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ORIGINAL ARTICLE

The onset of convection in a shallow box occupied by a heterogeneous porous medium with constant flux boundaries

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Abstract The effects of hydrodynamic and thermal heterogeneity, for the case of variation in both the horizontal and vertical directions, on the onset of convection in a horizontal layer of a saturated porous medium uniformly heated from below, are studied analytically using linear stability theory for the case of weak heterogeneity. Attention is focused on the case of constant flux upper and lower boundaries, a case for which the critical horizontal wavenumber is zero, and attention is also concentrated on the case of a shallow layer. It is found that the effect of such heterogeneity on the critical value of the Rayleigh number Ra based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable once the aspect ratio is taken into account, and to a first approximation are independent. The combination of permeability heterogeneity and conductivity heterogeneity can be either stabilizing or destabilizing for the present case.

Keywords Natural convection · Heterogeneity · Instability · Horton–Rogers–Lapwood problem

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Nomenclature

A	Aspect ratio (height to width)
c	Specific heat
H	Height of the enclosure
k	k^*/k_0
k^*	Overall (effective) thermal conductivity
k_0	Mean value of $k^*(x^*, y^*)$
K	K^*/K_0
K^*	Permeability
K_0	Mean value of $K^*(x^*, y^*)$
L	Width of the enclosure
P	Dimensionless pressure, $\frac{(\rho c)_f K_0}{\mu k_0} P^*$
P^*	Pressure
Ra	Rayleigh number, $\frac{(\rho c)_f \rho_0 g \beta K_0 L(T_1 - T_0)}{\mu k_0}$
t^*	Time
t	Dimensionless time, $\frac{k_0}{(\rho c)_m L^2} t^*$
T^*	Temperature
T_0	Temperature at the upper boundary
T_1	Temperature at the lower boundary
u	Dimensionless horizontal velocity, $\frac{(\rho c)_m L}{k_0} u^*$
$\mathbf{u}^*$	Vector of Darcy velocity, (u^*, v^*)
v	Dimensionless vertical velocity, $\frac{(\rho c)_m L}{k_0} v^*$
x	Dimensionless horizontal coordinate, x^*/L
x^*	Horizontal coordinate
y	Dimensionless upward vertical coordinate, y^*/H
y^*	Upward vertical coordinate

Greek symbols

β	Fluid volumetric expansion coefficient
θ	Dimensionless temperature, $\frac{T^* - T_0}{T_1 - T_0}$
μ	Fluid viscosity
ρ	Density
ρ_0	Fluid density at temperature T_0
σ	Heat capacity ratio, $\frac{(\rho c)_m}{(\rho c)_f}$
ψ	Streamfunction defined by Eqs. (10a,b)

Subscripts

f	Fluid
m	Overall porous medium

Superscripts

*	Dimensional variable

1 Introduction

The problem of the onset of convection in a horizontal layer of fluid heated uniformly from below is commonly called the Rayleigh–Bénard problem in the case of a fluid clear of solid material and the Horton–Rogers–Lapwood (HRL) problem for the case of a fluid-saturated porous medium. A feature of such convection is that it generally appears in the form of cells whose horizontal dimension is of the same order as their vertical dimension. (The critical dimensionless wavenumber a_c in the linear stability analysis turns out to have a value of about 3. In the HRL problem with conducting impervious boundaries $a_c = \pi$, a value that corresponds to rolls of square cross-section.) An exception occurs in the case of "insulating" (with respect to perturbation heat flux) boundaries. For this case $a_c = 0$, so that the convection occurs as a single cell. This exceptional result appeared as a limiting case in the results of Sparrow et al. (1964) and Hurle et al. (1967), but we believe that the earliest published explicit discussion of convection at zero wavenumber is that in an appendix of Nield (1968a). It was there pointed out that $a_c = 0$ applies if the layer extends to infinity in the horizontal direction, but in a practical situation the fluid will be bounded by lateral walls and the position of these will determine a small non-zero value of a_c.

Nield (1968a) provided the following physical explanation of this phenomenon. When the total heat flow across each boundary is kept constant, the perturbation heat flow out of each layer is zero. While surplus heat can still diffuse back into the body of the fluid, it cannot diffuse out across the boundary. A possible thermal stabilizing effect is thus absent and viscosity is then the dominant stabilizing factor. The favored configuration for convection is then that for which the viscous dissipation is least. This is a single cell.

Convection at zero wavenumber is also of mathematical interest, because one can then perform a perturbation analysis for small wavenumbers and thereby obtain a relatively simple analytical result. This situation was exploited by Nield (1975, 1977, 1987) and other workers.

