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Thermally Developing Forced Convection in a Porous Medium Occupied by a Rarefied Gas: Parallel Plate Channel or Circular Tube with Walls at Constant Heat Flux

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Abstract An adaptation of the classical Graetz methodology is applied to investigate the thermal development of forced convection in a parallel plate channel or a circular tube filled by a porous medium saturated by a rarefied gas, with walls held at constant heat flux. The Brinkman model is employed. The analysis leads to expressions for the local Nusselt number *Nu* as functions of the dimensionless longitudinal coordinate and the Darcy number. It is found that an increase in the velocity slip coefficient generally increases *Nu* by a small or moderate amount (but the circular tube at large Darcy number is an exception) while an increase in the temperature slip coefficient reduces *Nu* by a more substantial amount. These trends are uniform as the longitudinal coordinate varies.

Keywords Forced convection · Thermal development · Rarefied gas · Graetz problem · Parallel plate channel and circular tube

Nomenclature

c_P	Specific heat at constant pressure
C_0	Constant defined by Eq. 44
C_n	Coefficients defined by Eq. 38 for a channel and by Eq. 77 for a circular tube
Da	Darcy number defined as K/H^2 for a channel and K/r_0^2 for a circular tube
$f(r)$	Temperature perturbation function defined by Eq. 69 for a circular tube
$f(y)$	Temperature perturbation function defined by Eq. 30 for a channel

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G	Negative of the applied pressure gradient
H	Half channel width
k_{m}	Effective thermal conductivity of the porous medium
K	Permeability
Kn	Knudsen number
M	Viscosity ratio, $\tilde{\mu}/\mu$
Nu	Local Nusselt number defined as $\frac{2Hq''}{k_{\mathrm{m}}(T_{\mathrm{m}}^*-T_{\mathrm{w}}^*)}$ for a channel and $\frac{2r_0 q''}{k_{\mathrm{m}}(T_{\mathrm{m}}^*-T_{\mathrm{w}}^*)}$ for a circular tube
$\overline{Nu}$	Mean Nusselt number defined by Eq. 46
Pe	Péclet number defined as $\rho c_P HU^*/k_{\mathrm{m}}$ for a channel and $\rho c_P r_0 U^*/k_{\mathrm{m}}$ for a tube
q''	Wall heat flux
r	r^*/r_0
r^*	Radial coordinate
r_0	Circular tube radius
$R_n(y)$	Eigenfunctions for a circular tube
S	$(MDa)^{-1/2}$
T_{m}^*	Bulk mean temperature
T^*	Temperature
$\hat{T}$	$\frac{T^*-T_{\mathrm{w}}^*}{T_{\mathrm{m}}^*-T_{\mathrm{w}}^*}$
T_{IN}^*	Inlet temperature
T_{w}^*	Wall temperature
T^+	Perturbation temperature, $T^* - T_{\mathrm{FD}}^*$
u	$\tilde{\mu}u^*/GH^2$ for a channel and $\tilde{\mu}u^*/Gr_0^2$ for a circular tube
u^*	Filtration velocity
$\hat{u}$	u^*/U^*
U^*	Mean filtration velocity
$\tilde{x}$	x/Pe
x	x^*/H
x^*	Longitudinal coordinate
y	y^*/H
y^*	Transverse coordinate
$Y_n(y)$	Eigenfunctions for a channel

Greek symbols

α	Velocity slip coefficient
β	Temperature slip coefficient
γ	Parameter defined by Eq. 8 for a channel and by Eq. 51 for a circular tube
θ	Dimensionless temperature, defined by Eq. 10
θ^+	$\frac{T^+}{Hq''/k_{\mathrm{m}}}$ for a channel and $\frac{T^+}{r_0 q''/k_{\mathrm{m}}}$ for a circular tube
λ_n	Eigenvalues
μ	Fluid viscosity
$\tilde{\mu}$	Effective viscosity for the flow in the porous medium
ρ	Fluid density

Subscripts

FD	fully developed

1 Introduction

There has recently been renewed interest in the problem of forced convection in a porous medium channel or duct filled with a porous medium because of the use of hyperporous media in the cooling of electronic equipment. Until recently the only work done on the thermal development aspect was confined to the Darcy model, but Nield et al. (2004) extended this work to the Brinkman model. In this paper both the parallel channel and the circular tube were treated, but only for the case of isothermal walls. In this case the standard Graetz analysis can be readily applied, because a pair of boundary conditions for the dependent variable (a dimensionless temperature) is homogeneous, and so the method of separation of variables is immediately applicable. In the case of walls held at constant heat flux (the case treated in the present paper) a wall boundary condition is not homogeneous, and this means that the analysis has to proceed in two steps: first the fully developed solution to the problem must be found, and then the problem involving the perturbation temperature can be tackled using the method of separation of variables. This procedure was carried out by Nield et al. (2003a). Some extensions of the theory (local thermal non-equilibrium, viscous dissipation, longitudinal conduction) were treated by Nield et al. (2002, 2003b), and Kuznetsov et al. (2003).

In this paper we model in turn convection in a parallel plates channel, and in a circular duct, using the Brinkman model, but now assuming both limited velocity slip and temperature slip on the walls of the channel or duct, for the case of uniform heat flux at the walls. The fully developed situation was studied by Nield and Kuznetsov (2006). At the time of submission of that paper nothing had been published on convection in a porous medium occupied by a rarefied gas. Subsequently a number of papers by Haddad et al. (2005; 2006a,b; 2007a,b) have appeared, together with comments by Al-Nimr and Haddad (2007) and a response by Nield and Kuznetsov (2007). This is indicative of current interest in the topic. The emerging field of micro-scale heat transfer has opened up new applications involving slip-flow.

