.45	1.1929	0.00, 0.00, 0.05; 0.50, 050, 0.80
49	0.15, 0.50; 0.15, 0.50; 0.15, 0.50	1.1929	0.05, 0.05, 0.05; 0.55, 0.55, 0.80
50	0.00, 0.40; 0.00, 0.40; 0.00, 0.40	1.1531	0, 0, 0; 0.50, 0.50, 0.75
51	0.05, 0.45; 0.05, 0.45; 0.05, 0.45	1.1929	0, 0, 0; 0.60, 0.60, 0.90
52	0.10, 0.50; 0.10, 0.50; 0.10, 0.50	1.1929	0, 0, 0; 0.60, 0.60, 0.90
53	0.00, 0.45; 0.00, 0.45; 0.00, 0.45	1.1598	0, 0, 0; 0.50, 0.50, 0.75
54	0.05, 0.50; 0.05, 0.50; 0.05, 0.50	1.1929	0, 0, 0; 0.60, 0.60, 0.90
55	0.00, 0.50; 0.00, 0.50; 0.00, 0.50	1.1631	0, 0, 0; 0.60, 0.60, 0.90

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$.
Case $(\Lambda_x, \Lambda_y, \Lambda_z) = (1/3, 1/3, 1/3)$

Table 11 Results for localized tower variation: values of the normalized overall Rayleigh numbers (Ra_{NO}) for various values of the parameters (Λ_H, Λ_z) for the case $(\xi_x, \eta_x; \xi_y, \eta_y; \xi_z, \eta_z) = (0.2, 0.3; 0.2, 0.3; 0.2, 0.3)$ with grid-point settings

Row	λ_H	λ_z	Ra_{NO}
1	0	0	1.0000
2	0	0.1	1.0037
3	0	0.2	1.0076
4	0	0.3	1.0117
5	0.1	0	1.0051
6	0.1	0.1	1.0091
7	0.1	0.2	1.0134
8	0.1	0.3	1.0179
9	0.2	0	1.0140
10	0.2	0.1	1.0186
11	0.2	0.2	1.0235
12	0.2	0.3	1.0288
13	0.3	0	1.0278
14	0.3	0.1	1.0334
15	0.3	0.2	1.0395
16	0.3	0.3	1.0459

$N = 21$, $(M_{gx}, M_{gy}, M_{gz}) = (1, 1, 1)$

the ratio of peak $\tilde{K}$ to average $\tilde{K}$, and this depends on the degree of localization of the higher values of $\tilde{K}$. It is possible that grid size, and hence computation time, may ultimately limit how far we can go using the present approach, but the results so far with the tower variation make us optimistic that SEPSH will be useful in geologic settings.

Our immediate objective is to further explore periodic and localized variation. So far we have just considered periodic variation of simple harmonic type. Similarly, so far we have considered localized variation involving single towers, and we have in mind extensions to distributions of Manhattan type (involving multiple towers). Further, we plan to consider towers that are localized in both horizontal directions. Our preliminary results indicate that these lead to a stronger change in the value of the effective Raleigh number compared with towers localized in just one horizontal direction.

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