Our renewed interest in convection at zero wavenumber is a result of recent discussions about the effect of heterogeneity (of either permeability or thermal conductivity or both) on convection in a porous medium. In the case of strong heterogeneity, there can be dramatic effects (Simmons et al. 2001; Prasad and Simmons 2003; Nield and Simmons 2006). Even in the case of weak heterogeneity, it is of interest to investigate the combined effects of vertical heterogeneity (property variation in the vertical direction) and horizontal heterogeneity. This is the subject of the analysis of Nield and Kuznetsov (2006). The survey of the effects of heterogeneity in Nield and Bejan (2006, Sect. 6.13) indicates that this topic had not been considered previously. In their analytical study, Nield and Kuznetsov (2006) found that the effect of such heterogeneity on the critical value of the Rayleigh number Ra based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable and to a first approximation are independent. For the case of conducting impermeable top and bottom boundaries and a square box, the effects of permeability heterogeneity and conductivity permeability each cause a reduction in the critical value of Ra, while for the case of a tall box there can be either a reduction or an increase.

The question that then arises is whether these results are generic or whether they apply only to square or tall boxes with conducting top and bottom boundaries, the case previously studied. The present paper is a contribution to the answer of that question. The analysis of Nield and Kuznetsov (2006) is modified to treat a shallow box with constant-flux top and bottom boundaries.

2 Analysis

Single-phase flow in a saturated porous medium is considered. Asterisks are used to denote dimensional variables. We consider a rectangular box $0 \le x^* \le L$, $0 \le y^* \le H$, where the y^* axis is in the upward vertical direction. The side walls are taken as insulated, and constant uniform heat flux is imposed at the upper and lower boundaries, where the temperatures are denoted by T_0 and T_1, respectively. (It should be noted that T_0 and T_1 are regarded as constants for the purpose of specifying the basic temperature gradient and the reference temperature, but they are not constants in the context of perturbation quantities. It is the perturbation temperature gradients, and not the perturbation temperatures, that are assumed to be zero on the upper and lower boundaries.)

Within this box the permeability is $K^*(x^*, y^*)$ and the overall (effective) thermal conductivity is $k^*(x^*, y^*)$. The Darcy velocity is denoted by $\mathbf{u}^* = (u^*, v^*)$. The Oberbeck–Boussinesq approximation is invoked. The equations representing the conservation of mass, thermal energy, and Darcy's law take the form

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{1}$$

$$(\rho c)_{\mathrm{m}} \frac{\partial T^*}{\partial t^*} + (\rho c)_{\mathrm{f}} \left[u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right] = k^*(x^*, y^*) \left[\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} \right], \tag{2}$$

$$u^* = -\frac{K^*(x^*, y^*)}{\mu} \frac{\partial P^*}{\partial x^*}, \quad v^* = \frac{K^*(x^*, y^*)}{\mu} \left[-\frac{\partial P^*}{\partial y^*} + \rho_0 \beta g (T^* - T_0) \right]. \tag{3a,b}$$

Here, $(\rho c)_{\mathrm{m}}$ and $(\rho c)_{\mathrm{f}}$ are the heat capacities of the overall porous medium and the fluid, respectively, μ is the fluid viscosity, ρ_0 is the fluid density at temperature T_0, and β is the volumetric expansion coefficient.

In writing Eq. (2), we have made the assumption that the derivatives of $k^*(x^*, y^*)$ with respect to x^* and y^* are small compared with $k^*(x^*, y^*)$ itself. This is part and parcel of what we mean by weak heterogeneity of conductivity. The omitted terms are in fact identically zero for the piecewise constant distribution in the test case treated in detail below.

In this pioneering paper and in Nield and Kuznetsov (2006), the Darcy model has been employed for the sake of simplicity. A Forchheimer term does not affect the value of the critical Rayleigh number. A Brinkman term has a small effect of the value of the critical Rayleigh number if the Darcy number Da (defined as K/H^2 for a mean value of K) is small. (This is essentially because the Brinkman term is physically significant only within a thin boundary layer whose thickness is of order $\mathrm{Da}^{1/2} H$ near a rigid boundary.) The channelling effect is likewise small if Da is small. Also for simplicity, local thermal equilibrium is assumed here. The effects of local thermal non-equilibrium and a Brinkman term will be considered in a later paper.