2 Analysis for a Parallel Plate Channel

We start by considering a channel between two plane parallel walls a distance $2H$ apart, the boundaries being at $y = H$ and $y = -H$. For fully developed flow the velocity is $u(y)$ in the x-direction. We suppose that the governing momentum equation is

$$G = \frac{\mu u^*}{K} - \tilde{\mu}\frac{\mathrm{d}^2 u^*}{\mathrm{d}y^{*2}}. \tag{1}$$

Here the asterisks denote dimensional variables, and $-G$ is the applied pressure gradient in the x^*-direction. We define the dimensionless variables

$$x = \frac{x^*}{H}, \quad y = \frac{y^*}{H}, \quad u = \frac{\tilde{\mu} u^*}{GH^2}, \tag{2}$$

and write

$$M = \frac{\tilde{\mu}}{\mu}, \quad Da = \frac{K}{H^2}. \tag{3}$$

Thus M is the viscosity ratio and Da is the Darcy number.

Then Eq. 1 becomes

$$M\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \frac{u}{Da} + 1 = 0. \tag{4}$$

This equation is to be solved subject to the boundary/symmetry conditions

$$u = -\alpha \frac{\mathrm{d}u}{\mathrm{d}y} \quad \text{at } y = 1, \qquad \frac{\mathrm{d}u}{\mathrm{d}y} = 0 \quad \text{at } y = 0. \tag{5}$$

The value of α can be related to the value of Kn. One can write $\alpha = \kappa Kn$. Our knowledge about κ has been summarized by Harley et al. (1995). An argument due to Maxwell shows that the coefficient $\kappa = \kappa_o(2-F)/F$, where κ_o is a constant of $O(1)$ and F is a quantity called the momentum accommodation coefficient, defined as the fraction of diffusely reflected molecules (as distinct from specularly reflected molecules). In other words, F is the fraction of molecular tangential momentum lost through collisions with the solid surface. For a simple approximate argument which yields the above expression for κ in terms of F with $\kappa_o = 1$, the reader is referred to Schaaf and Chambre (1961). Thermal creep and thermal stress flow have been neglected. More generally, κ depends on temperature, surface roughness and type of gas.

The solution is

$$u = Da\left(1 - \frac{\cosh Sy}{\gamma \cosh S}\right), \tag{6}$$

where for convenience we introduce

$$S = \frac{1}{(MDa)^{1/2}}, \tag{7}$$

$$\gamma = 1 + \alpha S \tanh S. \tag{8}$$

We also introduce the mean velocity U^* and the bulk mean temperature T_m^* defined by

$$U^* = \frac{1}{H}\int_0^H u^* \mathrm{d}y^*, \quad T_\mathrm{m}^* = \frac{1}{HU^*}\int_0^H u^* T^* \mathrm{d}y^*. \tag{9}$$

We then define further dimensionless variables defined by

$$\hat{u} = \frac{u^*}{U^*}, \quad \theta = \frac{T^* - T_\mathrm{w}^*}{T_\mathrm{m}^* - T_\mathrm{w}^*}, \tag{10}$$

The Nusselt number is defined by

$$Nu = \frac{2Hq''}{k_\mathrm{m}(T_\mathrm{m}^* - T_\mathrm{w}^*)}. \tag{11}$$

Here T_w^* and q'' are the temperature and heat flux on the wall.

(The reader should note that we have followed Nield and Bejan (2006) and defined *Nu* in terms of the channel width rather than the hydraulic diameter. The Nusselt number defined in terms of the hydraulic diameter is twice *Nu*.)

Local thermal equilibrium is assumed. (The case of local thermal non-equilibrium requires a separate investigation.) It is also assumed that the Péclet number is sufficiently large for axial conduction to be neglected. Thermal dispersion is also neglected. The steady-state thermal energy equation is then

$$u^* \frac{\partial T^*}{\partial x^*} = \frac{k_\mathrm{m}}{\rho c_p} \frac{\partial^2 T^*}{\partial y^{*2}}. \tag{12}$$

Use of the first law of thermodynamics now leads to

$$\frac{\mathrm{d}T_\mathrm{m}^*}{\mathrm{d}x^*} = \frac{q''}{\rho c_p HU^*}. \tag{13}$$

This is here a constant.

In the case of fully developed convection, $\partial T^*/\partial x^* = \mathrm{d}T_\mathrm{m}^*/\mathrm{d}x$ and then Eq. 12 becomes, in non-dimensional form and for the fully developed case,

$$\frac{\mathrm{d}^2\theta_\mathrm{FD}}{\mathrm{d}y^2} = -\frac{1}{2} Nu_\mathrm{FD}\hat{u}. \tag{14}$$

For the Brinkman model, with u given by Eq. 6, we have

$$\hat{u} = \frac{S\gamma}{S\gamma - \tanh S}\left(1 - \frac{\cosh Sy}{\gamma \cosh S}\right), \tag{15}$$

The subscript FD distinguishes the solution of the fully developed problem from the solution for the thermally developing problem that we treat below.

Equation 14 is to be solved subject to the boundary conditions

$$\frac{\mathrm{d}\theta_\mathrm{FD}}{\mathrm{d}y}(0) = 0, \quad \beta\frac{\mathrm{d}\theta_\mathrm{FD}}{\mathrm{d}y} + \theta_\mathrm{FD}(1) = 0 \tag{16}$$

The temperature slip coefficient β can be written as $\beta = \kappa_\mathrm{T} Kn$ where κ_T was found by Weber to have the value 2.85 for air, 11.7 for hydrogen, and about 3 for most other common gases (Devine 1965).

