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RESEARCH ARTICLE

A stochastic model for filtration of particulate suspensions with incomplete pore plugging

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Abstract A population balance model for particulate suspension transport with capture of particles by porous medium accounting for complete and incomplete plugging of pores by retained particles is derived. The model accounts for pore space accessibility, due to restriction on finite size particle movement through the overall pore space, and for particle flux reduction, due to transport of particles by the fraction of the overall flux. The novel feature of the model is the residual pore conductivity after the particle retention in the pore and the possibility of one pore to capture several particles. A closed system of governing stochastic equations determines the evolution of size distributions for suspended particles and pores. Its averaging results in the closed system of hydrodynamic equations accounting for permeability and porosity reduction due to plugging. The problem of deep bed filtration of a single particle size suspension through a single pore size medium where a pore can be completely plugged by two particles allows for an exact analytical solution. The phenomenological deep bed filtration model follows from the analytical solution.

Keywords Deep bed filtration · Incomplete plugging · Suspension · Size distribution · Accessibility · Stochastic model · Averaging

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Nomenclature

a	constant in the solution of the pore size evolution equations in the discrete case
A	shape factor in the Kozeny–Carman model
b	constant in the solution of the pore size evolution equations in the discrete case
C	particle concentration distribution by sizes, L^{-4}
c	total suspended particle concentration, L^{-3}
D	decrease term in the pore size kinetics equation
D_c	concentration front velocity, LT^{-1}
f	distribution density, L^{-1}
$g(\sigma)$	dependence between the pore concentration and the concentration of the capturing particles
H	distribution of pore concentration, L^{-3} or L^{-4}
h	pore concentration, L^{-2} or L^{-3}
I	increase term in the pore size kinetics equation
$j(r_\mathrm{s}, r_\mathrm{p})$	correction to the particle velocity, dimensionless
k	permeability, L^2
k_1	permeability of a single pore, L^4
l	characteristic length, L
n	amount of particles of a given size per unit volume, L^{-4}
N	number of particles to be captured by one pore
P	pressure
$p(r_\mathrm{s}, r_\mathrm{p})$	capture probability, dimensionless
$p(r_\mathrm{s}, r_\mathrm{p} \to r'_\mathrm{p})$	distribution of the capture probability by r'_p, L^{-1}
$\bar{p}$	average entrapment factor, dimensionless
Q	total flow rate of the particles of a given size, $\mathrm{L}^{-3}\mathrm{T}^{-1}$
r	size of a particle or of a pore, L
s_1	pore cross-section, L^2
S_0	specific surface
t	time, T
T	characteristic time of the concentration relaxation, T
U	total velocity of the flux, LT^{-1}
q	flow rate through a single capillary, $\mathrm{L}^3\mathrm{T}^{-1}$
x	coordinate, L

Greek letters

α	average flux reduction factor, dimensionless
β_k	dimensionless constant in the permeability equation
β_p	dimensionless constant in the capturing factor equation
β_α	dimensionless constant in the entrapment factor equation
β_Γ	dimensionless constant in the porosity reduction factor equation
β_ϕ	dimensionless constant in the porosity equation
$\beta(\sigma)$	flux reduction coefficient
Γ	average porosity reduction factor, dimensionless
$\gamma(\sigma)$	average porosity reduction factor with regard to the initial porosity
$\phi(x, t)$	porosity, dimensionless
$\phi(r_\mathrm{s}, x, t)$	accessible porosity for a particle of the size r_s, dimensionless

$\chi(r_s, r_p)$	porosity correction, dimensionless
$\lambda(\sigma)$	filtration coefficient
μ	dynamic viscosity, $ML^{-1}T^{-1}$
σ	volumetric concentration of the captured particles, L^{-3}
Σ	size distribution of the concentration of the captured particles, L^{-4}
τ	tortuosity

Subscripts

s	suspended particle
p	pore
V	volumetric
0	initial condition

Superscripts

0	boundary condition

1 Introduction

Flow of particulate suspensions in porous media is usually accompanied by particle capture in the pores, causing permeability damage. This phenomenon is called deep bed filtration. Its understanding is essential for industrial and environmental technologies like water flooding of oil reservoirs, water filtration, wastewater treatment, separation processes in chemical engineering, and others.

In such processes as the produced water disposal in aquifers with offshore oil production, oil leaking from pipelines, propagation of viruses and bacteria in underground reservoirs, a number of the environmental processes, the most important parameter is the depth of the particle propagation until its capture. Another parameter, the permeability damage, is important to such processes as poor quality water injection/re-injection in oil reservoirs or invasion of drilling fluid into oil-bearing formations. For other processes of chemical and biochemical engineering more detailed characteristics, like particle size distributions after filtering, may become important. A model of the deep bed filtration, involving particle and pore size distributions and complex mechanisms of their interaction, is developed in the present paper.

The standard core-scale deep bed filtration model operates with averaged concentrations for suspended and deposited particles (Iwasaki 1937; Herzig et al. 1970). The model is widely used in industrial filtration (Payatakes et al. 1973, 1974; Khilar and Fogler 1998), in poor quality water injection into oil reservoirs (Pang and Sharma 1994; Wennberg and Sharma 1997; Logan 2000) and in environmental studies (Elimelech et al. 1995; Khilar and Fogler 1998). Several analytical solutions for direct prediction problems (Herzig et al. 1970) and for inverse data treatment problems (Wennberg and Sharma 1997; Bedrikovestky et al. 2001, 2003) were obtained. The model exhibits a good agreement with laboratory data and can be used for prediction purposes, like forecasts of well injectivity decline based on laboratory coreflood data.

Different physical mechanisms (electrical forces, size exclusion, diffusion, bridging, etc.) are responsible for the capture of suspended particles during fluid injection. However, the standard model does not distinguish between different mechanisms of particle capture. Moreover, the model operates with the phenomenological filtration coefficient and does not include real physical parameters like pore and particle sizes, ionic strength and electrical charges. Therefore, it cannot be used for diagnostic predictive purposes based on the physical parameters, for example, for forecast of the well injectivity decline based on the pore and particle size data.

The standard model predicts that the particle breakthrough occurs after injection of one pore volume. However, several cases where the breakthrough moment differs significantly from the moment of one pore volume injected have been reported in the literature for particulate and polymer suspensions (Dawson and Lantz 1972; Higgo et al. 1993; Kretzschmar et al. 1995; Bartelds et al. 1997; Veerapen et al. 2001; Massei et al. 2002).

If the particles are captured according to the size exclusion mechanism, the larger the particles and the smaller the pores are, the more intensive the capture is, resulting in major formation damage (Sharma and Yortsos 1987a, b, c). Proportional increase of the pore size and the particle sizes would not affect the capture rate, since it does not affect the mechanism of the size exclusion. Therefore, the filtration coefficient in the size exclusion case is a monotonically increasing function of the particle-pore size ratio. Nevertheless, several attempts to correlate the formation damage with sizes of particles and pores were unsuccessful (Oort et al. 1993; Bedrikovetsky et al. 2003). It means that either the size exclusion mechanism does not dominate in the discussed laboratory experiments, or the phenomenological model for average concentrations does not describe adequately the size exclusion filtration. One way around the problem is microscale modeling of each capture mechanism.

