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The Starting Flow in Ducts Filled with a Darcy–Brinkman Medium

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Abstract Analytical solutions are found for the transient starting flow due to a sudden pressure gradient in cylindrical, rectangular, and parallel plate ducts fill with a Darcy–Brinkman porous medium. It is found that, for all geometries, the initial velocity front is flat. It eventually becomes more parabolic for small porous media parameter s but remains flat for large s. The boundary layer thickness is of order $(1/s)$. The transient is also shorter (proportional to $\exp(-s^2 t)$) for large s.

Keywords Darcy–Brinkman · Porous media · Circular duct · Rectangular duct · Starting flow

Nomenclature

A_n, A_{mn}	Coefficients
b	Aspect ratio
G	Pressure gradient (N/m^3)
G_0	Constant pressure gradient (N/m^3)
H	Unit step function
I	Modified Bessel function
J	Bessel function
k_n, k_{mn}	Parameters defined by Eq. 12 or Eq. 23
K	Permeability (m^2)
L	Length scale (m)
n	Integer
Q	Normalized flow rate

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r	Normalized radial coordinate
s	Non-dimensional parameter $L\sqrt{\mu/K\mu_e}$
t	Normalized time
u	Normalized axial velocity
$\bar{u}$	Normalized steady-state velocity
$\tilde{u}$	Normalized transient velocity
x, y	Normalized Cartesian coordinates
λ_n	Zero of J_0
α_n	$(n-1/2)\pi$
β_m	$(n-1/2)\pi/b$
γ	$(n-1/4)\pi$
μ	Fluid viscosity (N s/m^2)
μ_e	Effective viscosity (N s/m^2)
ρ	Fluid density (kg/m^3)
θ	Polar angle coordinate
ϕ	Porosity
$'$	Dimensional quantity

1 Introduction

The flow in ducts filled with porous media is important in filtering equipment, chemical reactors, movement of ground water, oil and gas extraction, etc. See Kaviany (1991), Nield and Bejan (2006). At low velocities a saturated porous medium with high porosity is well described by the Darcy–Brinkman approximation, which takes into account the resistance of both the porous matrix and the confining boundary. Many authors have studied the steady flow in ducts filled with a Darcy–Brinkman medium; the earliest analytical works include Kaviany (1985) for parallel plate channels, Parang and Keyhani (1987) for circular ducts, Haji-Sheikh (2006) and Hooman and Merrikh (2006) for rectangular ducts.

There are few papers on the unsteady flow in a porous duct. Al-Nimr and Alkam (2000) considered the starting flow in a circular (and annular) duct, but their solutions are in error. Only recently Kuznetkov and Nield (2006) successfully solved the oscillatory flow in a parallel plate channel and in a circular tube. In the present paper we present the solutions to the transient starting flow in a porous channel, a circular duct, and a rectangular duct.

The unsteady Darcy–Brinkman equation is

$$\frac{\rho}{\phi}\frac{\partial u'}{\partial t'} = G + \mu_e \nabla^2 u' - \frac{\mu}{K}u' \tag{1}$$

Here G is the sudden pressure gradient which is zero for time negative and is a constant G_0 for time positive. Such a constant pressure gradient would be established almost at once due to the incompressibility of the fluid which saturates the highly porous medium. Normalize all lengths by a nominal characteristic length L, the velocity by (G_0L^2/μ_e), the time by $(\rho L^2/\phi\mu_e)$ and drop primes. Equation 1 become

$$u_t = H(t) + \nabla^2 u - s^2 u \tag{2}$$

where $H(t)$ is the unit step function and

$$s^2 = \frac{L^2\mu}{K\mu_e} \tag{3}$$

is the important non-dimensional parameter characterizing the porous medium. Let

$$u = \bar{u} - \tilde{u} \tag{4}$$

where $\bar{u}$ is the steady-state solution and $\tilde{u}$ is the transient which decays to zero for large times. Then Eq. 2 separates to

$$0 = 1 + \nabla^2 \bar{u} - s^2 \bar{u} \tag{5}$$

$$\tilde{u}_t = \nabla^2 \tilde{u} - s^2 \tilde{u} \tag{6}$$

The boundary conditions are that $\bar{u}$, $\tilde{u}$ are zero on the walls, and