The solution of Eq. 14 subject to Eq. 16 is

$$\theta_\mathrm{FD} = \frac{S\gamma Nu_\mathrm{FD}}{S\gamma - \tanh S}\left[\frac{1}{4}(1-y^2) - \frac{\cosh S - \cosh Sy}{2S^2\gamma\cosh S}\right] + \frac{\beta Nu_\mathrm{FD}}{2}. \tag{17}$$

The definition of the dimensionless temperature leads to the integral compatibility condition

$$\int_0^1 \hat{u}\theta_\mathrm{FD}\mathrm{d}y = 1. \tag{18}$$

Substitution from Eqs. 15 and 17 into 18 then leads to

$$Nu_\mathrm{FD} = 1 \Big/ \left\{\frac{\beta}{2} + \frac{2\gamma^2 S^3 + (12\gamma + 3)(\tanh S - S) + 3S\tanh^2 S}{12S(\gamma S - \tanh S)^2}\right\}. \tag{19}$$

For the case $\alpha = \beta = 0$ (and so $\gamma = 1$) this gives

$$Nu_\mathrm{FD} = \frac{12S(S - \tanh S)^2}{2S^3 - 15S + 15\tanh S + 3S\tanh^2 S}, \tag{20}$$

in agreement with Eq. 4.125 of Nield and Bejan (2006).

In order to tackle the thermally developing problem, it is convenient to work in terms of a new dimensionless temperature θ defined by

$$\frac{T^* - T_\mathrm{m}^*}{Hq''/k_\mathrm{m}} = \frac{2(1-\theta)}{Nu}. \tag{21}$$

From integration of Eq. 13,

$$\frac{T_\mathrm{m}^* - T_\mathrm{IN}^*}{Hq''/k_\mathrm{m}} = \tilde{x} \tag{22}$$

where

$$\tilde{x} = \frac{x}{Pe}, \tag{23}$$

and in turn the Péclet number Pe is now defined as

$$Pe = \frac{\rho c_P HU^*}{k_m}. \tag{24}$$

From Eqs. 21 and 22, by addition, and specification to the fully developed case, we have

$$\frac{T^*_{FD} - T^*_{IN}}{Hq''/k_m} = \tilde{x} + \frac{2(1-\theta_{FD})}{Nu_{FD}}. \tag{25}$$

Here T^*_{FD} is the dimensional temperature corresponding to the fully developed case.

We now introduce a perturbation temperature defined by

$$T^+ = T^* - T^*_{FD} \tag{26}$$

and define

$$\theta^+ = \frac{T^+}{Hq''/k_m}. \tag{27}$$

Since T^+ also satisfies Eq. 12, it follows that

$$\hat{u}\frac{\partial\theta^+}{\partial\tilde{x}} = \frac{\partial^2\theta^+}{\partial y^2}. \tag{28}$$

Also we have the boundary conditions

$$\frac{\partial\theta^+}{\partial y}(\tilde{x}, 0) = 0, \quad \frac{\partial\theta^+}{\partial y}(\tilde{x}, 1) = 0 \tag{29}$$

and the initial condition

$$\theta^+(0, y) = \frac{2(\theta_{FD} - 1)}{Nu_{FD}} \equiv -f(y). \tag{30}$$

It is interesting that $f(y)$ as defined by Eq. 30 is independent of β. When one substitutes in Eq. 30 from Eqs. 17 and 19 the terms in β cancel out.

Separation of variables, following the assumption that

$$\theta^+ = \Xi(\tilde{x})Y(y), \tag{31}$$

leads to two linear and homogeneous equations for Ξ and Y,

$$\Xi' + \lambda^2\Xi = 0, \tag{32}$$

$$Y'' + \lambda^2\hat{u}Y = 0. \tag{33}$$

Equation 33 together with the boundary conditions

$$Y'(0) = Y'(1) = 0 \tag{34}$$

defines an eigenvalue problem of Sturm-Liouville type with eigenvalues λ_n and corresponding eigenfunctions $Y_n(y)$ for $n = 1, 2, 3, \ldots$. In particular,

$$Y_n'' + \lambda_n^2\hat{u}Y_n = 0. \tag{35}$$

The general solution of Eqs. 28, 29 is the series

$$\theta^{+} = C_0 + \sum_{n=1}^{\infty} C_n Y_n(y) \exp(-\lambda_n^2 \tilde{x}), \tag{36}$$

where the constants $C_0, C_1, C_2, \ldots$ are determined by the condition (30) and the requirement that the contribution from the perturbation θ^{+} to the wall heat flux is zero. Since the eigenfunctions satisfy the orthogonality condition

$$\int_0^1 \hat{u} Y_m Y_n \mathrm{d}y = 0 \quad \text{if } m \neq n \tag{37}$$

it follows that

$$C_n = \frac{-\int_0^1 \hat{u} Y_n f(y)\, \mathrm{d}y}{\int_0^1 \hat{u} Y_n^2\, \mathrm{d}y} \text{ for } n = 1, 2, 3, \ldots \tag{38}$$

For example, in the Darcy limit one has

$$\hat{u} = 1, \quad Nu_{\mathrm{FD}} = 6, \quad \theta_{\mathrm{FD}} = \frac{3}{2}\left(1 - y^2\right), \quad f(y) = -\frac{1}{6}\left(1 - 3y^2\right). \tag{39}$$

$$Y_n(y) = \cos n\pi y, \quad C_n = \frac{2(-1)^{n-1}}{n^2\pi^2} \quad \text{for } n = 1, 2, 3, \ldots. \tag{40}$$

In the no-slip clear fluid limit one has

$$\hat{u} = \frac{3}{2}\left(1 - y^2\right), Nu_{\mathrm{FD}} = \frac{70}{17}, \theta_{\mathrm{FD}} = \frac{35}{136}\left(5 - 6y^2 + y^4\right),$$
$$f(y) = -\frac{1}{280}\left(39 - 210y^2 + 35y^4\right). \tag{41}$$