The intensity of capture due to the size exclusion mechanism is determined by the concentration of particles that are larger than the pores. Therefore, it is essential to consider the particle and the pore size distributions in modeling the size exclusion filtration. The stochastic approach involving size distributions was adopted to model the deep bed filtration in pore networks (Rege and Fogler 1987, 1988; Imdakm and Sahimi 1987, 1991; Sahimi et al. 1990; Sahimi and Imdakm 1991).

Phenomenological deep bed filtration models accounting for pore and particle size distributions are also found in the literature. The population balance equations have been derived by Sharma and Yortsos, (1987a, b, c), from the continuity equations for each particle size. The closed system of equations is presented. The model captures changes of pore and particle size distributions due to the size exclusion retention mechanism. Several analytical solutions have been obtained and interpreted physically. The model assumes that particles of any size move at the carrier water velocity and are transported by the overall water flux. The assumption results in deep bed filtration of all the particles, independently of their sizes, while in reality particles are present in the pores that are larger than the particles and are transported by the flux only via larger pores.

A population balance model for filtration with the size exclusion mechanism accounting for particle flux reduction and for pore accessibility due to restriction on large particle flow through small pores was considered by Santos and Bedrikovetsky, (2005). The derived nonlinear system allows for linearization in the case of low suspended particle concentration. An analytical solution for injection of low particle concentration suspension is presented. Averaging of the proposed population balance equations makes it possible to introduce the flux reduction and accessibility factors into the classical deep bed filtration model.

The models by Sharma and Yortsos (1987a, b, c) and Santos and Bedrikovetsky (2005) assume that the flux via pores stops after plugging the pores by the particles. This assumption is only valid for the pores and the particles of regular shapes fitting each other. In reality, the size exclusion plugging does not stop the flow. For example, the injected water flow in a reservoir continues after complete plugging of the inlet reservoir cross-section during the so-called transition time (Khatib 1994; Wennberg and Sharma 1997). The suspended particles form an external filter cake and do not enter porous rock after the transition time, while the water injection continues.

Therefore, an adequate model for deep bed filtration with injectivity decline should include incomplete plugging of the pores by the particles.

Incomplete pore plugging was incorporated into the network models, where sequential entrapment of several particles by a single pore was allowed (Rege and Fogler 1987, 1988; Siqueira et al. 2003).

The derivation of the population balance equations for deep bed filtration accounting for incomplete plugging of pores after particle retention is the main objective of the current paper. We introduce general microcharacteristics of the flow and of the capturing process, appropriate for description of various mechanisms of capturing. On the basis of these characteristics, we derive the closed system of the population balance equations, of the Boltzmann or Einstein–Smolukhovsky type. The capability of this system to describe realistic cases is demonstrated in the two ways. First, the averaged system of hydrodynamic equations for the suspension flow in porous media is derived in additional assumptions. Further, on the basis of the proposed model, the process of deep bed filtration is studied for several particular examples, with uniform particle sizes and a discrete pore size distribution. Equivalence to the (slightly modified) classical model of deep bed filtration is demonstrated for these cases.

The structure of the paper is as follows. In Section 2, the main assumptions of the model are formulated, and the basic model parameters are introduced. The derivation of the population balance equations is given in Section 3. The initial-boundary value problem for suspension injection is of the Goursat type. It makes it possible, in principle, to describe the particle and pore population dynamics in the inlet cross-section without solving the initial-boundary value problem. Section 4 contains averaging of the microscopic equations and derivation of the system of the phenomenological equations, including the differential equations for permeability and porosity reduction due to trapping. Section 5 discusses a special example of one-sized particle suspension flowing in a porous medium with single-sized pores where a pore is completely plugged after several entrapments. Exact solutions for this case are obtained, and a comparison with the extended classical phenomenological model is carried out. The conservation law for the total number of pores and the connection between the capture probability and the filtration coefficient are discussed in appendices.

2 Parameters describing the deep bed filtration

We consider the flow of a homogeneous liquid, containing solid particles, through a porous medium. In the first approximation, we assume that the particles are spherical, and the porous medium is a network of cylindrical (although, may be, tortuous) pores. Alternatively, one could think of the network of capillaries or channels connecting large voids (pores). In the second approach, the capillary sizes are comparable to the particle sizes, while the pores are much larger than the particles. Since both approaches produce similar results, we will restrict our discussion with the first approach. The pores and the particles possess certain size distributions. The sizes (effective radii) of the pores will be denoted by r_{p} and those of the particles by r_{s}.

The described model of the filtration process has been used previously (Sharma and Yortsos 1987a, b; Santos and Bedrikovetsky 2005, 2006). Obviously, it is too restrictive for description of the realistic process of particle capturing. In this process, not only sizes, but also shapes of the particles and pores may be important. Moreover, other parameters may also be influential (like characteristics of the structure of the internal porous surface or of the particle surface). Most of the results obtained in the present paper remain valid if r_s and r_p are not interpreted as sizes, but as other parameters (parameters of particle shapes, electrical charge...) or, even, as sets of such parameters. However, in order to make the discussion more simple and pictorial, in the following we talk about particle and capillary sizes.

One size is not sufficient for characterization of the pore geometry even for a cylindrical capillary, which, apart from the radius r_p, is characterized by length l. If a realistically shaped (that is, rather disordered) pore is modeled by a cylindrical capillary, orientation of the capillary should be chosen in order to identify r_p and l with particular geometrical sizes of the pore. In the present paper discussing one-dimensional filtration flows we assume that there is a specified direction of the flow parallel to axis x. By r_p we mean the pore size across the flow direction, while l is parallel to this direction.

Another assumption, which we introduce from very beginning, is that all the porous space is connected and accessible to flow. As other assumptions introduced above, this assumption is not restrictive and is only introduced for simplicity of the discussion. The problem with the dead-end pores is that, while they do not participate in the flow, they may still catch particles. The model described in this paper may be extended onto such pores.

Let us consider individual characteristics of the particle–pore interactions. The particles of different sizes may flow through the pores or may be captured by them (Fig. 1). Capturing may occur if the particle size is larger than the pore size (the so-called pore size exclusion). Another possibility is capturing a particle on the internal surface of a pore, due to mechanical, electrical or other types of interactions (Elimelech et al. 1995; Veerapen et al. 2001). We introduce the dimensionless coefficient $p\left(r_s, r_p\right)$ expressing the probability that a particle of the size r_s is captured in a pore of the size r_p. Moreover, the value of $p(r_s, r_p \rightarrow r_p')dr_p'$ is determined as a probability of the event that after particle capturing the pore size is changed from r_p to r_p'. The distribution function $p(r_s, r_p \rightarrow r_p')$ possesses the following properties:

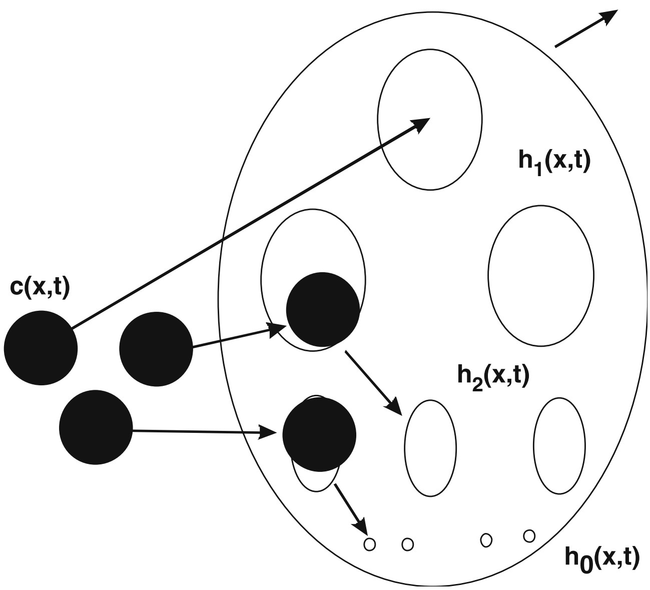

Fig. 1 Dynamics of three pore populations during particle capture

$$
\begin{aligned}
p\left(r_{\mathrm{s}}, r_{\mathrm{p}} \to r_{\mathrm{p}}'\right) &= 0: \quad r_{\mathrm{p}} \le r_{\mathrm{p}}'; \\
\int_0^\infty p\left(r_{\mathrm{s}}, r_{\mathrm{p}} \to r_{\mathrm{p}}'\right) \mathrm{d}r_{\mathrm{p}}' &= p\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right).
\end{aligned} \tag{1}
$$

The first equation means that plugging always leads to decrease of the pore size. From the second equation it follows, in particular, that the dimension of $p(r_{\mathrm{s}}, r_{\mathrm{p}} \to r_{\mathrm{p}}')$ is L^{-1}.

Another parameter determining particle behavior in a pore is the porosity correction factor $\chi(r_{\mathrm{s}}, r_{\mathrm{p}})$: an effective part of the pore volume, which is accessible to a particle. This parameter has to be introduced due to the fact that a particle has a finite size, and its center may not occupy all the possible positions inside a pore. Moreover, there may be the pores, which are totally inaccessible for the particles: $\chi(r_{\mathrm{s}}, r_{\mathrm{p}}) = 0$ if $r_{\mathrm{s}} > r_{\mathrm{p}}$. For a spherical particle in a cylindrical pore, we have $\chi = (1 - r_{\mathrm{s}}/r_{\mathrm{p}})^2$. Generally, if entrapment is only governed by the characteristic particle and pore sizes, it follows from the dimensionality considerations that the value $\chi(r_{\mathrm{s}}, r_{\mathrm{p}})$ does not depend on the particle and the pore sizes separately, but on their ratio $r_{\mathrm{s}}/r_{\mathrm{p}}$. However, in cases where entrapment is governed by other mechanisms, this assumption cannot be applied.

The next characteristic of the particle behavior in a pore is the average correction to the particle velocity, $j(r_{\mathrm{s}}, r_{\mathrm{p}})$. This correction was introduced by Dawson and Lantz (1972), in connection with the experimental observation that the large particles move faster in porous media than the small particles and, even, faster than the average interstitial velocity of the carrying flow (Higgo et al. 1993; Kretzschmar et al. 1995; Bartelds et al. 1997; Veerapen et al. 2001; Massei et al. 2002). This effect was explained by the fact that the liquid velocities inside pores are distributed, being the highest in the pore centers and approaching zero on the pore walls. The centers of the large particles are, on average, closer to the centers of the pores, which leads to higher than average particle velocity in a pore. Moreover, large particles tend to move in larger pores where the liquid velocity is higher. On the other hand, the paths consisting of large pores may be highly tortuous, which may result in slowing down the large particles. A detailed discussion of the tortuosity effect on the particle flow may be carried out in the framework of the percolation theory (Seljakov and Kadet 1996), which is outside the scope of the present work.

For simple geometries of particles and pores, it is reasonable to assume that $j = j(r_{\mathrm{s}}/r_{\mathrm{p}})$, similarly to the case of $\chi(r_{\mathrm{s}}, r_{\mathrm{p}})$. It should be noted that the value of j does not become zero in cases where $r_{\mathrm{s}} > r_{\mathrm{p}}$, that is, where complete plugging takes place and the particle cannot move in the pore. In this case $j(r_{\mathrm{s}}, r_{\mathrm{p}})$ should be understood as the correction to the particle velocity at the pore entrance.

The number of particles is characterized by their concentration $c(x, t)$ per unit of pore volume. We consider a more detailed characteristic, the concentration distribution $C(r_{\mathrm{s}}, x, t)$ of the particles by their sizes r_{s} (Sharma and Yortsos 1987a, b, c). The value of $C(r_{\mathrm{s}}, x, t)\mathrm{d}r_{\mathrm{s}}$ is the concentration of the particles of sizes varying from r_{s} to $r_{\mathrm{s}} + \mathrm{d}r_{\mathrm{s}}$. The following equality takes place:

$$
\int_0^\infty C\left(r_{\mathrm{s}}, x, t\right) \mathrm{d}r_{\mathrm{s}} = c(x, t). \tag{2}
$$

The dimension of $c(x, t)$ is L^{-3} (the number of particles per unit volume). Correspondingly, the dimension of the distribution function $C(r_{\mathrm{s}}, x, t)$ is L^{-4}.

The probabilistic particle size distribution density is

$$f_{\mathrm{s}}\left(r_{\mathrm{s}}, x, t\right)={ }^{C\left(r_{\mathrm{s}}, x, t\right)} / c(x, t) ; \quad \int_0^{\infty} f_{\mathrm{s}}\left(r_{\mathrm{s}}, x, t\right) \mathrm{d} r_{\mathrm{s}}=1 .$$

The liquid flow depends on the number of pores in a unit of the cross-section of a porous medium, which are open for the flow (are not totally plugged), Fig. 1. The number of vacant pores in a unit cross-section (at a certain time moment t at a certain macroscopic point x) will be denoted by $h(x, t)$, and its concentration distribution by pore sizes r_{p} by $H(r_{\mathrm{p}}, x, t)$. The value of $H(r_{\mathrm{p}}, x, t)\mathrm{d}r_{\mathrm{p}}$ is the concentration of pores of sizes varying from r_{p} to $r_{\mathrm{p}}+\mathrm{d}r_{\mathrm{p}}$. The dimension of $H(r_{\mathrm{p}}, x, t)$ is L^{-3} (number of pores divided by cross-section area and by pore size). It is convenient to consider also the number $H(0, x, t)$ of totally closed pores, "of the pores of the zero size." With this addition, we can prove the conservation law for the total number of pores—see Appendix A.

