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Thermal Non-Equilibrium Porous Convection with Heat Generation and Density Maximum

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Abstract Linear stability criterion for the onset of natural convection in a fluid saturated porous medium with uniform internal heat generation and density maximum is determined. The porous medium is not in local thermal equilibrium (LTE) and we follow a two-field model for the energy equation. It is found that both the heat generation and density maximum have an additive effect in advancing the onset condition. In general the destabilising effect of density maximum increases for large values of the fluid heat generation parameter. This effect becomes prominent even for small values of the fluid heat generation parameter when the flow is of Darcy type and LTE is not valid.

Keywords Density maximum · Internal heat generation · Nonlocal thermal equilibrium

1 Introduction

Porous convection is of considerable geophysical and technical interest as it may occur in geothermal areas, through aquifers, oil reservoirs, snow layers, packed beds, mechanically agitated vessels in the food processing and chemical industries, etc. Natural convection in a horizontal fluid saturated porous layer which is heated from within by a uniform distribution of internal heat sources has become important in the study of the flow of magma in the Earth's crust. Many studies on convective instabilities in packed beds with or without mass flux in the presence of internal heat sources are available in the open literature (see for example Rudraiah et al. 1982; Yoon et al. 1998; Saravanan and Kandaswamy 2003).

There are fluids such as water, liquid helium, aqueous ammonia, and molten HgCdTe alloy for which the fluid density is not a monotonic function of temperature. It reaches a maximum value at a specific temperature and decreases while deviating from that temperature. This property, known as density maximum, is of scientific and engineering importance and can significantly change the onset of instability, the resulting flow field and heat transport in

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a fluid domain. For example melting a frozen moist soil is used as a part of underground digging. Having this in mind the stability of an internally heated water layer near its temperature of maximum density was studied by McKay (1990) and more recently by Saravanan and Kandaswamy (2002). Unsteady laminar natural convection flow of water subject to density inversion in a rectangular cavity formed by isothermal vertical walls with internal heat generation was investigated by Hossain and Rees (2005).

Most of the works on convective instability in porous media have been investigated mainly under the assumption that the fluid and solid phases are everywhere in local thermal equilibrium (LTE). Nevertheless in many practical applications, such as at sufficiently large velocities the LTE will break down so that the temperatures in the solid and fluid phases are no longer identical. In such a nonlocal thermal equilibrium (NLTE) situation instead of having a single energy equation which describes the common temperature of the saturated medium, two equations are used to model the fluid and solid phases separately. Rees and his coworkers (see Banu and Rees 2002; Postelnicu and Rees 2003) were the first to employ this treatment to analyse the onset of convection and recovered the classical LTE results. To obtain analytical results stress-free condition was imposed on the boundaries in their work. In both the LTE and NLTE limits the critical wavenumber approached the known value π and remained above this elsewhere. Very recently stability analysis of the conductive state in a horizontal porous layer which is caused by uniform heat generation in either the fluid or solid phase has been investigated by Borujerdi et al. (2007).

In this article, we study the onset of porous convection with internal heat generation and maximum density. We assume the porous medium to be in NLTE. We now consider the porous medium to be confined between two rigid boundaries, a more realistic situation. Our objective here is to find the smallest critical Rayleigh number using a method that eliminates elaborate numerical computation.

2 Mathematical Formulation

We consider an isotropic and homogeneous porous layer bounded between two rigid and conducting surfaces $z = 0$ and $z = d$ and saturated with a fluid near its temperature of maximum density. The lower surface of the layer is held at a temperature T_l while the upper surface is at T_u ($T_\mathrm{u} < T_\mathrm{l}$). Both the fluid and solid phases generate heat at a uniform rate q'''. We assume that the solid and fluid phases of the medium are not in LTE and use a two-field model for temperature. It is also assumed that at the bounding surfaces the solid and fluid phases have identical temperatures so that the problem becomes well posed. As the basic state is quiescent the Forchheimer drag is not considered. Taking into account the presence of boundary effects the basic governing equations are

$$\nabla \cdot \bar{q} = 0 \tag{1}$$

$$\mu\bar{q} + K(\nabla p - \rho\bar{g}) - K\widetilde{\mu}\nabla^2\bar{q} = 0 \tag{2}$$