There is no simple analytical expression for $Y_n(y)$ or C_n.
At the wall,

$$\frac{T_{\mathrm{w}}^* - T_{\mathrm{IN}}^*}{Hq''/k_{\mathrm{m}}} = \tilde{x} + f(1) + C_0 + \sum_{n=1}^{\infty} C_n Y_n(1) \exp(-\lambda_n^2 \tilde{x}). \tag{42}$$

It follows from Eqs. 11, 22, 25 and 42 that

$$Nu = \frac{2}{f(1) + C_0 + \sum_{n=1}^{\infty} C_n Y_n(1) \exp(-\lambda_n^2 \tilde{x})}. \tag{43}$$

The constant C_0 is determined by the requirement that $Nu \to Nu_{\mathrm{FD}}$ as $\tilde{x} \to \infty$. One finds that

$$C_0 = \frac{2}{Nu_{\mathrm{FD}}} - f(1) = \frac{2\theta_{\mathrm{FD}}(1)}{Nu_{\mathrm{FD}}} = \beta. \tag{44}$$

When C_0 is eliminated one has

$$Nu = \left\{ \frac{1}{Nu_{\mathrm{FD}}} + \frac{1}{2} \sum_{n=1}^{\infty} C_n Y_n(1) \exp(-\lambda_n^2 \tilde{x}) \right\}^{-1}. \tag{45}$$

It is noteworthy that $C_0 = 0$ when $\beta = 0$, that is in the absence of velocity slip. For this reason Nield et al. (2003a,b) had no need to introduce the C_0 and they were able to compute Nu from Eq. 43 with C_0 absent.

Equation 45 gives the local Nusselt number. The mean Nusselt number, averaged over a length $\tilde{x}$, is

$$\overline{Nu} = \frac{1}{\tilde{x}} \int_0^{\tilde{x}} Nu d\tilde{x}. \tag{46}$$

3 Calculations: Parallel Plate Channel

It is convenient to express the second order Eq. 35 as a system of two first order ones, by writing $y_1 = Y$, $y_2 = Y'$, where a prime denotes a derivative with respect to x. (We drop the tilde). Then

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\lambda^2 \hat{u} y_1. \end{aligned} \tag{47}$$

These equations may be solved by a shooting procedure. Each eigenfunction may be normalized by the requirement that it satisfies the condition $Y(0) = 1$. Then we have

$$y_1(0) = 1, \quad y_2(0) = 0. \tag{48}$$

Starting with an estimate for the value of an eigenvalue, one can step forward from $x = 0$ to $x = 1$ and vary the value of λ to satisfy the condition $y_2(1) = 0$. This yields the precise eigenvalue, and the corresponding function $y_1(x)$ is the required eigenfunction. Once the eigenvalues and eigenfunctions have been obtained, the coefficients C_n can be obtained by simple numerical integration of the integrals that are involved, and the solution is readily completed.

We checked our calculated eigenvalues with known results for the case of slug flow (very small *Da*) (when the λ_n are multiples of π) and for the case of plane Poiseuille flow with no slip (very large *Da*), given by Shah and London (1978).

4 Results and Discussion: Parallel Plate Channel

The plots of Nusselt number versus longitudinal coordinate are presented as Figs. 1–3. Figure 1, for the case $MDa = 1$, is typical for a hyperporous medium and approximates the case of a fluid clear of solid material. It is seen that an increase in the velocity slip coefficient α leads to a moderate increase in *Nu*, and this is so across the board. An increase in the temperature slip coefficient β produces a more substantial reduction in *Nu*, and at large values of β the curves develop an inflection point. The increase of *Nu* with increasing α is expected since increasing the velocity slip facilitates the flow and hence aids the convective heat transfer. The decrease of *Nu* with increasing β is also expected since increasing the temperature slip decreases the coupling between the wall temperature and the temperature within the bulk of the fluid. A reduction in temperature gradient leads to a reduction in heat transfer.

Figures 2 and 3 illustrate the cases $MDa = 10^{-2}$ and 10^{-4}, respectively. In general, a decrease in Darcy number leads to an increase in *Nu*. (One recalls that for fully developed convection without slip one has $Nu = 70/17$ in the clear fluid limit and $Nu = 6$ in the Darcy limit). For small values of the Darcy number, the effect of variation in the value of α is small. This is particularly the case when β is large; this is especially exemplified by the coalescence of curves in Fig. 3c. This is as expected since then the velocity profile is close to that of slug