The pore size distribution density is defined as

$$f_{\mathrm{p}}\left(r_{\mathrm{p}}, x, t\right)={ }^{H\left(r_{\mathrm{p}}, x, t\right)} / h(x, t) ; \quad \int_0^{\infty} f_{\mathrm{p}}\left(r_{\mathrm{p}}, x, t\right) \mathrm{d} r_{\mathrm{p}}=1 .$$

Let us introduce the flow rate through a single capillary $q(r_{\mathrm{p}}, x, t)$, $[q]=\mathrm{L}^3\mathrm{T}^{-1}$. Assume that the flow rate may on average be expressed as

$$q=-\frac{k_1\left(r_{\mathrm{p}}\right)}{\mu} \frac{\partial P}{\partial x} . \tag{3}$$

Then the overall flow rate U per unit area (or the superficial velocity), and the permeability k of the porous medium may be expressed in terms of H:

$$U(t)=\int_0^{\infty} q\left(r_{\mathrm{p}}, x, t\right) H\left(r_{\mathrm{p}}, x, t\right) \mathrm{d} r_{\mathrm{p}}, \tag{4}$$

$$k(x, t)=\int_0^{\infty} k_1\left(r_{\mathrm{p}}\right) H\left(r_{\mathrm{p}}, x, t\right) \mathrm{d} r_{\mathrm{p}} . \tag{5}$$

Here μ is the flow viscosity, $k_1(r_{\mathrm{p}})$ is the hydraulic conductance of a given capillary possessing a dimension of L^4. For example, for the Poiseuille flow, we have $k_1(r_{\mathrm{p}})=\pi r_{\mathrm{p}}^4/8$ (Landeau and Lifshitz 1987). This definition is in agreement with the correct dimensions of the overall flow rate and permeability, $[U]=\mathrm{LT}^{-1}$ and $[k]=\mathrm{L}^2$. The flow rate in a single capillary u_1 may vary, since the pressure gradient $\partial P/\partial x$ may change as the local permeability decreases due to particle capturing. The dimensions of the permeabilities for one pore k_1 and for the porous medium k are different, since the conductance of a single pore is determined with regard to the flow rate, while k is with regard to the flow rate per unit area. The fact that the total velocity $U(t)$ is independent of x follows from the incompressibility and one-dimensional character of the flow (Sharma and Yortsos 1987a, b, c). The value of $U(t)$ is supposed to be known from the boundary conditions.

Strictly speaking, Eqs. (3) to (5) are only valid for the systems of parallel capillaries. Equation (3) may alter depending on the complex pore interconnectivity. This interconnectivity does not influence Eqs. (4) and (5), since they are written for a cross-section. However, still, these equations may not be precise, since they assume implicitly that the flows in the capillaries are orthogonal to the cross-section. Additionally, the dead-end pores are not taken into account. Introduction of necessary corrections to these equations may be discussed (Sharma and Yortsos 1987b, c; Seljakov and Kadet 1996; Siqueira et al. 2003), however, it does not affect the results qualitatively. We take the assumptions necessary for obtaining Eqs. (3) to (5) as additional assumptions of the model.

Comparison of Eqs. (3) to (5) and application of the Darcy law

$$U = -\frac{k}{\mu}\frac{\partial P}{\partial x} \tag{6}$$

makes it possible to express the flow rate in a single pore in terms of the overall flow velocity:

$$q(r_{\mathrm{p}}, x, t) = \frac{k_1(r_{\mathrm{p}})}{k(x,t)}U = \frac{k_1(r_{\mathrm{p}})U}{\int_0^\infty k_1\left(r_{\mathrm{p}}\right) H\left(r_{\mathrm{p}}, x, t\right)\mathrm{d}r_{\mathrm{p}}}. \tag{7}$$

The porosity of the medium and the effective porosity available for a particle of the size r_{s} (accessible pore volume) may be found as

$$\phi(x,t) = \int_0^\infty s_1\left(r_{\mathrm{p}}\right) H\left(r_{\mathrm{p}}, x, t\right)\mathrm{d}r_{\mathrm{p}}; \tag{8}$$

$$\phi(r_{\mathrm{s}}, x, t) = \int_0^\infty \chi\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right) s_1\left(r_{\mathrm{p}}\right) H\left(r_{\mathrm{p}}, x, t\right)\mathrm{d}r_{\mathrm{p}}. \tag{9}$$

Here $s_1(r_{\mathrm{p}}) \sim r_{\mathrm{p}}^2$ is the effective area of the pore cross-section.

Apart from the number of pores per unit of the cross-section $H(r_{\mathrm{p}}, x, t)$, we will also consider their volumetric distribution $H_{\mathrm{V}}(r_{\mathrm{p}}, x, t)$. The values of $H(r_{\mathrm{p}}, x, t)$ and $H_{\mathrm{V}}(r_{\mathrm{p}}, x, t)$ possess different dimensions: $[H_{\mathrm{V}}(r_{\mathrm{p}}, x, t)] = [H(r_{\mathrm{p}}, x, t)]\mathrm{L}^{-1}$. The relation between these two distributions is non-trivial and depends on the distribution of the pores by lengths. For simplicity we postulate the existence of such a characteristic length l that

$$H_{\mathrm{V}}(r_{\mathrm{p}}, x, t) = H(r_{\mathrm{p}}, x, t)/l. \tag{10}$$

This assumption may be substantiated if the distributions of pores by sizes and lengths are independent. In this case l is identical with the average value of the introduced above pore length along the flow direction.

The last function to be introduced is the amount of the captured particles per unit volume. It will be denoted by $\sigma(x,t)$ and its distribution by particle sizes by $\Sigma(r_{\mathrm{s}}, x, t)$, correspondingly:

$$\sigma(x,t) = \int_0^\infty \Sigma(r_{\mathrm{s}}, x, t)\mathrm{d}r_{\mathrm{s}}.$$

In the next section we derive the closed system of equations, from which the distributions $C(r_{\mathrm{s}}, x, t)$, $H(r_{\mathrm{s}}, x, t)$ and $\Sigma(r_{\mathrm{s}}, x, t)$ may be found provided that the initial distributions, the flow parameters and the capture probability p are known.

3 The system of kinetic equations

In the derivations of the kinetic equations, we will proceed from an assumption similar to the Boltzmann assumption about "molecular chaos (stosszahlansatz)" (McQuarrie 1976, 18–4; Landeau and Lifshitz 1980). According to this assumption, the particles coming to the pores are being distributed between them independently of relations between the particle and the pore sizes and the pre-history of the pore filling. Before "collision" between the particles and the pores, their behavior is totally uncorrelated. Of course, after a particle meets a pore, the possibility of capturing introduces some correlation between their sizes, which is reflected by our function $p(r_{\mathrm{s}}, r_{\mathrm{p}} \to r'_{\mathrm{p}})$. In a similar way, in the Boltzmann theory the molecular velocities are uncorrelated before collision, while after the collision the correlation may be established.

On the assumption about independence of particle and pore sizes before collision, the number of particles of a given size r_s meeting the capillaries of the size r_p in a unit cross-section ("the number of particle–pore collisions") may be expressed as

$$C(r_s, x, t)\, j\left(r_s, r_p\right) q\left(r_p, x, t\right) H\left(r_p, x, t\right). \tag{11}$$

The assumption about "particle–pore chaos" is oversimplified for some kinds of the suspension flow, since it excludes the experimentally observed correlated behavior of particles in the pores (like "bridging"). It will serve as a first approximation to a more realistic picture, since it makes it possible to obtain a closed system of kinetic equations for the process of filtration. The assumption about the "particle–pore chaos" looks valid, at least, for the cases where the sizes of the particles are much smaller than the pore or, opposite, if the particle and the pore sizes are comparable and the flow of multiple particles in the same pore is improbable.