We introduce dimensionless variables by defining

$$(\hat{x}, \hat{y}) = \frac{1}{H}(x^*, y^*), \quad (u, v) = \frac{(\rho c)_{\mathrm{m}} H}{k_0}(u^*, v^*), \quad t = \frac{k_0}{(\rho c)_{\mathrm{m}} H^2} t^*,$$

$$\theta = \frac{T^* - T_0}{T_1 - T_0}, \quad P = \frac{(\rho c)_{\mathrm{f}} K_0}{\mu k_0} P^*, \tag{4a,b,c,d,e}$$

where k_0 is the mean value of $k^*(x^*, y^*)$ and K_0 is the mean value of $K^*(x^*, y^*)$. We also define a Rayleigh number Ra by

$$\mathrm{Ra} = \frac{(\rho c)_\mathrm{f}\rho_0 g\beta K_0 H(T_1 - T_0)}{\mu k_0} \tag{5}$$

and the heat capacity ratio

$$\sigma = \frac{(\rho c)_\mathrm{m}}{(\rho c)_\mathrm{f}}. \tag{6}$$

The governing equations then take the form

$$\frac{\partial u}{\partial \hat{x}} + \frac{\partial v}{\partial \hat{y}} = 0, \tag{7}$$

$$\frac{\partial \theta}{\partial \tau} + \frac{1}{\sigma}\left[u\frac{\partial \theta}{\partial \hat{x}} + v\frac{\partial \theta}{\partial \hat{y}}\right] = k(\hat{x}, \hat{y})\left[\frac{\partial^2 \theta}{\partial \hat{x}^2} + \frac{\partial^2 \theta}{\partial \hat{y}^2}\right], \tag{8}$$

$$u = -K(\hat{x}, \hat{y})\frac{\partial P}{\partial \hat{x}}, \quad v = K(\hat{x}, \hat{y})\left[-\frac{\partial P}{\partial \hat{y}} + \sigma \mathrm{Ra}\theta\right], \tag{9}$$

where $k(\hat{x}, \hat{y}) = k^*(x^*, y^*)/k_0$ and $K(\hat{x}, \hat{y}) = K^*(x^*, y^*)/K_0$.

We introduce a streamfunction ψ so that

$$u = \sigma \mathrm{Ra}\frac{\partial \psi}{\partial \hat{y}}, \quad v = -\sigma \mathrm{Ra}\frac{\partial \psi}{\partial \hat{x}}. \tag{10a,b}$$

We also eliminate P. In doing this we assume that, in accordance with the assumption of weak heterogeneity, that the maximum variation of K over the domain is small compared with the mean value of K, so we can approximate $\partial(u/K)/\partial\hat{x}$ by $(1/K)\partial u/\partial\hat{x}$, etc. The result is

$$\frac{\partial^2 \psi}{\partial \hat{x}^2} + \frac{\partial^2 \psi}{\partial \hat{y}^2} = -K(\hat{x}, \hat{y})\frac{\partial \theta}{\partial \hat{x}}, \tag{11}$$

$$\frac{\partial \theta}{\partial \tau} + \mathrm{Ra}\left[\frac{\partial \psi}{\partial \hat{y}}\frac{\partial \theta}{\partial \hat{x}} - \frac{\partial \psi}{\partial \hat{x}}\frac{\partial \theta}{\partial \hat{y}}\right] = k(\hat{x}, \hat{y})\left[\frac{\partial^2 \theta}{\partial \hat{x}^2} + \frac{\partial^2 \theta}{\partial \hat{y}^2}\right]. \tag{12}$$

We introduce the aspect ratio A defined by

$$A = H/L, \tag{13}$$

and rescale the spatial coordinates by the transformation

$$x = A\hat{x}, \quad y = \hat{y}, \tag{14}$$

and write $K(x, y) = \hat{K}(\hat{x}, \hat{y})$ and $k(x, y) = \hat{k}(\hat{x}, \hat{y})$. This is done so that in the new coordinates the domain of interest is a square, and this simplifies the subsequent algebra.

We then have

$$A^2\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -K(x, y)A\frac{\partial \theta}{\partial x}, \tag{15}$$

$$\frac{\partial \theta}{\partial \tau} + \mathrm{Ra}A\left[\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y}\right] = k(x, y)\left[A^2\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right]. \tag{16}$$

The conduction solution is given by

$$\psi = 0, \quad \theta = 1 - y. \tag{17a,b}$$

The perturbed solution is given by

$$\psi = \varepsilon\psi', \quad \theta = 1 - y + \varepsilon\theta'. \tag{18a,b}$$

To first order in the small constant ε, we get

$$A^2 \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + K(x, y) A \frac{\partial \theta'}{\partial x} = 0, \tag{19}$$

$$\frac{\partial \theta'}{\partial \tau} + \mathrm{Ra} A \frac{\partial \psi'}{\partial x} - k(x, y) \left[A^2 \frac{\partial^2 \theta'}{\partial x^2} + \frac{\partial^2 \theta'}{\partial y^2} \right] = 0. \tag{20}$$

Since there is a single agent causing the instability, for the onset of convection we can invoke the "principal of exchange of stabilities" and hence take the time derivative in Eq. (16) to be zero.