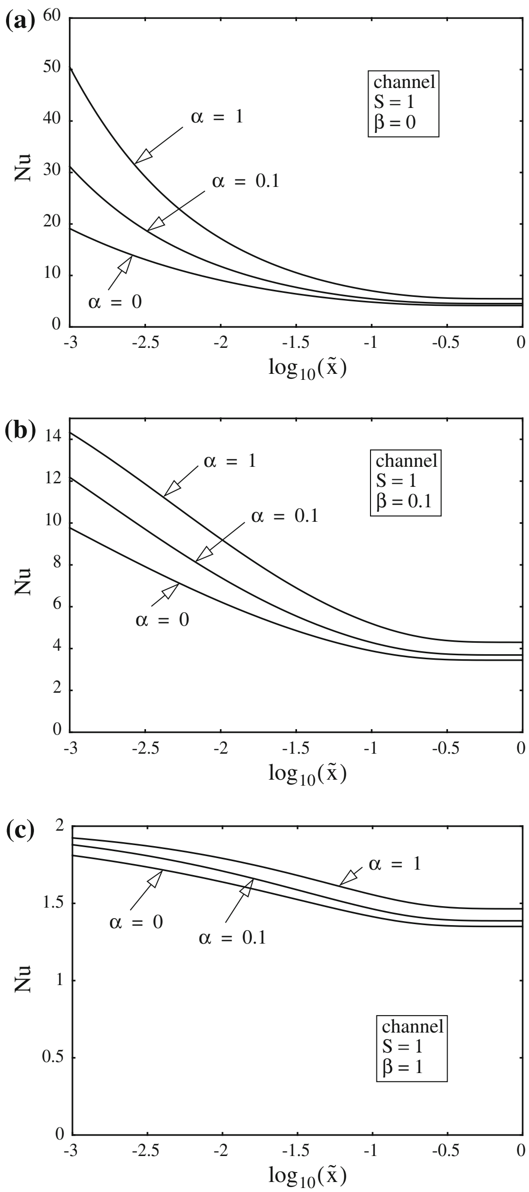


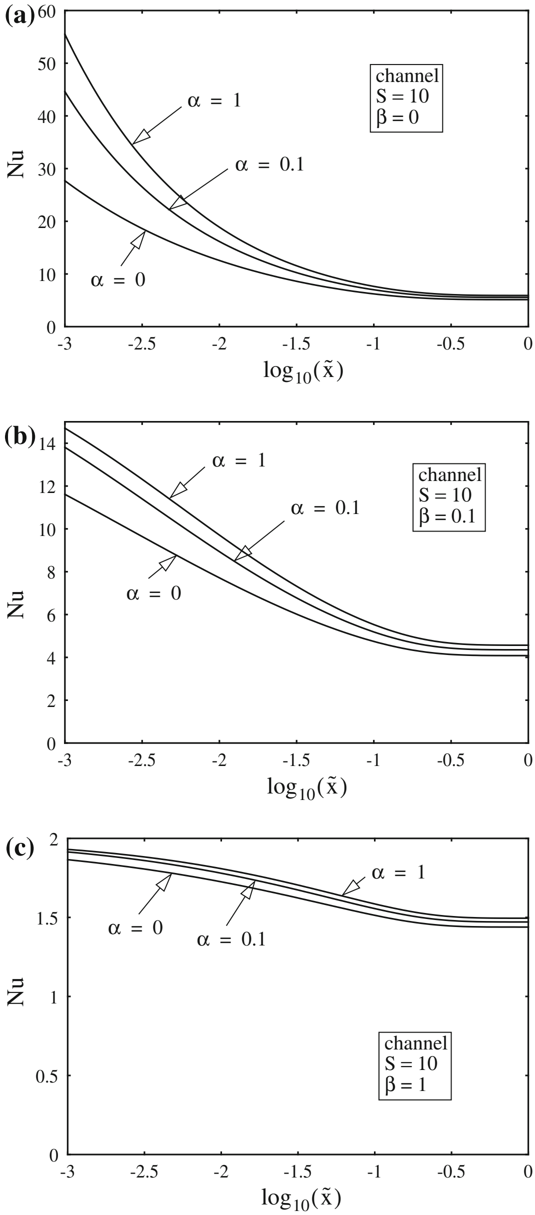
Fig. 1 Plots of local Nusselt number versus longitudinal coordinate for the parallel plate channel problem, for the case $MDa = 1$ ($S = 1$), for various values (0, 0.1, 1.0) of the velocity slip coefficient α and the temperature slip coefficient β; (**a**) $\beta = 0$, (**b**) $\beta = 0.1$, (**c**) $\beta = 1.0$

flow in the bulk of the channel and variation of α just modifies the flow in thin boundary layers near the walls. An increase in the value of β continues to produce a substantial reduction in Nu.

5 Analysis: Circular Tube

The analysis is much the same as that for the parallel plate channel, so we briefly note the changes. We consider a tube of radius r_0, so the boundary is at $r^* = r_0$, and r_0 replaces H as the length scale in the definitions of the Darcy number Da and the Nusselt number Nu.

Fig. 2 As for Fig. 1, but now for $MDa = 10^{-2}$ ($S = 10$)



The dimensionless form of the momentum equation is now

$$M\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) - \frac{u}{Da} + 1 = 0, \tag{49}$$

and the solution for the rescaled velocity $\hat{u}$, subject to the hydrodynamic slip boundary condition, is now

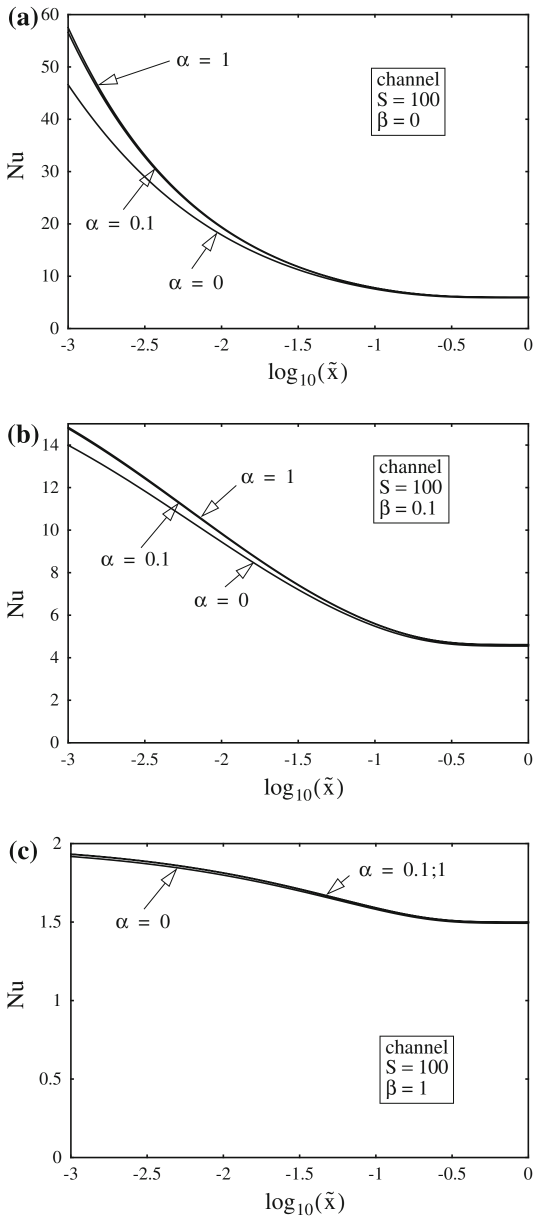


Fig. 3 As for Fig. 1, but now for $MDa = 10^{-4}$ ($S = 100$)