$$\tilde{u}\,|_{t=0} = \bar{u} \tag{7}$$

2 The Circular Duct

Let the diameter be $2L$ (Fig. 1a). Equation 5 in axisymmetric polar coordinates is

$$0 = 1 + \bar{u}_{rr} + \bar{u}_r / r - s^2 \bar{u} \tag{8}$$

The solution which is bounded on the axis and zero at $r = 1$ is

$$\bar{u} = \frac{1}{s^2}\left(1 - \frac{I_0(sr)}{I_0(s)}\right) \tag{9}$$

The steady-state solution agrees with that of Parang and Keyhani (1987). Note that for pure fluids, $s \to 0$ and Eq. 9 reduces to the parabolic Poiseuille flow $\bar{u} = (1 - r^2)/4$. For the transient flow, let

$$\tilde{u} = \sum_{n=1}^{\infty} A_n e^{-k_n t} J_0(\lambda_n r) \tag{10}$$

where λ_n are the zeroes of J_0. The first five zeroes are 2.4048, 5.5201, 8.6537, 11.7915, 14.9309. For $n > 5$ the zeroes are well approximated by (Abramowitz and Stegun 1965)

$$\lambda_n = \gamma + \frac{1}{8\gamma} - \frac{31}{96\gamma^3} \tag{11}$$

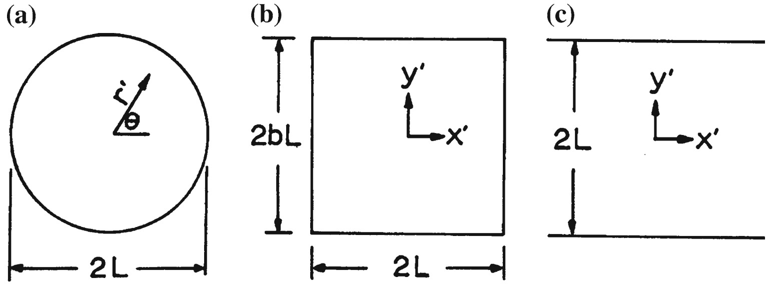

Fig. 1 (**a**) The circular duct, (**b**) the rectangular duct and (**c**) the parallel plate duct

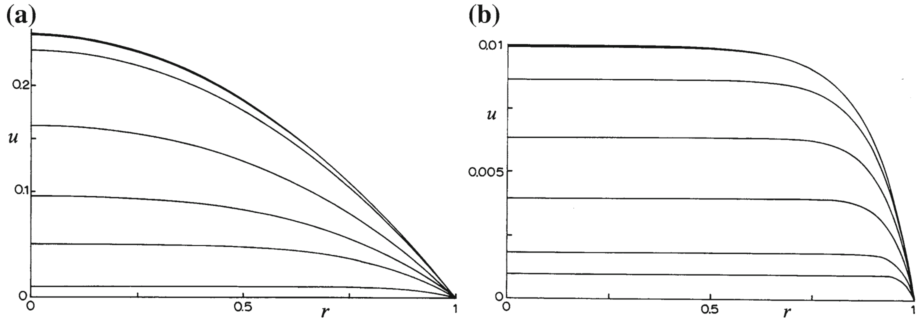

Fig. 2 Velocity profiles for the circular duct (**a**) $s = 0.1$. From bottom: $t = 0.01, 0.05, 0.1, 0.2, 0.5, 1, 10$. (**b**) $s = 10$. From bottom: $t = 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 1$

where $\gamma = (n - 1/4)\pi$. Substitution of Eq. 10 into Eq. 6 gives

$$k_n = \lambda_n^2 + s^2 > 0 \tag{12}$$

The initial condition Eq. 7 yields

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = \frac{1}{s^2}\left(1 - \frac{I_0(sr)}{I_0(s)}\right) \tag{13}$$

Multiplying Eq. 13 by $r J_0(\lambda_n r)$ and integrating from zero to one, and utilizing orthogonality gives the coefficients

$$A_n = \frac{2}{\lambda_n J_1(\lambda_n)(s^2 + \lambda_n^2)} \tag{14}$$

Figure 2 shows the development of the velocity profiles. For small s initially the velocity is almost flat, but approaches parabolic shape for large times. For large s the velocity front remains flat all the time. There is also a boundary layer near the wall.