It should be remarked that in previous works (Sharma and Yortsos 1987a, b, c; Santos and Bedrikovetsky 2005, 2006) the system of kinetic equations was not derived in terms of the capture probability p, but in terms of the filtration coefficient λ. The connection between the two coefficients is discussed in Sects. 4 and 5.

3.1 Pore size kinetics

The variation of the pores of a given size r_p in a cross-section due to particle deposition may be represented as a difference between the "increase" and "decrease" terms. The value of $H\left(r_p, x, t\right)$ increases when a pore of a size larger than r_p captures a particle and acquires the size r_p. The value $H\left(r_p, x, t\right)$ decreases after a pore of the size r_p captures the particle and acquires a smaller size:

$$\frac{\partial H\left(r_p, x, t\right)}{\partial t} = I\left(r_p, x, t\right) - D\left(r_p, x, t\right). \tag{12}$$

In terms of the capture probability $p(r_s, r_p' \to r_p)$ introduced above and by taking into account Eq. (11), the "increase" and the "decrease" terms in Eq. (12) are expressed as

$$\begin{aligned} I\left(r_p, x, t\right) &= \int_{r_p}^{\infty} dr_p' \int_0^{\infty} dr_s \left\{p(r_s, r_p' \to r_p) j(r_s, r_p') q(r_p', x, t) H\left(r_p', x, t\right) C(r_s, x, t)\right\},\\ D\left(r_p, x, t\right) &= \int_0^{\infty} dr_s \int_0^{r_p} dr_p' \left\{p\left(r_s, r_p \to r_p'\right) j(r_s, r_p) q\left(r_p, x, t\right) H\left(r_p, x, t\right) C(r_s, x, t)\right\}. \end{aligned} \tag{13}$$

The last term is simplified by integration over r_p' by taking into account Eq. (1):

$$D\left(r_p, x, t\right) = q\left(r_p, x, t\right) H\left(r_p, x, t\right) \int_0^{\infty} dr_s p\left(r_s, r_p\right) j(r_s, r_p) C\left(r_s, x, t\right).$$

Finally, the equation for $H\left(r_p, x, t\right)$ assumes the form of

$$\begin{aligned} &\frac{\partial H\left(r_p, x, t\right)}{\partial t} \\ &= \int_0^{\infty} dr_s \left[C\left(r_s, x, t\right) \left(\begin{array}{l} \int_{r_p}^{\infty} dr_p' \left\{p\left(r_s, r_p' \to r_p\right) j\left(r_s, r_p'\right) q\left(r_p', x, t\right) H\left(r_p', x, t\right)\right\} - \\ p\left(r_s, r_p\right) j\left(r_s, r_p\right) q\left(r_p, x, t\right) H\left(r_p, x, t\right) \end{array} \right) \right]. \end{aligned} \tag{14}$$

The last equation for the pore size kinetics is not independent of the suspended ensemble dynamics, since it contains the particle concentration distribution $C\,(r_{\mathrm{s}}, x, t)$. The equation for $C\,(r_{\mathrm{s}}, x, t)$ may be obtained considering the particle flow, as derived in the next subsection.

3.2 Particle flow kinetics

The conservation law (the continuity equation) for the number of particles of a given size $n\,(r_{\mathrm{s}}, x, t)$ is obtained by equating the time variation of the number of particles in a unit volume to the influx $Q\,(r_{\mathrm{s}}, x, t)$ of the particles to the volume (we assume that no aggregation/splitting of the particles takes place). By performing ordinary transition from the integral to the differential equations in the one-dimensional case, the continuity equation is expressed in the common divergent form:

$$\frac{\partial n\,(r_{\mathrm{s}}, x, t)}{\partial t} + \frac{\partial Q\,(r_{\mathrm{s}}, x, t)}{\partial x} = 0. \tag{15}$$

The amount of particles in the unit volume $n\,(r_{\mathrm{s}}, x, t)$ consists of the free particles $\phi(r_{\mathrm{s}}, x, t)C\,(r_{\mathrm{s}}, x, t)$ (cf. Eq. (9)) and the particles captured in the unit volume $\Sigma\,(r_{\mathrm{s}}, x, t)$:

$$n\,(r_{\mathrm{s}}, x, t) = \phi(r_{\mathrm{s}}, x, t)C\,(r_{\mathrm{s}}, x, t) + \Sigma\,(r_{\mathrm{s}}, x, t)\,.$$

The flow of the particles $Q\,(r_{\mathrm{s}}, x, t)$ may be expressed as the total flow of the particles through all the pores at a unit surface. The particles of the size r_{s} may only flow through the part $\chi\,(r_{\mathrm{s}}, r_{\mathrm{p}})$ of the pores. Applying the velocity correction factor j, we express this flow as

$$Q = C\big(r_{\mathrm{s}}, x, t\big) \int_0^{\infty} \chi\big(r_{\mathrm{s}}, r_{\mathrm{p}}\big) j\big(r_{\mathrm{s}}, r_{\mathrm{p}}\big) q\,\big(r_{\mathrm{p}}, x, t\big)\, H\,\big(r_{\mathrm{p}}, x, t\big)\, \mathrm{d}r_{\mathrm{p}}.$$

This expression for Q differs from Eq. (11) by factor χ, which has to be introduced into the last equation, since we do not consider the particles "colliding" with the capillaries, but the particles flowing through them.

Substituting the last two equations into Eq. (15), we reduce it to the form of

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\phi\,(r_{\mathrm{s}}, x, t)\, C\,(r_{\mathrm{s}}, x, t)\right] \\
&\quad + \frac{\partial}{\partial x}\left[C\,(r_{\mathrm{s}}, x, t) \int_0^{\infty} \chi\,(r_{\mathrm{s}}, r_{\mathrm{p}}) j\,(r_{\mathrm{s}}, r_{\mathrm{p}}) q\,\big(r_{\mathrm{p}}, x, t\big)\, H\,\big(r_{\mathrm{p}}, x, t\big)\, \mathrm{d}r_{\mathrm{p}}\right] \\
&= -\frac{\partial \Sigma\,(r_{\mathrm{s}}, x, t)}{\partial t}.
\end{aligned} \tag{16}$$

In order to close the system of Eqs. (14), (16), we have to derive the expression for the particle capturing kinetics $\partial \Sigma\,(r_{\mathrm{s}}, x, t)\,/\partial t$.