$$\epsilon(\rho c)_\mathrm{f}\frac{\partial T_\mathrm{f}}{\partial t} + (\rho c)_\mathrm{f}(\bar{q}\cdot\nabla)T_\mathrm{f} = \epsilon k_\mathrm{f}\nabla^2 T_\mathrm{f} + h(T_\mathrm{s} - T_\mathrm{f}) + \epsilon q_\mathrm{f}''' \tag{3}$$

$$(1-\epsilon)(\rho c)_\mathrm{s}\frac{\partial T_\mathrm{s}}{\partial t} = (1-\epsilon)k_\mathrm{s}\nabla^2 T_\mathrm{s} - h(T_\mathrm{s} - T_\mathrm{f}) + (1-\epsilon)q_\mathrm{s}''' \tag{4}$$

where $\bar{q}$ is the velocity vector, p the pressure, ρ is the density, ϵ the porosity, $\bar{g}$ the gravitational acceleration, μ the fluid viscosity, T the temperature, h the inter-phase heat transfer coefficient, c the specific heat, K the permeability of the porous matter, $\tilde{\mu}$ the effective viscosity, k the thermal conductivity, and the subscript f and s refer to the fluid and the solid phases, respectively. We recover the Darcy model in the absence of the viscous resistance term $K\tilde{\mu}\nabla^2\bar{q}$. To model the density maximum property a simple non-linear approximation is used and the equation of state takes the form

$$\rho = \rho_0\left[1 - \beta_1(T_\mathrm{f} - T_\mathrm{u}) - \beta_2(T_\mathrm{f} - T_\mathrm{u})^2\right] \tag{5}$$

where $\beta_1 > 0$ and $\beta_2 > 0$ are coefficients of thermal expansion.

We eliminate the pressure from the momentum equation and render the resulting equation and the energy equations for fluid and solid phases dimensionless by using the scales d for distance, $\epsilon k_\mathrm{f}/((\rho c)_\mathrm{f} d)$ for velocity, $k_\mathrm{f}\mu/((\rho c)_\mathrm{f} K)$ for pressure, $T_\mathrm{l} - T_\mathrm{u}$ for temperature and $(\rho c)_\mathrm{f}/(k_\mathrm{f} d^2)$ for time. Hence we obtain

$$\nabla^2 w = Da\nabla^4 w + R_1\nabla_1^2 T_\mathrm{f} + R_2\nabla_1^2 T_\mathrm{f}^2 \tag{6}$$

$$\frac{\partial T_\mathrm{f}}{\partial t} + (\bar{q}\cdot\nabla)T_\mathrm{f} = \nabla^2 T_\mathrm{f} + H(T_\mathrm{s} - T_\mathrm{f}) + Q_\mathrm{f} \tag{7}$$

$$\alpha\frac{\partial T_\mathrm{s}}{\partial t} = \nabla^2 T_\mathrm{s} - \gamma H(T_\mathrm{s} - T_\mathrm{f}) + Q_\mathrm{s} \tag{8}$$

where $R_1 = \rho_0 g\beta_1(T_\mathrm{l} - T_\mathrm{u})Kd/(\mu\epsilon\kappa_\mathrm{f})$, $R_2 = \rho_0 g\beta_2(T_\mathrm{l} - T_\mathrm{u})^2 Kd/(\mu\epsilon\kappa_\mathrm{f})$ the Rayleigh numbers based on the properties of the fluid phase, $Da = K\tilde{\mu}/(\mu d^2)$ the Darcy number, $H = hd^2/(\epsilon k_\mathrm{f})$ the nondimensional inter-phase heat transfer coefficient, $Q_\mathrm{f} = q_\mathrm{f}'''d^2/(k_\mathrm{f}(T_\mathrm{l} - T_\mathrm{u}))$ the fluid-heat generation parameter, $Q_\mathrm{s} = q_\mathrm{s}'''d^2/(k_\mathrm{s}(T_\mathrm{l} - T_\mathrm{u}))$ the solid-heat generation parameter, $\alpha = k_\mathrm{f}(\rho c)_\mathrm{s}/(k_\mathrm{s}(\rho c)_\mathrm{f})$ the diffusivity ratio and $\gamma = \epsilon k_\mathrm{f}/((1-\epsilon)k_\mathrm{s})$ the porosity-modified conductivity ratio. Here R_2 serves as a measure of the density maximum property. Da assumes values less than 10^{-3} for the Darcy model and exceeds 10^{-3} for the Brinkmann model (Walker and Homsy 1977). The basic state is assumed to be quiescent with the corresponding temperature distribution