The boundary conditions are

$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial y = 0 \text{ on } y = 0, \tag{21a,b}$$
$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial y = 0 \text{ on } y = 1, \tag{22a,b}$$
$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial x = 0 \text{ on } x = 0, \tag{23a,b}$$
$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial x = 0 \text{ on } x = 1. \tag{24a,b}$$

We now drop the primes and let

$$\psi_T = A_{11}\psi_{11} + A_{12}\psi_{12} + A_{21}\psi_{21} + A_{22}\psi_{22}, \tag{25}$$
$$\theta_T = B_{11}\theta_{11} + B_{12}\theta_{12} + B_{21}\theta_{21} + B_{22}\theta_{22}. \tag{26}$$

We take as trial functions (satisfying the boundary conditions and having appropriate symmetry with respect to the midlines of the square, and hence satisfying desirable orthogonality properties) for the set

$$\begin{aligned} \psi_{11} &= (x - x^2)(y - y^2), \\ \psi_{12} &= (x - x^2)(y - 3y^2 + 2y^3), \\ \psi_{21} &= (x - 3x^2 + 2x^3)(y - y^2), \\ \psi_{22} &= (x - 3x^2 + 2x^3)(y - 3y^2 + 2y^3). \end{aligned} \tag{27}$$

$$\begin{aligned} \theta_{11} &= 1 - 6x^2 + 4x^3, \\ \theta_{12} &= (1 - 6x^2 + 4x^3)(1 - 6y^2 + 4y^3), \\ \theta_{21} &= x^2 - 2x^3 + x^4, \\ \theta_{22} &= (x^2 - 2x^3 + x^4)(1 - 6y^2 + 4y^3). \end{aligned} \tag{28}$$

(Note that $\theta' = 1$ satisfies the boundary conditions (21b)–(24b) but we have rejected this as a trial function because it does not contribute to the residual. If one tries to use unity as a trial function one faces a mathematical singularity. It is plausible that one can eliminate consideration of this type of perturbation on physical grounds because a uniform jump in temperature leads to a zero net buoyancy effect in the domain occupied by fluid and so does not produce a convective circulation.)

In the Galerkin method, the expression (25) is substituted into the left-hand side of Eq. (19) and the resulting residual is made orthogonal to the separate trial functions

$\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}$ in turn. Likewise the residual on the substitution of the expression (26) into Eq. (20) is made orthogonal to $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$ in turn. We use the notation

$$\langle f(x,y)\rangle = \int_0^1 \int_0^1 f(x,y)\mathrm{d}x\,\mathrm{d}y, \tag{29}$$

We write

$$\begin{aligned} &L_1 \equiv A^2 \tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}, \quad L_2 = K(x,y)A\tfrac{\partial}{\partial x}, \\ &L_3 = \mathrm{Ra}A\tfrac{\partial}{\partial x}, \quad L_4 = -k(x,y)\left[A^2 \tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right], \end{aligned} \tag{30a,b,c,d}$$

We define the residuals

$$\begin{aligned} R_1 = L_1\psi_T + L_2\theta_T &= A_{11}L_1\psi_{11} + A_{12}L_1\psi_{12} + A_{21}L_1\psi_{21} + A_{12}L_1\psi_{22} \\ &\quad + B_{11}L_2\theta_{11} + B_{12}L_2\theta_{12} + B_{21}L_2\theta_{21} + B_{22}L_2\theta_{22}, \\ R_2 = L_3\psi_T + L_4\theta_T &= A_{11}L_3\psi_{11} + A_{12}L_3\psi_{12} + A_{21}L_3\psi_{21} + A_{22}L_3\psi_{22} \\ &\quad + B_{11}L_4\theta_{11} + B_{12}L_4\theta_{12} + B_{21}L_4\theta_{21} + B_{22}L_4\theta_{22}. \end{aligned} \tag{31a,b}$$

We set $\langle R_1\psi_{mn}\rangle = 0, \langle R_2\theta_{mn}\rangle = 0$ for $m,n = 1,2$.

We note that $\langle k(x,y)\rangle = 1$ and $\langle K(x,y)\rangle = 1$.