3.3 Particle capturing kinetics

For proper determination of the particle capturing kinetics it must be realized that the value of $\Sigma\,(r_{\mathrm{s}}, x, t)$ expresses the number of particles captured in the volume rather than on the surface. Therefore, in Eq. (11) the value of $H\,(r_{\mathrm{s}}, x, t)$ should be substituted by $H_{\mathrm{V}}\,(r_{\mathrm{s}}, x, t)$. Application of assumption (10) and employment of the capture probability $p\,\big(r_{\mathrm{s}}, r_{\mathrm{p}}\big)$ result in the following:

$$\frac{\partial \Sigma\big(r_{\mathrm{s}}, x, t\big)}{\partial t} = C\big(r_{\mathrm{s}}, x, t\big)\frac{1}{l} \int_0^{\infty} p\,\big(r_{\mathrm{s}}, r_{\mathrm{p}}\big)\, j\big(r_{\mathrm{s}}, r_{\mathrm{p}}\big) q\,\big(r_{\mathrm{p}}, x, t\big)\, H\,\big(r_{\mathrm{p}}, x, t\big)\, \mathrm{d}r_{\mathrm{p}}. \tag{17}$$

Equations (14), (16), (17) coupled with Eqs. (7) and (9) form the closed system, from which the three distributions determining flow and filtration of the particulate suspensions in porous media may be found: the particle concentration distribution $C\,(r_{\mathrm{s}}, x, t)$, the pore concentration distribution $H\left(r_{\mathrm{p}}, x, t\right)$ and the concentration distribution for the captured particles $\Sigma\left(r_{\mathrm{s}}, x, t\right)$.

Substitution of the particle retention kinetics (17) into the population balance Eq. (16) makes it possible to exclude the retained particle size distribution Σ from the system of kinetic equations:

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\phi\left(r_{\mathrm{s}}, x, t\right) C\left(r_{\mathrm{s}}, x, t\right)\right]\\&\quad+\frac{\partial}{\partial x}\left[C\left(r_{\mathrm{s}}, x, t\right) \int_{0}^{\infty} \chi\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right) j\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right) q\left(r_{\mathrm{p}}, x, t\right) H\left(r_{\mathrm{p}}, x, t\right) \mathrm{d} r_{\mathrm{p}}\right]\\&=-C\left(r_{\mathrm{s}}, x, t\right) \frac{1}{l} \int_{0}^{\infty} p\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right) j\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right) q\left(r_{\mathrm{p}}, x, t\right) H\left(r_{\mathrm{p}}, x, t\right) \mathrm{d} r_{\mathrm{p}}.\end{aligned} \tag{18}$$

The two Eqs. (14) and (18), together with Eqs. (7) and (9), form a closed system for unknowns $C\,(r_{\mathrm{s}}, x, t)$ and $H\left(r_{\mathrm{p}}, x, t\right)$. There is a similarity between the equations of this system and the classical Boltzmann equation (Kampen 1984): the right-hand sides of Eqs. (14) and (18) may be interpreted as the “collision integrals” between the particles and the pores.

3.4 Initial-boundary value problem for the suspension injection process

Let us consider the injection of particulate suspension into a porous reservoir.

The injected particle size distribution at the inlet is a given function:

$$x=0:\; C=C^{0}\left(r_{\mathrm{s}}, t\right). \tag{19}$$

Initially, all pore and particle size distributions are given:

$$t=0:\; C=C_{0}\left(r_{\mathrm{p}}, x\right), \quad H=H_{0}\left(r_{\mathrm{p}}, x\right). \tag{20}$$

An important particular case corresponds to a constant injected suspension composition, absence of particles in the reservoir before the injection, and homogeneity of the porous medium:

$$\begin{aligned}&x=0\text{:}\;\; C=C^{0}\left(r_{\mathrm{s}}\right),\\&t=0\text{:}\;\; C=0,\, H=H_{0}\left(r_{\mathrm{p}}\right).\end{aligned}$$

Let us substitute Eq. (19) into the pore plugging kinetics Eq. (14). We obtain the equation for the pore size distribution at the inlet:

$$\begin{aligned}&\frac{\partial H\left(r_{\mathrm{p}}, 0, t\right)}{\partial t}\\&=\int_{0}^{\infty} \mathrm{d} r_{\mathrm{s}}\left[C^{0}\left(r_{\mathrm{s}}, t\right)\left(\begin{array}{l}\int_{r_{\mathrm{p}}}^{\infty} \mathrm{d} r_{\mathrm{p}}^{\prime}\left\{p\left(r_{\mathrm{s}}, r_{\mathrm{p}}^{\prime} \rightarrow r_{\mathrm{p}}\right) j\left(r_{\mathrm{s}}, r_{\mathrm{p}}^{\prime}\right) q\left(r_{\mathrm{p}}^{\prime}, 0, t\right) H\left(r_{\mathrm{p}}^{\prime}, 0, t\right)\right\}- \\ p\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right) j\left(r_{\mathrm{s}}, r_{\mathrm{p}}\right) q\left(r_{\mathrm{p}}, 0, t\right) H\left(r_{\mathrm{p}}, 0, t\right)\end{array}\right)\right].\end{aligned}$$

The last integral-differential equation determines the pore plugging dynamics $H\left(r_{\mathrm{p}}, x=0, t\right)$ at the inlet independently of the solution of the deep bed filtration problem for $x>0$.

4 Averaging the system of flow equations

The system of equations in the previous section makes it possible to determine, in principle, the distributions C, H, Σ at each point of space and at each time moment, provided that the initial distributions and the total flux U are given. However, solution of the non-linear integral–differential system of Eqs. (14), (16), (17) is a difficult task. Moreover, the initial and boundary distributions H_0, C^0 are not always known.

Often, the detailed information about the particle and the sizes contained in the distributions $C(r_s, x, t)$, $H(r_p, x, t)$ and $\Sigma(r_s, x, t)$ is not required. The only values of practical interest are the particle concentration $c(x, t)$ and the evolution of the average parameters of the porous medium $k(x, t)$, $\phi(x, t)$. Our next goal is to derive, on some additional assumptions, the complete system of hydrodynamic equations for these variables.

Derivation of such a system may be carried out in the two different ways. The first way is direct averaging of the complete system of integral–differential equations derived above by their integration over r_s and/or r_p. In order to get a closed system of equations on the basis of such an averaging, it is necessary to impose additional, rather strong assumptions about the different flow characteristics introduced above. This way is described in the present section, with introduction of the weakest necessary assumptions.

An alternative way may be imposing some (also rather strong) assumptions about the particle and the pore size distributions: that they are uni- or bimodal, etc. Then the constituting system of the integral–differential equations becomes strongly simplified, allowing for an analytical solution. Subsequent averaging becomes an almost trivial task. In fact, this method is applied for derivation of the phenomenological models, since it is often implicitly assumed in such models, for example, that the particle size difference is of no importance (Santos and Bedrikovetsky 2005, 2006). We follow this way in the next section.

It is shown in Appendix A that $\partial h(x, t)/\partial t = 0$. This is the (trivial) averaged equation for the variable h.