$$T_\mathrm{mb} = -\frac{Q_\mathrm{m}}{2}z^2 + \left(\frac{Q_\mathrm{m}}{2} - 1\right)z + 1 \tag{9}$$

where the subscript m stands for either s or f. It is to be noted that (9) is obtained with $H = 0$.

The basic state is perturbed and the quantities in the perturbed state are given by

$$(u, v, w) = (u', v', w'), \quad T_\mathrm{f} = T_\mathrm{fb} + \theta, \quad T_\mathrm{s} = T_\mathrm{sb} + \phi \tag{10}$$

where the perturbations are very small and unsteady quantities. Substituting these in Eqs. 6–8 and using the basic state solutions, we obtain the following linearized equations for the perturbed quantities as

$$\nabla^2 w = Da\nabla^4 w + R_1\nabla_1^2\theta + 2R_2\nabla_1^2\left[-\frac{Q_\mathrm{f}}{2}z^2 + \left(\frac{Q_\mathrm{f}}{2} - 1\right)z + 1\right]\theta \tag{11}$$

$$\frac{\partial\theta}{\partial t} + w\left[Q_\mathrm{f}\left(\frac{1}{2} - z\right) - 1\right] = \nabla^2\theta + H(\phi - \theta) \tag{12}$$

$$\alpha\frac{\partial\phi}{\partial t} = \nabla^2\phi - \gamma H(\phi - \theta) \tag{13}$$

The suitable conditions at the bounding surfaces are

$$w = \frac{dw}{dz} = \theta = \phi = 0 \ \text{ at } \ z = 0, 1 \tag{14}$$

We use the normal mode expansions for the dependent variables in the form $[w, \theta, \phi] = [W(z), \Theta(z), \Phi(z)]e^{i(lx+my)+ct}$ where $a = \sqrt{l^2 + m^2}$ is the horizontal wave number and c the complex frequency of perturbations. Then 11–13 lead to the amplitude equations

$$\begin{aligned} &Da\left[W''''(z) - 2a^2W''(z) + a^4W(z)\right] - W''(z) + a^2W(z) \\ &-R_1a^2\Theta(z) - 2R_2a^2\left[-\frac{Q_\mathrm{f}}{2}z^2 + \left(\frac{Q_\mathrm{f}}{2} - 1\right)z + 1\right]\Theta = 0 \end{aligned} \tag{15}$$

$$\Theta''(z) - (c + a^2 + H)\Theta(z) - W(z)\left[Q_\mathrm{f}\left(\frac{1}{2} - z\right) - 1\right] + H\Phi(z) = 0 \tag{16}$$

$$\Phi''(z) - (c + a^2 + \gamma H)\Phi(z) + \gamma H\Theta(z) = 0 \tag{17}$$

The eigenvalue problem defined by (15, 16), and (17) together with the boundary conditions is a two-point boundary value problem with variable co-efficients and can be solved to obtain a relation between parameters.