The total flow rate, normalized by $(G_0 L^3/\mu_e)$, is

$$Q = 2\pi \int_0^1 (\bar{u} - \tilde{u}) r dr = 2\pi \left[\frac{1}{s^2}\left(\frac{1}{2} - \frac{I_1(s)}{s I_0(s)}\right) - \sum_{n=0}^{\infty} A_n e^{-k_n t} \frac{J_1(\lambda_n)}{\lambda_n}\right] \tag{15}$$

The transient increase in flow rate is shown in Fig. 3. We see that the steady state is reached much sooner for large s. The reason is due to the parameter $k_n$in the time exponent that is larger for large s.

3 The Rectangular Duct

Consider the rectangular duct shown in Fig. 1b. Let

$$\alpha_n = (n - 1/2)\pi, \quad \beta_m = (m - 1/2)\pi/b \tag{16}$$

Expand unity in a cosine series

$$1 = \sum_{n=1}^{\infty} a_n \cos(\alpha_n x) \tag{17}$$

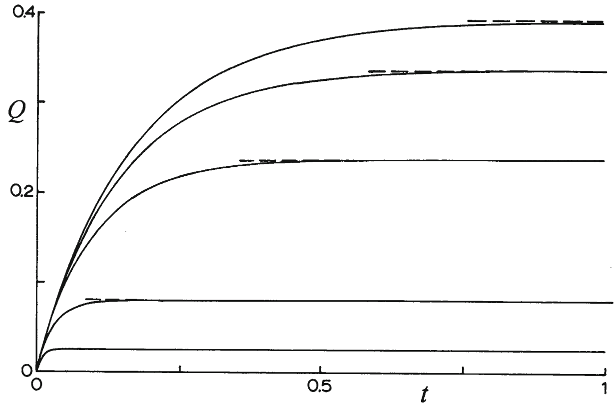

Fig. 3 Flow rate in the circular duct as function of time. From bottom: $s = 10, 5, 2, 1, 0.1$. Dashed lines indicate steady-state values

Fourier inversion shows

$$a_n = \frac{2(-1)^{n+1}}{\alpha_n} \tag{18}$$

Let

$$\bar{u} = \sum_{n=1}^{\infty} \cos(\alpha_n x) f_n(y) \tag{19}$$

Equation 5 gives

$$f_n'' - (\alpha_n^2 + s^2) f_n = -a_n \tag{20}$$

The solution which is even in y and zero on the boundary is

$$f_n = \frac{a_n}{(\alpha_n^2 + s^2)} \left[1 - \frac{\cosh\left(\sqrt{\alpha_n^2 + s^2}y\right)}{\cosh\left(\sqrt{\alpha_n^2 + s^2}b\right)} \right] \tag{21}$$

The steady-state solution agrees with that of Hooman and Merrikh (2006). For the transient we let

$$\tilde{u} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos(\alpha_n x) \cos(\beta_m y) e^{-k_{mn} t} \tag{22}$$

Equation 6 gives

$$k_{mn} = \alpha_n^2 + \beta_m^2 + s^2 \tag{23}$$

The initial condition yields

$$\bar{u} = \sum_{n=1}^{\infty} \cos(\alpha_n x) f_n(y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \cos(\alpha_n x) \cos(\beta_m y) \tag{24}$$