The output of the Galerkin procedure is a set of 8 homogeneous linear equations in the 8 unknown constants $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22}$. Eliminating these constants, we get

$$\det \mathbf{M} = 0, \tag{32}$$

where the matrix $\mathbf{M}$ takes the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \tag{33}$$

where

$$\mathbf{M}_{11} = \begin{bmatrix} \langle\psi_{11}L_1\psi_{11}\rangle & \langle\psi_{11}L_1\psi_{12}\rangle & \langle\psi_{11}L_1\psi_{21}\rangle & \langle\psi_{11}L_1\psi_{22}\rangle \\ \langle\psi_{12}L_1\psi_{11}\rangle & \langle\psi_{12}L_1\psi_{12}\rangle & \langle\psi_{12}L_1\psi_{21}\rangle & \langle\psi_{12}L_1\psi_{22}\rangle \\ \langle\psi_{21}L_1\psi_{11}\rangle & \langle\psi_{21}L_1\psi_{12}\rangle & \langle\psi_{21}L_1\psi_{21}\rangle & \langle\psi_{21}L_1\psi_{22}\rangle \\ \langle\psi_{22}L_1\psi_{11}\rangle & \langle\psi_{22}L_1\psi_{12}\rangle & \langle\psi_{22}L_1\psi_{21}\rangle & \langle\psi_{22}L_1\psi_{22}\rangle \end{bmatrix}, \tag{34}$$

$$\mathbf{M}_{12} = \begin{bmatrix} \langle\psi_{11}L_2\theta_{11}\rangle & \langle\psi_{11}L_2\theta_{12}\rangle & \langle\psi_{11}L_2\theta_{21}\rangle & \langle\psi_{11}L_2\theta_{22}\rangle \\ \langle\psi_{12}L_2\theta_{11}\rangle & \langle\psi_{12}L_2\theta_{12}\rangle & \langle\psi_{12}L_2\theta_{21}\rangle & \langle\psi_{12}L_2\theta_{22}\rangle \\ \langle\psi_{21}L_2\theta_{11}\rangle & \langle\psi_{21}L_2\theta_{12}\rangle & \langle\psi_{21}L_2\theta_{21}\rangle & \langle\psi_{21}L_2\theta_{22}\rangle \\ \langle\psi_{22}L_2\theta_{11}\rangle & \langle\psi_{22}L_2\theta_{12}\rangle & \langle\psi_{22}L_2\theta_{21}\rangle & \langle\psi_{22}L_2\theta_{22}\rangle \end{bmatrix}, \tag{35}$$

$$\mathbf{M}_{21} = \begin{bmatrix} \langle\theta_{11}L_3\psi_{11}\rangle & \langle\theta_{11}L_3\psi_{12}\rangle & \langle\theta_{11}L_3\psi_{21}\rangle & \langle\theta_{11}L_3\psi_{22}\rangle \\ \langle\theta_{12}L_3\psi_{11}\rangle & \langle\theta_{12}L_3\psi_{12}\rangle & \langle\theta_{12}L_3\psi_{21}\rangle & \langle\theta_{12}L_3\psi_{22}\rangle \\ \langle\theta_{21}L_3\psi_{11}\rangle & \langle\theta_{21}L_3\psi_{12}\rangle & \langle\theta_{21}L_3\psi_{21}\rangle & \langle\theta_{21}L_3\psi_{22}\rangle \\ \langle\theta_{22}L_3\psi_{11}\rangle & \langle\theta_{22}L_3\psi_{12}\rangle & \langle\theta_{22}L_3\psi_{21}\rangle & \langle\theta_{22}L_3\psi_{22}\rangle \end{bmatrix}, \tag{36}$$

$$\mathbf{M}_{22} = \begin{bmatrix} \langle\theta_{11}L_4\theta_{11}\rangle & \langle\theta_{11}L_4\theta_{12}\rangle & \langle\theta_{11}L_4\theta_{21}\rangle & \langle\theta_{11}L_4\theta_{22}\rangle \\ \langle\theta_{12}L_4\theta_{11}\rangle & \langle\theta_{12}L_4\theta_{12}\rangle & \langle\theta_{12}L_4\theta_{21}\rangle & \langle\theta_{12}L_4\theta_{22}\rangle \\ \langle\theta_{21}L_4\theta_{11}\rangle & \langle\theta_{21}L_4\theta_{12}\rangle & \langle\theta_{21}L_4\theta_{21}\rangle & \langle\theta_{21}L_4\theta_{22}\rangle \\ \langle\theta_{22}L_4\theta_{11}\rangle & \langle\theta_{22}L_4\theta_{12}\rangle & \langle\theta_{22}L_4\theta_{21}\rangle & \langle\theta_{22}L_4\theta_{22}\rangle \end{bmatrix}. \tag{37}$$

In the general case, the integrals in Eqs. (34) – (37) can be obtained by quadrature. The eigenvalue equation, Eq. (32) can then be solved to give the critical Rayleigh number.