3 Solution Procedure

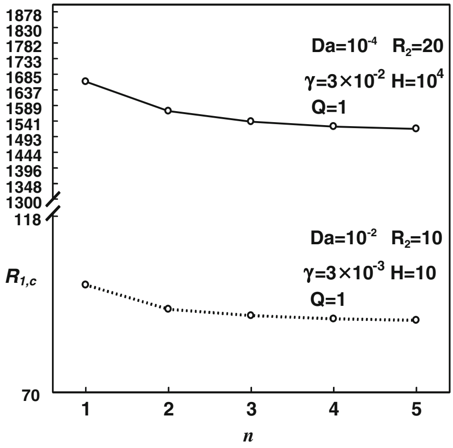
The Galerkin method that has the advantage of dealing with a large parameter space economically was used to solve the eigenvalue problem (see Finlayson 1972). The trial functions $W_i = z^{i+2}(1 - z)^2$ and $\Theta_i = \Phi_i = z^i(1 - z)$ were used to approximate W, Θ, and Φ, respectively. The critical Rayleigh number $R_{1,c}$ is obtained by minimizing R_1 with respect to a and that a corresponding $R_{1,c}$ is the critical wavenumber a_c. Before doing elaborate computations $R_{1,c}$ was calculated for a few representative cases and are plotted in Fig. 1 as a function of n, the number of trial functions. It can be seen that $R_{1,c}$ becomes independent of n as n increases. However, the monotonic convergence of $R_{1,c}$ implies that no qualitative changes can occur in $R_{1,c}$ even when n is less. Hence we restrict the unknowns to have a one-term expansion in this study.

The above amplitude equations together with the appropriate boundary conditions cannot be expressed into a self-adjoint form and hence the principle of exchange of stabilities does not hold in general. However, the computations using the one-term expansion indicated that oscillatory mode is not responsible for the instability for all possible values of the parameters and hence we report here the procedure only for the onset of stationary convection ($c = 0$). Accordingly multiplying (15) by $W(z)$, (16) by $\Theta(z)$, (17) by $\Phi(z)$, integrating across the layer and replacing $W(z)$ by $A_1W_1(z)$, $\Theta(z)$ by $B_1\Theta_1(z)$, $\Phi(z)$ by $C_1\Phi_1(z)$, we have

$$\begin{aligned} &A_1\left[Da\left(\left\langle(D^2W_1)^2\right\rangle + 2a^2\left\langle(DW_1)^2\right\rangle + a^4\left\langle W_1^2\right\rangle\right) + \left\langle(DW_1)^2\right\rangle + a^2\left\langle W_1^2\right\rangle\right] \\ &\quad -B_1[a^2R_1\langle\Theta_1W_1\rangle + 2R_2a^2(\langle\Theta_1W_1\rangle + \left(Q_\mathrm{f}/2 - 1\right)\langle z\Theta_1W_1\rangle \\ &\quad\quad -Q_\mathrm{f}/2\left\langle z^2\Theta_1W_1\right\rangle)] = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} &B_1[(a^2 + H)\left\langle\Theta_1^2\right\rangle + \left\langle(D\Theta_1)^2\right\rangle] \\ &\quad - A_1[Q_\mathrm{f}\langle zW_1\Theta_1\rangle - (Q_\mathrm{f}/2 - 1)\langle W_1\Theta_1\rangle] - HC_1\langle\Phi_1\Theta_1\rangle = 0 \end{aligned} \tag{19}$$

Fig. 1 $R_{1,c}$ as a function of n



$$C_1\left[(a^2+\gamma H)\left\langle\Phi_1^2\right\rangle+\left\langle(D\Phi_1)^2\right\rangle\right]-\gamma H B_1\left\langle\Theta_1\Phi_1\right\rangle=0 \tag{20}$$

where the bracket $\langle\ldots\rangle$ denotes the integration with respect to z. Eliminating A_1, B_1, and C_1 from the above equations, we get

$$R_1=\frac{\begin{aligned}&\left(\gamma H^2\left\langle\Theta_1\Phi_1\right\rangle^2-\left[(a^2+H)\left\langle\Theta_1^2\right\rangle+\left\langle(D\Theta_1)^2\right\rangle\right]\right.\\&\quad\times\left[(a^2+\gamma H)\left\langle\Phi_1^2\right\rangle+\left\langle(D\Phi_1)^2\right\rangle\right]\Big)\left(\left\langle(DW_1)^2\right\rangle+a^2\left\langle W_1^2\right\rangle\right.\\&\quad+Da\left[\left\langle(D^2W_1)^2\right\rangle+2a^2\left\langle(DW_1)^2\right\rangle+a^4\left\langle W_1^2\right\rangle\right]\Big)\\&\quad-2R_2a^2\left(\left\langle\Theta_1W_1\right\rangle+(Q_f/2-1)\left\langle z\Theta_1W_1\right\rangle-Q_f/2\left\langle z^2\Theta_1W_1\right\rangle\right)\\&\quad\left[(a^2+\gamma H)\left\langle\Phi_1^2\right\rangle+\left\langle(D\Phi_1)^2\right\rangle\right]\left[(Q_f/2-1)\left\langle W_1\Theta_1\right\rangle-Q_f\left\langle zW_1\Theta_1\right\rangle\right]\end{aligned}}{\begin{aligned}&a^2\left\langle\Theta_1W_1\right\rangle\left[(a^2+\gamma H)\left\langle\Phi_1^2\right\rangle+\left\langle(D\Phi_1)^2\right\rangle\right]\Big[(Q_f/2-1)\left\langle W_1\Theta_1\right\rangle\\&\quad-Q_f\left\langle zW_1\Theta_1\right\rangle\Big]\end{aligned}} \tag{21}$$