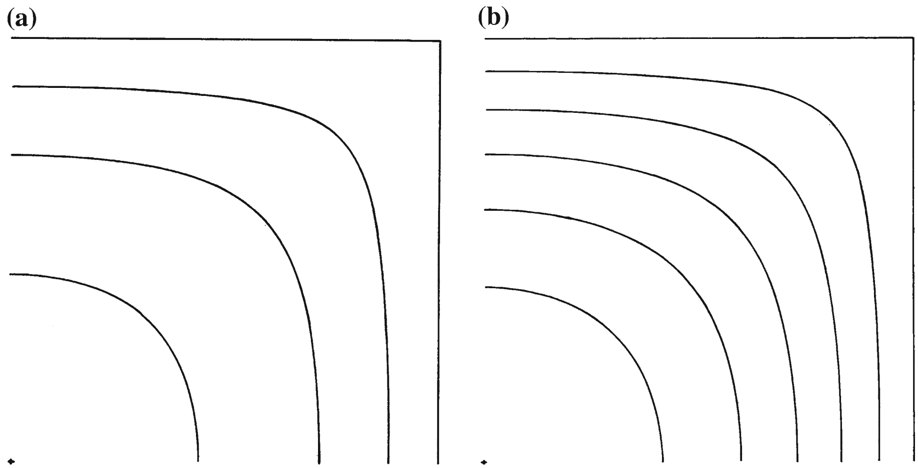

Fig. 4 Constant velocity lines for the square duct (first quadrant) for $s = 0.1$ (**a**) $t = 0.2$, $\Delta u = 0.05$. Maximum velocity at cross: 0.1724 (**b**) $t = 1$, $\Delta u = 0.05$. Maximum velocity at cross: 0.2940

or

$$f_n(y) = \sum_{m=1}^{\infty} A_{mn} \cos(\beta_m y) \tag{25}$$

Multiplying by $\cos(\beta_m y)$ and integrating from zero to b result in

$$A_{mn} = \frac{2a_n(-1)^{m+1}}{b\beta_m(\alpha_n^2 + \beta_m^2 + s^2)} \tag{26}$$

The constant velocity lines for a square duct are shown in Fig. 4 for small s. The velocity increases with time, but retains a somewhat paraboloidal shape. For large s (Fig. 5) the interior is almost flat with boundary layers near the walls. The normalized flow rate is

$$Q = 4\int_0^b \int_0^1 (\bar{u} - \tilde{u})dxdy = 4\sum_n \frac{a_n(-1)^{n+1}}{\alpha_n(\alpha_n^2 + s^2)} \left(b - \frac{\tanh\left(\sqrt{\alpha_n^2 + s^2}b\right)}{\sqrt{\alpha_n^2 + s^2}} \right)$$
$$-4\sum_n \sum_m A_{mn} \frac{(-1)^{m+n}}{\alpha_n \beta_m} e^{-k_{mn}t} \tag{27}$$

Figure 6 shows the increase in flow with time. We see that the transient is shorter for larger s, consistent with the exponent Eq. 23. On the other hand, the transient is longer for larger aspect ratio b.

4 Discussion

Al-Nimr and Alkam (2000) attempted to solve the transient circular tube problem by an expansion in modified Bessel functions. But these functions have neither eigenvalues nor orthogonality. Thus their solution is incorrect. The correct solution, presented here, should be expressed in terms of Bessel functions J_0.

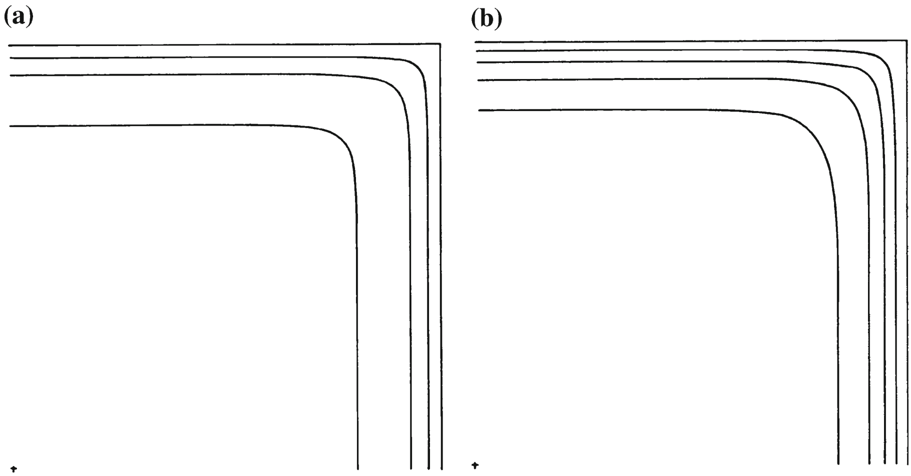

Fig. 5 Constant velocity lines for the square duct (first quadrant) for $s = 10$ (**a**) $t = 0.01$, $\Delta u = 0.002$. Maximum velocity at cross: 0.00632 (**b**) $t = 0.2$, $\Delta u = 0.002$. Maximum velocity at cross: 0.00997