3 Results and discussion

3.1 First order results

As a test case, we consider a quartered rectangle with a piecewise constant distribution of property values. We consider the case (a quartered square in the scaled domain) with

$$\begin{aligned}
&K(x,y) = 1-\delta_H-\delta_V, \quad && k(x,y) = 1-\varepsilon_H-\varepsilon_V, \quad && \text{for } 0<x<1/2,\ 0<y<1/2;\\
&K(x,y) = 1+\delta_H-\delta_V, && k(x,y) = 1+\varepsilon_H-\varepsilon_V, && \text{for } 1/2<x<1,\ 0<y<1/2;\\
&K(x,y) = 1-\delta_H+\delta_V, && k(x,y) = 1-\varepsilon_H+\varepsilon_V, && \text{for } 0<x<1/2,\ 1/2<y<1;\\
&K(x,y) = 1+\delta_H+\delta_V, && k(x,y) = 1+\varepsilon_H+\varepsilon_V, && \text{for } 1/2<x<1,\ 1/2<y<1.
\end{aligned} \tag{38a,b,c,d}$$

This case approximates a general case in which each slowly varying quantity is approximated by a piecewise-constant distribution. The mean value of the quantity is approximated by its value at center of the main square:

$$\bar{f} = f(0.5, 0.5).$$

In each quarter, the function is approximated by its value at the center of that quarter, and a truncated Taylor series expansion is used to approximate this factor. For example, in the region $1/2 < x < 1, 1/2 < y < 1, f(x,y)$ is approximated by $f(0.75,0.75)$ and then by $f(0.5,0.5) + 0.25f_x(0.5,0.5) + 0.25f_y(0.5,0.5)$.

Thus,

$$\begin{aligned}
\delta_H &= \frac{1}{4}\left[\frac{1}{K}\frac{\partial K}{\partial x}\right]_{(1/2,1/2)}, \quad & \delta_V &= \frac{1}{4}\left[\frac{1}{K}\frac{\partial K}{\partial y}\right]_{(1/2,1/2)},\\
\varepsilon_H &= \frac{1}{4}\left[\frac{1}{k}\frac{\partial k}{\partial x}\right]_{(1/2,1/2)}, & \varepsilon_V &= \frac{1}{4}\left[\frac{1}{k}\frac{\partial k}{\partial y}\right]_{(1/2,1/2)}.
\end{aligned} \tag{39}$$

A criterion for weak heterogeneity is that the magnitude of each of these four quantities is less than unity.

The order-one Galerkin method (using a single trial function for each of ψ and θ) yields the eigenvalue equation

$$\det\begin{bmatrix} \langle\psi_{11}L_1\psi_{11}\rangle & \langle\psi_{11}L_2\theta_{11}\rangle \\ \langle\theta_{11}L_3\psi_{11}\rangle & \langle\theta_{11}L_4\theta_{11}\rangle \end{bmatrix} = \det\begin{bmatrix} \frac{-(1+A^2)}{90} & -\frac{A}{15} \\ \frac{A\mathrm{Ra}}{15} & \frac{24A^2}{5} \end{bmatrix} = 0, \tag{40}$$

which gives

$$\mathrm{Ra} = 12(1+A^2). \tag{41}$$

From Eq. (41), we can conclude that to the present order of approximation the value of the critical Rayleigh number is not affected by any heterogeneity provided that Ra is defined in terms of mean properties. We also conclude that as the height-to-width aspect number A tends to zero, Ra tends to 12 from above. The critical Rayleigh number value 12 was shown by Nield (1968b) to be the exact value for the homogeneous problem with an infinite layer. (A perturbation analysis for small wavenumber shows that it arises as 4!/2!)

3.2 Second order results

At the second-order Galerkin approximation and for the quartered square test case, the eigenvalue equation takes the form, after some elementary row and column manipulations (multiplication of rows 1,2,3,4,7,8 and columns 7,8 by −1),

$$\det\begin{bmatrix} \frac{(1+A^2)}{90} & 0 & 0 & 0 & \frac{A}{15} & \frac{-A\delta_V}{30} & \frac{-A\delta_H}{288} & 0 \\ 0 & \frac{(21+5A^2)}{3150} & 0 & 0 & \frac{-A\delta_V}{40} & \frac{3A}{175} & 0 & \frac{-A\delta_H}{1120} \\ 0 & 0 & \frac{(5+21A^2)}{3150} & 0 & \frac{-A\delta_H}{48} & 0 & \frac{A}{630} & \frac{-A\delta_V}{1260} \\ 0 & 0 & 0 & \frac{(1+A^2)}{1050} & 0 & \frac{-3A\delta_H}{560} & \frac{-A\delta_V}{1680} & \frac{A}{2450} \\ \frac{A\mathrm{Ra}}{15} & 0 & 0 & 0 & \frac{24A^2}{5} & \frac{-(102+105A^2)\varepsilon_V}{35} & \frac{-A^2\varepsilon_H}{4} & 0 \\ 0 & \frac{3A\mathrm{Ra}}{175} & 0 & 0 & -3A^2\varepsilon_V & \frac{408(1+A^2)}{175} & 0 & \frac{(77-136A^2)\varepsilon_H}{1120} \\ 0 & 0 & \frac{A\mathrm{Ra}}{630} & 0 & \frac{-A^2\varepsilon_H}{8} & 0 & \frac{2A^2}{105} & \frac{-(4+5A^2)\varepsilon_V}{420} \\ 0 & 0 & 0 & \frac{A\mathrm{Ra}}{2450} & 0 & \frac{(77+68A^2)\varepsilon_H}{1120} & \frac{-A^2\varepsilon_V}{420} & \frac{28+34A^2}{3675} \end{bmatrix} = 0 \tag{42}$$