Choosing the trial functions as $W_1=z^3(1-z)^2$ and $\Theta_1=\Phi_1=z(1-z)$, we have

$$\begin{aligned}R_1&=\frac{280}{a^2C}\left[\left(\frac{\gamma H^2}{900}-AB\right)\left[\frac{a^2}{2310}+\frac{2}{315}+Da\left(\frac{a^4}{2310}+\frac{4a^2}{315}+\frac{12}{35}\right)\right]\right.\\&\quad\left.-2R_2a^2C\left[\frac{1}{280}-\frac{Q_f}{1680}+\left(\frac{Q_f}{2}-1\right)\frac{1}{504}\right]\right]\end{aligned} \tag{22}$$

where $A=(a^2+H)/30+1/3$, $B=(a^2+\gamma H)/30+1/3$, and $C=(Q_f/2-1)/280-Q_f/504$.

The results obtained for $Q_f=R_2=0$ and $Da=0, 10^{-3}, 1$ were compared with those in Postelnicu and Rees (2003) for stress-free boundaries. It was found that $R_{1,c}$ in the case of rigid boundaries are greater than those of the stress-free boundaries. Thus the effect of rigid boundaries is to inhibit the onset of convection, as expected from the physical grounds.

4 Discussion

The Rayleigh number given by Eq. 22 is evaluated for various values of the physical parameters to know their effects on setting up of convection. We found that a_c remains unchanged irrespective of the values of Q_{f} and R_2 similar to the result previously reported in Rudraiah et al. (1982). In all cases the value of a_c is 3.4 and hence there is no change in the cell size at the onset condition. The marginal stability curves are plotted in Fig. 2 for different values of Q_{f} and $R_2 = 20$. They all exhibit the well-known shape of marginal curves of convective instability problems with a single destabilizing mechanism (Saravanan and Kandaswamy 2002).

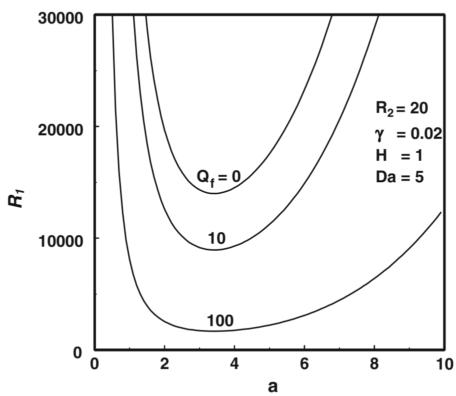
Figure 3 shows the effect of Q_{f} on the critical curves for different values of H. We see that $R_{1,c}$ decreases against Q_{f} monotonically and becomes zero at some finite value of Q_{f}. This shows that Q_{f} advances the onset of convection and the heat generation alone can induce the instability for sufficiently large values of Q_{f}. This is quite natural due to a net increase in the buoyancy force, the only destabilizing agency. Also we observe that $R_{1,c}$ decreases drastically as $Da \to 0$. This is because Da is the ratio of viscous resistance near the rigid boundaries and bulk resistance in the core region and hence a decrease in Da causes a reduction in the boundary layer thickness which in turn allows the fluid to move with less hindrance. Also we observe that an increase in H advances the appearance of convection.