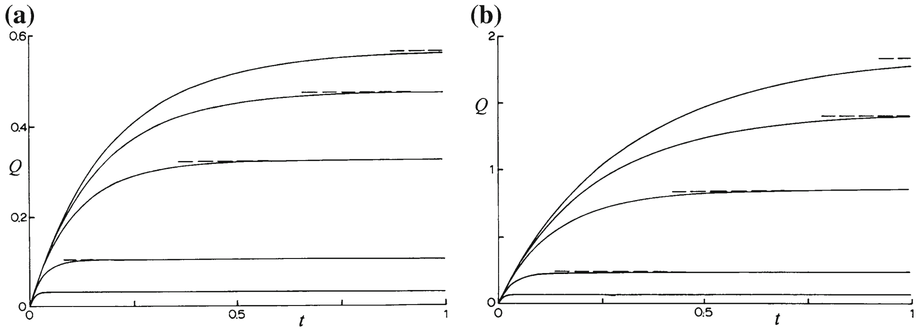

Fig. 6 Flow rate in the rectangular duct as function of time. From bottom: $s = 10, 5, 2, 1, 0.1$. Dashed lines indicate steady-state values. (**a**) $b = 1$ (square duct), (**b**) aspect ratio $b = 2$

We have successfully solved for the transient starting flow in circular, rectangular, and parallel plate ducts filled with a Darcy–Brinkman medium. It is found that, for all geometries, the initial velocity front is flat. It eventually becomes more parabolic for small s but remains flat for large s. From Eq. 2 the boundary layer thickness is of order $(1/s)$. The transient is also shorter (proportional to $\exp(-s^2 t)$) for large s.

The stopping flow (sudden removal of a pressure gradient) is represented by the transient solution $\tilde{w}$. Since the equations are linear, superposition is possible, such as the response due to a pulsed pressure gradient. Our basic analytical results would serve as a check for numerical solutions of more complicated problems.

Appendix: The Parallel Plate Channel

For a channel consisting of two plates $2L$ apart, let Cartesian axes be placed at mid-channel (Fig. 1c). Equation 5 becomes

$$0 = 1 + \bar{u}_{yy} - s^2 \bar{u} \tag{A1}$$

The solution is

$$\bar{u} = \frac{1}{s^2}\left(1 - \frac{\cosh(sy)}{\cosh(s)}\right) \tag{A2}$$

Equation 6 is

$$\tilde{u}_t = \tilde{u}_{yy} - s^2 \tilde{u} \tag{A3}$$

The solution is

$$\tilde{u} = \sum_{n=1}^{\infty} A_n e^{-k_n t} \cos(\alpha_n y) \tag{A4}$$

Here

$$\alpha_n = (n - 1/2)\pi, k_n = \alpha_n^2 + s^2 \tag{A5}$$

Using the initial condition Eq. 7 the coefficients are found to be

$$A_n = \frac{2(-1)^{n+1}}{\alpha_n(\alpha_n^2 + s^2)}$$

The normalized flow rate is

$$Q = \frac{2}{s^2}\left(1 - \frac{\tanh s}{s}\right) - 2\sum_n A_n \frac{(-1)^{n+1}}{\alpha_n} e^{-k_n t} \tag{A6}$$

The properties of the parallel plate channel are similar to those of the rectangle with large aspect ratio and will not be discussed here.

References

Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1965)

Al-Nimr, M.A., Alkam, M.K.: Basic fluid flow problems in porous media. J. Porous Media **3**, 45–49 (2000)

Haji-Sheikh, A.: Fully developed heat transfer to fluid flow in rectangular passages filled with porous materials. J. Heat Transfer **128**, 550–556 (2006)

Hooman, K., Merrikh, A.A.: Analytical solution of forced convection in a duct of rectangular cross section saturated by a porous medium. J. Heat Transfer **128**, 596–600 (2006)

Kaviany, M.: Laminar flow through a porous channel bounded by isothermal parallel plates. Int. J. Heat Mass Transfer **28**, 851–858 (1985)

Kaviany, M.: Principles of Heat Transfer in Porous Media. Springer, New York (1991)

Kuznetkov, A., Nield, D.A.: Forced convection with laminar pulsating flow in a saturated porous channel or tube. Transp. Porous Media **65**, 505–523 (2006)

Nield, D.A., Bejan, A.: Convection in Porous Media, 3rd edn. Springer, New York (2006)

Parang, M., Keyhani, M.: Boundary effects in laminar mixed convection flow through an annular porous medium. J. Heat Transfer **115**, 506–510 (1987)