$$\hat{u} = \frac{S\{\gamma I_0(S) - I_0(Sr)\}}{\gamma S I_0(S) - 2I_1(S)}, \tag{50}$$

where

$$\gamma = 1 + \frac{\alpha S I_1(S)}{I_0(S)}. \tag{51}$$

Here I_0 and I_1 are modified Bessel functions of orders zero and one, respectively.

The Nusselt number Nu is now defined as

$$Nu = \frac{2r_0 q''}{k_\mathrm{m}(T_\mathrm{w}^* - T_\mathrm{m}^*)}. \tag{52}$$

The steady-state thermal energy equation is now

$$u^* \frac{\partial T^*}{\partial x^*} = \frac{k_\mathrm{m}}{\rho c_p}\left(\frac{\partial^2 T^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial T^*}{\partial r^*}\right). \tag{53}$$

Use of the first law of thermodynamics now leads to

$$\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}x^*} = \frac{2q''}{\rho c_p r_0 U^*}. \tag{54}$$

The dimensionless form of the thermal energy equation for fully developed convection is now

$$\frac{\mathrm{d}^2\theta_\mathrm{FD}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\theta_\mathrm{FD}}{\mathrm{d}r} = -Nu\hat{u}, \tag{55}$$

and the solution of this equation, subject to the temperature slip boundary condition, is

$$\theta_\mathrm{FD} = \frac{Nu\gamma S I_0(S)}{\gamma S I_0(S) - 2I_1(S)}\left\{\frac{1}{4}(1-r^2) - \frac{I_0(S) - I_0(Sr)}{\gamma S^2 I_0(S)}\right\} + \frac{Nu\beta}{2}. \tag{56}$$

Substitution in the integral compatibility condition

$$2\int_0^1 r\hat{u}\theta_\mathrm{FD}\mathrm{d}r = 1 \tag{57}$$

yields

$$Nu_\mathrm{FD} = 1 \Big/ \left\{\frac{\beta}{2} + \frac{\{\gamma^2 S^3 - 8(\gamma+2)S\}[I_0(S)]^2 + 16(\gamma+2)I_0(S)I_1(S) + 8S[I_1(S)]^2}{8S\{\gamma S I_0(S) - 2I_1(S)\}^2}\right\}. \tag{58}$$

For the case $\alpha = \beta = 0$, and so $\gamma = 1$, this reduces to

$$Nu_\mathrm{FD} = \frac{8S\{SI_0(S) - 2I_1(S)\}^2}{(S^3 - 24S)[I_0(S)]^2 + 48I_0(S)I_1(S) + 8S[I_1(S)]^2}, \tag{59}$$

in agreement with Eq. 53 of Nield et al. (2003) and Eq. 4.119 of Vafai (2005).

In order to tackle the thermally developing problem, it is convenient to work in terms of a new dimensionless temperature

$$\frac{T^* - T_m^*}{r_0 q''/k_m} = \frac{2(1-\theta)}{Nu}. \tag{60}$$

From integration of Eq. 54,

$$\frac{T_\mathrm{m}^* - T_\mathrm{IN}^*}{r_0 q''/k_\mathrm{m}} = 2\tilde{x}, \tag{61}$$

where

$$\tilde{x} = \frac{x}{Pe}, \tag{62}$$

and in turn the Peclet number Pe is now defined as

$$Pe = \frac{\rho c_P r_0 U^*}{k_{\mathrm{m}}}. \tag{63}$$

From Eqs. 60 and 61, by addition, and specialization to the fully developed case,

$$\frac{T^*_{\mathrm{FD}} - T^*_{\mathrm{IN}}}{r_0 q''/k_{\mathrm{m}}} = 2\tilde{x} + \frac{2(1-\theta_{\mathrm{FD}})}{Nu_{\mathrm{FD}}}. \tag{64}$$

We now introduce a perturbation temperature defined by

$$T^+ = T^* - T^*_{\mathrm{FD}} \tag{65}$$

and define

$$\theta^+ = \frac{T^+}{r_0 q''/k_{\mathrm{m}}}. \tag{66}$$

Since T^+ also satisfies Eq. 52, it follows that

$$\hat{u}\frac{\partial \theta^+}{\partial \tilde{x}} = \frac{\partial^2 \theta^+}{\partial r^2} + \frac{1}{r}\frac{\partial \theta^+}{\partial r}. \tag{67}$$

Also we have the boundary conditions

$$\frac{\partial \theta^+}{\partial r}(x,0) = 0, \quad \frac{\partial \theta^+}{\partial r}(x,1) = 0 \tag{68}$$

and the initial condition

$$\theta^+(0,r) = \frac{2(\theta_{\mathrm{FD}} - 1)}{Nu_{\mathrm{FD}}} \equiv -f(r). \tag{69}$$

Separation of variables, following the assumption that

$$\theta^+ = \Xi(\tilde{x})R(r), \tag{70}$$

leads to two linear and homogeneous equations for Ξ and R,

$$\Xi' + \lambda^2 \Xi = 0, \tag{71}$$

$$R'' + (1/r)R' + \lambda^2 \hat{u} R = 0. \tag{72}$$

Equation 71 together with the boundary conditions

$$R'(0) = R'(1) = 0 \tag{73}$$

defines an eigenvalue problem of Sturm-Liouville type with eigenvalues λ_n and corresponding eigenfunctions $R_n(y)$ for $n = 1, 2, 3, \ldots$. In particular,

$$R''_n + (1/r)R'_n + \lambda_n^2 \hat{u} R_n = 0. \tag{74}$$

The general solution of Eqs. 62 and 63 is the series

$$\theta^+ = C_0 + \sum_{n=1}^{\infty} C_n R_n(y) \exp(-\lambda_n^2 \tilde{x}) \tag{75}$$

where the constants $C_0, C_1, C_2, \ldots$ are determined by the condition (69) and the requirement that the contribution from the perturbation θ^+ to the wall heat flux is zero.