In Fig. 4, we display the effect of R_2 for small and large values of H in the presence of heat generation. It is to be noticed that small values of H correspond to a situation in which the two phases remain almost without any interaction and hence only the fluid phase takes part in convection. On the other extreme, for large values of H both the solid and fluid phases interact well and behave as a single phase. We observe that R_2 produces a destabilizing effect and its influence increases with Q_{f}. We also notice that the effect of R_2 is visible even for small values of Q_{f} only when the porous medium is in NLTE and the flow is of Darcy type.

Figure 5 shows the effect of γ for small and large values of H. We note that an increase in γ may be due to an increase in either the porosity or the relative fluid–solid conductivity ratio $k_{\mathrm{f}}/k_{\mathrm{s}}$. We find that the destabilizing effect of γ becomes clear only when H is increased as expected physically.

Figure 6 shows the effect of H on $R_{1,c}$, $\gamma R_{1,c}/(1+\gamma)$, and a_c when $R_2 = Q_{\mathrm{f}} = 10$ and $Da = 1$. It is interesting to note that $\gamma R_{1,c}/(1+\gamma)$ is the critical Rayleigh number based on the mean properties of the porous medium which is used when assuming LTE. Both $R_{1,c}$ and $\gamma R_{1,c}/(1+\gamma)$ are close to 1760 and are independent of γ in the small H and large H limits. Both the critical Rayleigh numbers increase monotonically as H increases with γ fixed. For

Fig. 2 Marginal curves



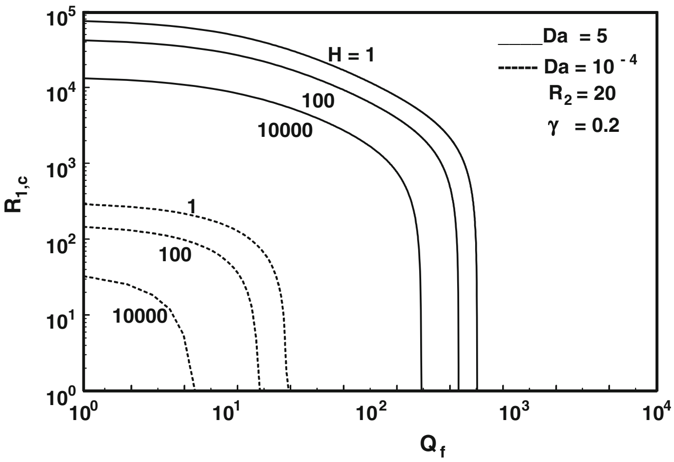


Fig. 3 $R_{1,c}$ against Q_f for different values of H

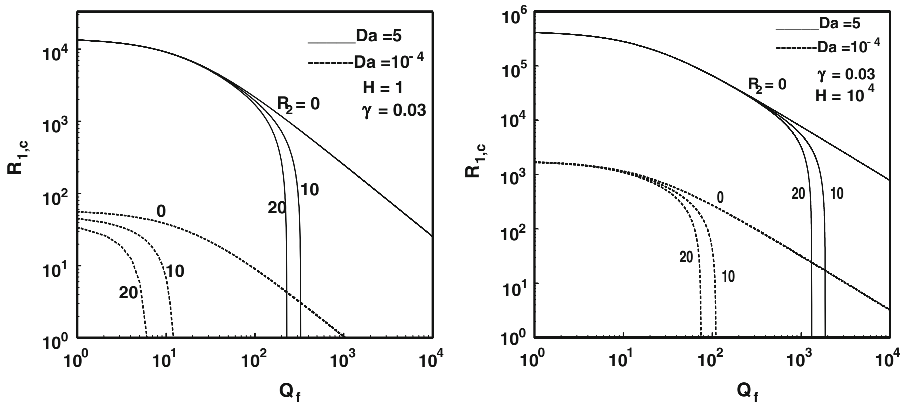


Fig. 4 $R_{1,c}$ against Q_f for different values of R_2: (**a**) small and (**b**) large H