This expands to give a quartic equation in Ra, and the smallest root (eigenvalue) is sought. As we have noted, for the homogeneous case this smallest root is $\mathrm{Ra} = 12(1 + A^2)$. For the case where $\delta_H, \delta_V, \varepsilon_H, \varepsilon_V$ are all small compared with unity, we anticipate that

$$\mathrm{Ra} = 12(1 + A^2)(1 + S), \tag{43}$$

where S is small compared with unity, and one might expect that one could simply substitute the expression (43) into (42), linearize with respect to S, and solve for S.

However, on examination it is found that in the limit as A tends to zero the two smallest eigenvalues of Eq. (42) coalesce.

It appears that the coalescence is a result of the fact that the eigenvector corresponding to the pair (ψ_{21}, θ_{21}) and that corresponding to the pair (ψ_{11}, θ_{11}) differ by functions of x only and the terms involving the x- derivatives in the operators L_1, L_2, L_3, L_4 vanish in the limit as A tends to zero. In the limit, the quartic equation for Ra factorizes into linear factors, each of which gives Ra as a fraction with the product of two double integrals in the numerator and two in the denominator. In two of the fractions, the double integrals factorize into x-integrals that cancel and y-integrals that are the same in the two fractions.

As a result of this coalescence of the eigenvalues, there is a singularity in the limit as A tends to zero. One can get around the singularity by substituting $\mathrm{Ra} = 12(1 + A^2)(1 + S)$ in M(5,1) and M(7,3) and $\mathrm{Ra} = 12(1 + A^2)$ in M(6,2) and M(8,4), and then expanding the determinant as a quadratic in S, solving the quadratic equation using the usual formula and taking the mean of the two roots. The reason for the last step is that the individual roots each contain a term proportion to $1/A^2$ but this singular term cancels when the sum is taken. In this way, one obtains the expression

$$S = \frac{1}{7168}\left[1225(4\delta_H^2 - \delta_H\varepsilon_H) + 160(9\delta_V\varepsilon_V - 35\varepsilon_V^2)\right]. \tag{44}$$

This leads to the critical value

$$\begin{aligned}\mathrm{Ra} &= 12(1+A^2)\left\{1+\frac{1}{7168}\left[1225(4\delta_H^2-\delta_H\varepsilon_H)+160(9\delta_V\varepsilon_V-35\varepsilon_V^2)\right]\right\}.\\ &\approx 12(1+A^2)\left\{1+0.684\delta_H^2-0.171\delta_H\varepsilon_H+0.201\delta_V\varepsilon_V-0.781\varepsilon_V^2\right\}.\end{aligned} \quad (45)$$

A number of conclusions can be drawn. The effects of weak horizontal heterogeneity and vertical heterogeneity are each of second order in the property deviations. Their combined contribution is of the order of the variances of the distributions for permeability and conductivity (which are here equal to $\delta_H^2+\delta_V^2$ and $\varepsilon_H^2+\varepsilon_V^2$, respectively.) The vertical heterogeneity and horizontal heterogeneity act independently at this order of approximation. (Product terms like $\delta_H\delta_V$ are absent in the last expression.) The effects of the horizontal and vertical contributions are immediately comparable if one uses the total amount of variation across the box as a the measure of heterogeneity. If one uses the rate of variation with distance as the criterion, one has to consider the fact that the x- and y-coordinates have been differently scaled, by a factor A. For example, in terms of quantities evaluated at the center of the box,

$$\frac{\delta_V}{\delta_H}=\frac{\partial K/\partial y}{\partial K/\partial x}=\frac{H}{L}\frac{\partial K^*/\partial y^*}{\partial K^*/\partial x^*}=A\frac{\partial K^*/\partial y^*}{\partial K^*/\partial x^*}. \quad (46)$$

Thus, if A is small then the horizontal heterogeneity has a greater impact than the vertical permeability, other things being equal.

The heterogeneities of permeability and conductivity interact with each other even when treated as separate parameters. (In a practical situation, of course, a heterogeneity of porosity leads to a heterogeneity of both permeability and conductivity.) The sensitivities of the critical Rayleigh number to conductivity heterogeneity and permeability heterogeneity are much the same.)