Since the eigenfunctions satisfy the orthogonality condition

$$\int_0^1 \hat{u} R_m R_n r \mathrm{d}r = 0 \quad \text{if } m \neq n \tag{76}$$

it follows that

$$C_n = \frac{-\int_0^1 \hat{u} R_n r f(r) \mathrm{d}r}{\int_0^1 \hat{u} R_n^2 r \mathrm{d}r} \quad \text{for } n = 1, 2, 3, \ldots \tag{77}$$

For example, in the Darcy limit one has

$$\begin{aligned} &\hat{u} = 1, Nu_{\mathrm{FD}} = 8, \quad \theta_{\mathrm{FD}} = \tfrac{1}{4}\left(1 - r^2\right), \quad f(r) = \tfrac{1}{4}\left(2r^2 - 1\right), \\ &R(r) = J_0(\lambda_n r), \end{aligned} \tag{78}$$

where the λ_n are the roots of $J_1(\lambda) = 0$. This leads to

$$C_n = \frac{2J_2(\lambda_n)}{\lambda_n^2 \left[J_0(\lambda_n)\right]^2} \text{ for } n = 1, 2, 3, \ldots \tag{79}$$

At the wall,

$$\frac{T_{\mathrm{w}}^* - T_{\mathrm{IN}}^*}{r_0 q''/k_{\mathrm{m}}} = 2\tilde{x} + f(1) + C_0 + \sum_{n=1}^{\infty} C_n R_n(1) \exp(-\lambda_n^2 \tilde{x}). \tag{80}$$

It follows from Eqs. 51, 61, 64 and 80 that

$$Nu = \frac{2}{f(1) + C_0 + \sum_{n=1}^{\infty} C_n R_n(1) \exp(-\lambda_n^2 \tilde{x})}. \tag{81}$$

The constant C_0 is determined by the requirement that $Nu \to Nu_{FD}$ as $\tilde{x} \to \infty$. When that constant (which has the value β) is eliminated one has

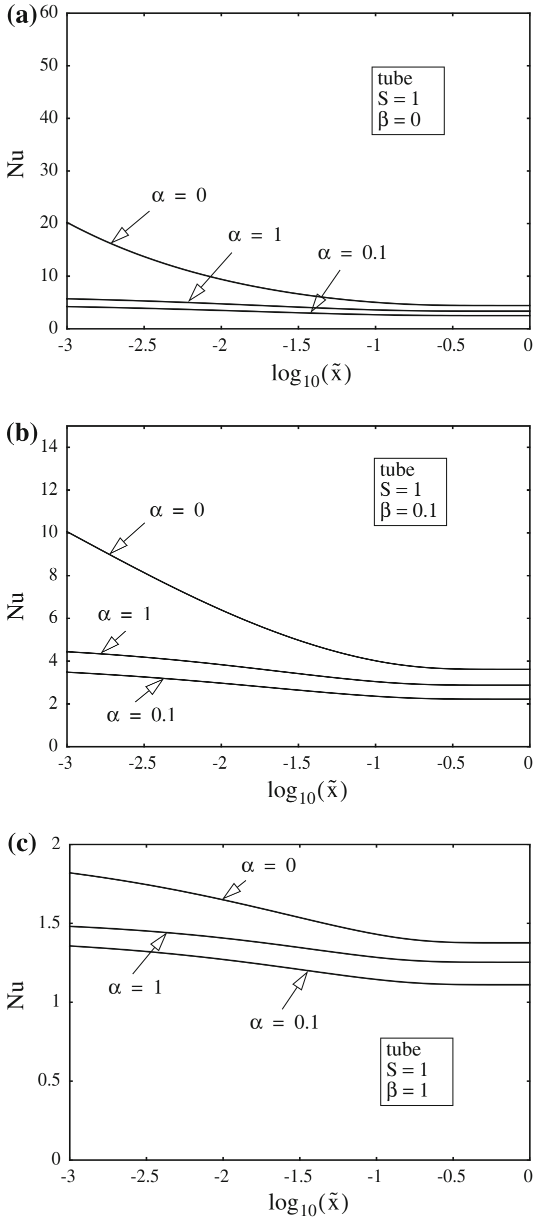
$$Nu = \left\{ \frac{1}{Nu_{\mathrm{FD}}} + \frac{1}{2} \sum_{n=1}^{\infty} C_n R_n(1) \exp(-\lambda_n^2 \tilde{x}) \right\}^{-1}. \tag{82}$$

6 Results and Discussion: Circular Tube

The results for the circular tube are presented in Figs. 4–6. On the whole, the patterns in this set are similar to those in the set presented in Figs. 1–3 for the parallel plate channel. However, there are some differences, particularly for the case $MDa = 1$, approximating the clear fluid limit. In the case of the circular tube, Nu does not increase monotonically as α increases; rather, it decreases to a minimum value before increasing. For the case of small *MDa, Nu* does increase monotonically as α increases.

Also for the case of small *MDa*, the effect of a change of geometry from parallel plate to circular tube is not large.