4.1 The averaged flow equation

Let us introduce the following averaged variables:
the porosity reduction factor:

$$\Gamma = \frac{\int_0^\infty \mathrm{d}r_s \phi\left(r_s, x, t\right) C\left(r_s, x, t\right)}{\phi c(x, t)}, \tag{21}$$

the flux reduction factor:

$$\begin{aligned}\alpha &= \frac{\int_0^\infty \mathrm{d}r_s \int_0^\infty \mathrm{d}r_p \left\{\chi\left(r_s, r_p\right) j\left(r_s, r_p\right) q\left(r_p\right) C\left(r_s, x, t\right) H\left(r_p, x, t\right)\right\}}{Uc(x, t)} \\ &= \frac{\int_0^\infty \mathrm{d}r_s \int_0^\infty \mathrm{d}r_p \left\{\chi\left(r_s, r_p\right) j\left(r_s, r_p\right) k_1\left(r_p\right) C\left(r_s, x, t\right) H\left(r_p, x, t\right)\right\}}{kc(x, t)},\end{aligned} \tag{22}$$

and the entrapment factor:

$$\overline{p} = \frac{\int_0^\infty \mathrm{d}r_\mathrm{s} \int_0^\infty \mathrm{d}r_\mathrm{p} \left\{ p\left(r_\mathrm{s}, r_\mathrm{p}\right) j\left(r_\mathrm{s}, r_\mathrm{p}\right) q\left(r_\mathrm{p}\right) C\left(r_\mathrm{s}, x, t\right) H\left(r_\mathrm{p}, x, t\right) \right\}}{Uc\left(x, t\right)}$$
$$= \frac{\int_0^\infty \mathrm{d}r_\mathrm{s} \int_0^\infty \mathrm{d}r_\mathrm{p} \left\{ p\left(r_\mathrm{s}, r_\mathrm{p}\right) j\left(r_\mathrm{s}, r_\mathrm{p}\right) k_1\left(r_\mathrm{p}\right) C\left(r_\mathrm{s}, x, t\right) H\left(r_\mathrm{p}, x, t\right) \right\}}{kc\left(x, t\right)}. \tag{23}$$

The porosity reduction factor shows how the porosity is reduced due to the fact that the porous space is not completely accessible to the particles. The flux reduction factor shows the difference between the average flow rate of the particles and the average flow rate of the liquid. The entrapment factor is, simply, the probability of capturing averaged over all the particles. The factors Γ, α, $\overline{p}$ are dimensionless.

Generally, the values of the introduced factors depend on the local distributions of pores and particles. However, these factors become constants in the model case where the values of $\chi\left(r_\mathrm{s}, r_\mathrm{p}\right)$, $p\left(r_\mathrm{s}, r_\mathrm{p}\right)$ and $j\left(r_\mathrm{s}, r_\mathrm{p}\right)$ are constants. In this case $\Gamma = \chi$, $\alpha = \chi j$ and $\overline{p} = pj$. These equalities serve to clarification of the physical meaning of the introduced parameters. Of course, this constancy assumption is oversimplified and is only used in order to demonstrate the physical significance of the introduced factors. They may somehow be substantiated for the particles, which are much smaller than pores. However, in this limit factors j and χ are probably close to unity.

In order to average the flow Eq. (16), it is integrated by r_s. By application of Eqs. (2), (4) and the definitions (21) to (23) of the dimensionless factors, the following equation is obtained:

$$\frac{\partial}{\partial t}\phi\Gamma c\left(x, t\right) + \frac{\partial}{\partial x}U\alpha c\left(x, t\right) = -\frac{\partial}{\partial t}\sigma\left(x, t\right). \tag{24}$$

In order to find the right-hand side of Eq. (24), the particle capturing kinetics Eq. (17) is integrated in r_s. By application of the definitions (22), (23), it is obtained that

$$\frac{\partial}{\partial t}\sigma\left(x, t\right) = \frac{\overline{p}}{l}Uc\left(x, t\right). \tag{25}$$

The last equation makes it possible to identify the ratio $\overline{p}/l$ with the standard filtration coefficient λ (Herzig et al. 1970).

It follows from the system (24), (25) that particle deposition leads to the two effects. Apart from the capture term on the right-hand side of Eq. (24), there is also a change of the particle flow velocity compared to the velocity of the carrying liquid. If, for simplicity, it is assumed that the porosity and the correction factors are constants (or are very slowly changing values), then the particle velocity may be estimated to be $U_\mathrm{p} = U\alpha/\phi\Gamma$. This value is to be compared with the interstitial velocity $U_l = U/\phi$ of the liquid. As discussed above, for the particles which are much smaller than the pores, $\alpha = j\Gamma$, and the ratio U_p/U_l is simply equal to j. In this case the particles are likely to move faster than the carrying flow, in agreement with the observations by Higgo et al. (1993), Kretzschmar et al. (1995), Massei et al. (2002).

If the particles are much smaller than the pores, j and χ approach unity, as well as α and Γ. If the particle and the pore sizes are comparable, and the size exclusion mechanisms dominate, the relation between α and Γ may be more complicated, resembling the fractional flow function in the Buckley–Leverett theory (Santos and Bedrikovetsky 2005).

4.2 Evolution of porosity, permeability and correction factors

The values of permeability and porosity entering Eqs. (3) to (5) as well as (24), (25) also vary, due to the particle capturing and the corresponding decrease of the pore sizes. Moreover, the correction factors vary. This variation may be evaluated on the basis of the kinetic Eqs. (14), (16), (17), although additional assumptions are needed for derivation of the equations governing the evolution of permeability, porosity and correction factors.

The equation for permeability variation may be derived on the basis of the permeability definition (5) and Eq. (14) for H. Multiplication of both sides of Eq. (14) by $k_1(r_\mathrm{p})$ and integration over r_p, after some transformations, result in

$$\frac{\partial k(x,t)}{\partial t} = -\int_0^\infty \mathrm{d}r_\mathrm{s} C(r_\mathrm{s},x,t) \int_0^\infty \mathrm{d}r_\mathrm{p} \int_0^\infty \mathrm{d}r'_\mathrm{p} \left[k_1\left(r'_\mathrm{p}\right) - k_1\left(r_\mathrm{p}\right)\right] \times \left\{p\left(r_\mathrm{s}, r'_\mathrm{p} \to r_\mathrm{p}\right) j(r_\mathrm{s}, r'_\mathrm{p}) u_1\left(r'_\mathrm{p}\right) H\left(r'_\mathrm{p}, x, t\right)\right\} = -I. \quad (26)$$

Integral I on the right-hand side of this equation is positive, since $r'_\mathrm{p} > r_\mathrm{p}$. This integral is difficult to calculate exactly. We suggest an approximate expression for the integral, based on the following considerations. The difference $A = k_1\left(r'_\mathrm{p}\right) - k_1\left(r_\mathrm{p}\right)$ is of the order of magnitude of the fourth degree of the characteristic pore size. This size is usually estimated to be $\sqrt{k/\phi}$, within a proportionality multiplier of the order of unity. Thus, we assume that the value of A may roughly be estimated to be $\beta_k (k/\phi)^2$, where β_k is a dimensionless constant. The rest of the integral I is exactly $\overline{p}Uc$ (cf. Eq. (23)).

Thus, the permeability equation may approximately be represented in the form of

$$\frac{\partial k(x,t)}{\partial t} = -\beta_k \overline{p} U c(x,t) \frac{k^2(x,t)}{\phi^2(x,t)}. \quad (27)$$

This expression might be obtained, simply, by definition of the value of β_k as $I\phi^2/\overline{p}Uck^2$ and its substitution into Eq. (26). The value of β_k introduced in such a way is dimensionless. The preceding considerations show that this value is of the order of unity. We assume that, within the experimental accuracy, the value of β_k may be set to constant.