The above conclusions were also obtained by Nield and Kuznetsov (2006) for the conducting boundaries problem.

The following conclusions are specific to the insulating boundaries problem. In the absence of conductivity heterogeneity, the effect of permeability heterogeneity is stabilizing (to increase the critical Rayleigh number) and comes from the horizontal variation only. In the absence of permeability heterogeneity, the effect of conductivity permeability is destabilizing and comes from the vertical variation only. The combination of conductivity heterogeneity and permeability heterogeneity can be either stabilizing of destabilizing, whether due to horizontal variation or vertical variation or both.

The move from the first-order Galerkin approximation to a second order one was an important step because it broke the symmetry, with respect to the vertical and horizontal midlines of the enclosure, of the overall trial functions. We would not expect a move from a second-order approximation to a third-order one to introduce anything dramatically new. Rather, we would expect it to lead to a refinement of the values of the various coefficients that appear in Eq. (45).

4 Conclusions

We have continued a linear stability study begun by Nield and Kuznetsov (2006) of the relationships between the effects of horizontal and vertical heterogeneities on the onset of convection in a porous medium. In the previous study, we considered conducting boundaries. In that case the favored convection pattern is in rolls of square cross-section, and it was not clear to what extent our results would generalize to the case where the convection pattern was substantially different. In the present study, we have treated the contrasting case of insulating boundaries, a case for which the critical wavenumber is small for an infinite layer. This means that the convection pattern is in then in the form of very wide cells.

In the previous study, we employed an approximate analysis to reach some general conclusions of the case of weak heterogeneity. It was shown that a Rayleigh number based on mean properties is a good basis for the prediction of the onset of instability. It was shown that piecewise-constant variation leads to effects that enter at second order in small variations. It was also shown that the effects of horizontal heterogeneity and vertical heterogeneity were comparable once account of the aspect ratio was taken into account, and to a first approximation are independent.

Our basic conclusion from the present study is that the change of convection pattern does not greatly change the effect of the heterogeneity, once account has been made of the change in aspect ratio. Some minor differences showed up. For the case of conducting boundaries, we found that the heterogeneities lead to a reduction in the critical value of Ra for all combinations of horizontal and vertical heterogeneities and all combinations of permeability and conductivity heterogeneities. In the present case, that of constant flux boundaries, the situation is more complicated. Now the combination of vertical heterogeneity and horizontal heterogeneity can be either stabilizing or destabilizing, and there are differences between the effects of permeability heterogeneity and conductivity heterogeneity.

The cases of moderate or strong heterogeneity remain as challenges for future work. We believe that it is likely that moderate heterogeneity can be treated by numerical methods along roughly the same lines as the present work. Strong heterogeneity may require a more radical treatment.

References

Hurle, D.T.J., Jakeman, E., Pike, E.R.: On solution of Bénard problem with boundaries of finite conductivity. Proc. Roy. Soc. Lond. A **296**, 469–475 (1967)

Nield, D.A.: The thermohaline Rayleigh-Jeffreys problem. J. Fluid. Mech. **29**, 545–558 (1968a)

Nield, D.A.: Onset of thermohaline convection in a porous medium. Water Resour. Res. **11**, 553–560 (1968b)

Nield, D.A.: The onset of transient convective instability. J. Fluid Mech. **71**, 441–454 (1975)

Nield, D.A.: Onset of convection in a fluid layer overlying a layer of a porous medium. J. Fluid Mech. **81**, 513–522 (1977)

Nield, D.A.: Convective instability in porous media with throughflow. AIChE J. **33**, 1222–1224 (1987)

Nield, D.A., Bejan, A.: Convection in Porous Media, 3rd edn. Springer, New York (2006)

Nield, D.A., Kuznetsov, A.V.: The effects of combined horizontal and vertical heterogeneity on the onset of convection in a porous medium. Int. J. Heat Mass Transfer submitted (2006)

Nield, D.A., Simmons, C.T.: A discussion on the effect of heterogeneity on the onset of convection in a porous medium. Transp. Porous Med. submitted (2006)

Prasad, A., Simmons, C.T.: Unstable density-driven flow in heterogeneous porous media: a stochastic study of the Elder [1967b] "short heater" problem. Water Resour. Res. **39**(1), #1007 (2003)

Simmons, C.T., Fenstemaker, T.R., Sharp, J.M.: Variable-density flow and solute transport in heterogeneous porous media: approaches, resolutions and future challenges. J. Contam. Hydrol. **52**, 245–275 (2001)

Sparrow, E.M., Goldstein, R.J., Jonsson, V.K.: Thermal instability in a horizontal fluid layer: effect of boundary conditions and non-linear temperature profile. J. Fluid Mech. **18**, 513–528 (1964)