Fig. 4 Plots of local Nusselt number versus longitudinal coordinate for the circular tube problem, for the case $MDa = 1$ ($S = 1$), for various values (0, 0.1, 1.0) of the velocity slip coefficient α and the temperature slip coefficient β; (**a**) $\beta = 0$, (**b**) $\beta = 0.1$, (**c**) $\beta = 1.0$



7 Conclusions

We have obtained an analytic solution for the velocity profile, temperature profile and Nusselt number for thermally developing forced convection in either a parallel plates channel or a circular duct occupied by a hyperporous medium saturated by a rarefied gas, appropriate for the Knudsen slip-flow regime, for the case of uniform heat flux on the boundary walls. The results are presented in terms of a velocity slip coefficient α and a temperature slip coefficient

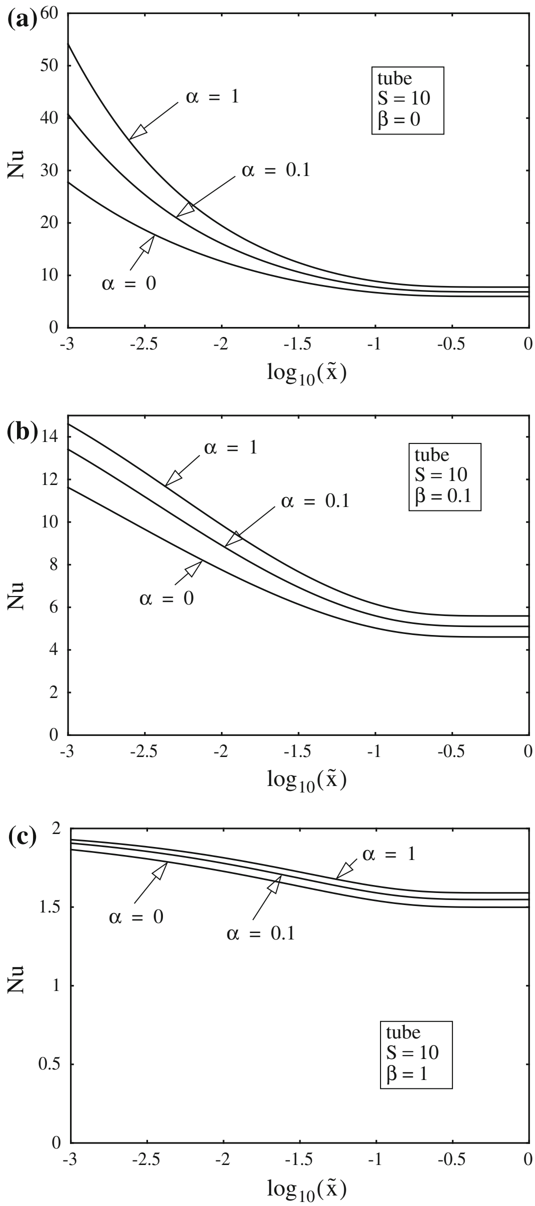


Fig. 5 As for Fig. 4, but now for $MDa = 10^{-2}$ ($S = 10$)

β, each of which is proportional to the Knudsen number. It has been shown that the Nusselt number decreases as β increases. This trend was expected. For the parallel plates channel, the Nusselt number increases as α increases. For the circular duct, the variation with α is more complex in the case of large Darcy number. The complexity was not anticipated.

The corresponding problem, with walls at constant uniform temperature rather than at constant heat flux, has not yet been investigated.

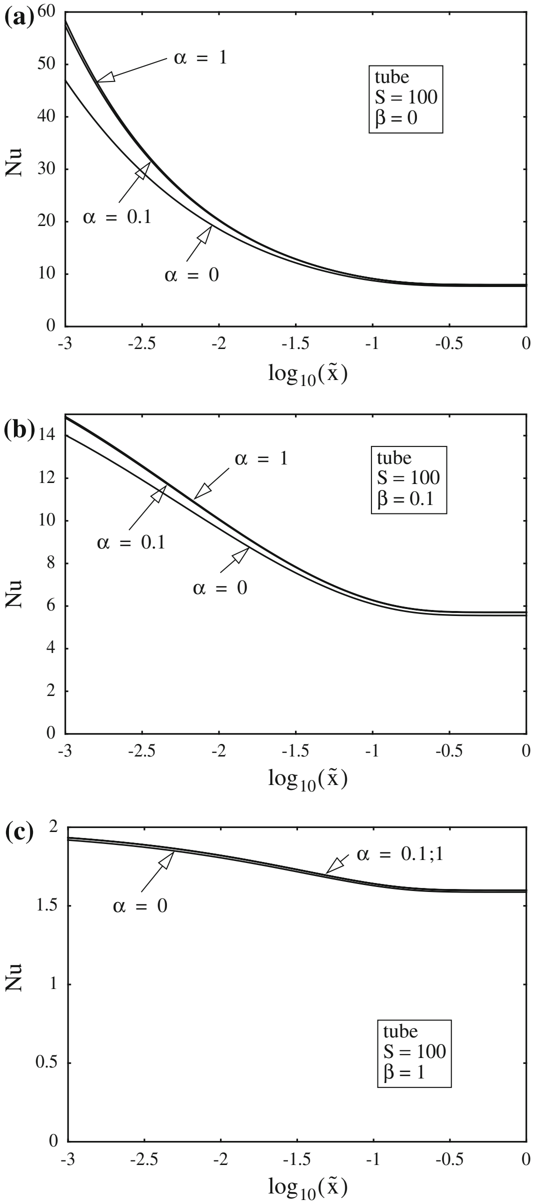


Fig. 6 As for Fig. 4, but now for $MDa = 10^{-4}$ ($S = 100$)

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