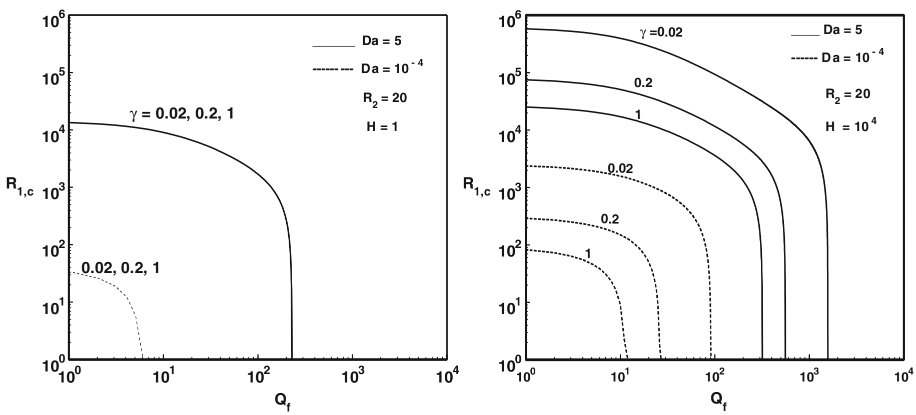


Fig. 5 $R_{1,c}$ against Q_f for different values of γ: (**a**) small and (**b**) large H

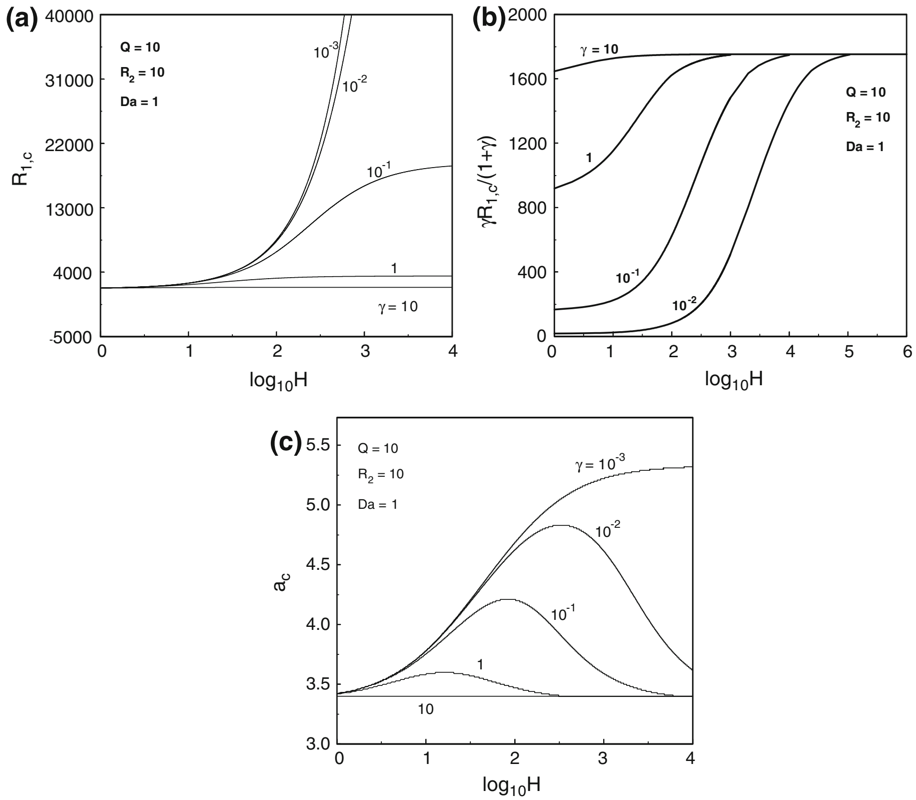


Fig. 6 (**a**) $R_{1,c}$, (**b**) $\gamma R_{1,c}/(1+\gamma)$, and (**c**) a_c against H

the intermediate values of H, a_c attains the maximum value, a typical feature of this type of problem. It is also noticed that large values of γ with H fixed restore the LTE limit.

5 Conclusion

The onset of stationary convection in a fluid saturated porous layer of NLTE type is investigated in the presence of internal heat sources and density maximum. The layer is assumed to be confined between rigid planes in order to include the contribution of the boundary effects. It is found that Da stabilizes the system whereas both Q_f and R_2 destabilize it. The effect of R_2 increases with Q_f and is felt even for small values of Q_f when both Da and H approach zero. The results of LTE are recovered when either H or γ is sufficiently large.

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