The equation for porosity variation may be derived in a similar way, by assuming that the cross-section $s_1(r_\mathrm{p})$ for a pore size r_p is proportional to r_p^2 (like, for example, for a cylindrical or other regularly shaped pore). The resulting equation is

$$\frac{\partial \phi(x,t)}{\partial t} = -\beta_\phi \overline{p} c(x,t) U \frac{k(x,t)}{\phi(x,t)}. \quad (28)$$

Similar considerations may be applied to derivation of the equations for the correction factors $\gamma, \alpha, \overline{p}$. For example, multiplication of both parts of Eq. (14) by $\chi(r_\mathrm{s}, r_\mathrm{p}) s_1(r_\mathrm{p})$, integration over r_p, comparison to Eqs. (9), (21) and application of considerations similar to those applied to the derivation of Eqs. (27), (28) result in:

$$\frac{\partial \phi \Gamma}{\partial t} = -\beta_\Gamma \Gamma \overline{p} c(x,t) U \frac{k(x,t)}{\phi(x,t)}.$$

Comparison with Eq. (28) gives

$$\frac{\partial \Gamma}{\partial t} = -(\beta_\Gamma - \beta_\phi) \Gamma \overline{p} c(x,t) U \frac{k(x,t)}{\phi^2(x,t)}. \quad (29)$$

A similar equation for α reads

$$\frac{\partial \alpha}{\partial t} = -(\beta_\alpha - \beta_k)\alpha \overline{p} U c\,(x,t)\,\frac{k\,(x,t)}{\phi^2\,(x,t)}. \tag{30}$$

Finally, the equation for $\overline{p}$ may be obtained by similar processing of Eq. (23):

$$\frac{\partial \overline{p} k}{\partial t} = -\beta_\mathrm{p} \overline{p}^2 U c\,(x,t)\,\frac{k^2\,(x,t)}{\phi^2\,(x,t)},$$

or, accounting for Eqs. (27), (30),

$$\frac{\partial \overline{p}}{\partial t} = -(\beta_\mathrm{p} - \beta_k)\overline{p}^2 U c\,(x,t)\,\frac{k\,(x,t)}{\phi^2\,(x,t)}, \tag{31}$$

Equations (24), (25) and (27) to (31) coupled with the Darcy law (6) form a closed system of equations for variables c, P, k, ϕ, α, Γ, $\overline{p}$. The parameters β_i in these equations are supposed to be of the order of unity. Their more precise evaluation is a matter of experiment.

4.3 Power law for porosity-permeability dependence

An important consequence of Eqs. (27), (28) may be obtained on the assumption about constancy of β_k and β_ϕ. By division of Eq. (27) by Eq. (28), the ordinary differential equation with regard to k, ϕ is obtained:

$$\frac{\mathrm{d}k}{\mathrm{d}\phi} = \frac{\beta_k}{\beta_\phi}\frac{k}{\phi}.$$

Solution of this equation results in

$$\frac{k}{k_0} = \left(\frac{\phi}{\phi_0}\right)^{\beta_k/\beta_\phi}. \tag{32}$$

Here k_0, ϕ_0 are initial values of permeability and porosity.

Equation (32) may be used in the system of hydrodynamic equations instead of one of the Eqs. (27) or (28). Verification of this power dependence on the basis of the available experimental data may confirm or disprove the hypothesis about constancy of β_k, β_ϕ for a given porous medium.

Similar relations between other parameters may be obtained. It should be remarked, however, that all these relations are rather approximate. They are based on the assumptions about the relations between the pore conductivities/volumes and the pore sizes of the type of $k_1 \sim (k/\phi)^2$, which may or may not be valid for certain types of the porous media. Moreover, the statement at the beginning of Sect. 2 that the values r_s, r_p are not necessarily pore and particle sizes, but may be other parameters, does not hold for these derivations.

Let us assume, for example, that in the whole course of porosity reduction it obeys the famous Carman–Kozeny model (Dullien, 1979):

$$k = K\phi^3/(1-\phi)^2, \quad K = 1/A\tau^2 S_0^2.$$

Assuming that coefficient K is constant, we obtain that the value of β_k/β_ϕ from Eq. (32) is equal to $(3-\phi)/(1-\phi)$. Evidently, this value varies with porosity. However, if, for example, in the course of the filtration process porosity reduces from 0.2 to zero, the value of β_k/β_ϕ varies from 3.5 to 3, which is well within the accuracy of the proposed averaging.

Of course, this example is brought here for purely illustrative purposes. It is not obvious, whether the Carman–Kozeny theory holds in the framework of the deep bed filtration process. Additionally, both shape factor A, tortuosity τ, and the specific surface S_0, entering coefficient K, may vary in this process.

5 Example: a one-particle size and two-pore size model

5.1 Mathematical description of the model

The goal of this section is to demonstrate the properties of the proposed model on relatively simple characteristic examples allowing for analytical study. As mentioned above, some of such examples provide a background intuition behind empirical models of deep bed filtration. Derivation of the hydrodynamic flow equations on the basis of the simplified models of the porous medium and simplified particle distributions is an approach alternative to direct averaging described in the previous section. Much more complex examples may be considered in the framework of the same formalism. Their detailed study is a subject to separate work. The analysis below is intended to demonstrate the capacity of the formalism developed and a possible regular way of working within it. More general conclusions relevant to all the models of such a type are also formulated.

In the simplest example considered below, we assume that all the particles are of the same size r_{s0}, so that

$$C\,(r_s, x, t) = c(x, t)\delta(r_s - r_{s0}). \tag{33}$$

Moreover, initially all the pores are of the same size r_{p1}. When a particle passes through a pore of the initial size, it may be captured with the probability p_0. In this case the pore size changes from r_{p1} to r_{p2}. If a particle meets a pore of the size r_{p2}, it is captured with the probability 1, and the pore disappears (its size becomes zero). Thus, a pore may capture up to two particles, and complete pore plugging happens after capture of these particles. A schematic picture of the capturing process is presented in Fig. 1. Figure 2 presents evolution of the pores of different sizes and of the particle concentration.

It should be noted that, in accordance to the comment at the beginning of Sect. 2, in this example r_{p1} and r_{p2} do not necessarily mean actual pore sizes. They may indicate different types of interactions between a particle and a pore. It is actually assumed that after capturing a particle a pore is "switched" to another state where its probability to catch another particle strongly increases. This may be due to physical sizes, as well as due to other factors (shapes, force interactions between particles, flow redistribution etc.) In this interpretation, the value of r_p is a "switch" parameter showing whether the pore has captured the particle. The same remark is valid about all the examples of this section.

The described model corresponds to the following function $p(r_s, r'_p \to r_p)$:

$$p(r_{s0}, r_{p1} \to r_p) = p_0\delta(r_p - r_{p2}); \quad p(r_{s0}, r_{p2} \to r_p) = \delta(r_p). \tag{34}$$

According to the model, the solution for $H(r_p, x, t)$ is looked upon in the following form:

$$H(r_p, x, t) = h_1(x, t)\delta(r_p - r_{p1}) + h_2(x, t)\delta(r_p - r_{p2}) + h_0(x, t)\delta(r_p). \tag{35}$$

Let us substitute this form of solution into Eq. (14) for the pore size kinetics. In view of Eq. (33), integrating this equation over r_s results in obtaining $c(x, t)$ instead of $C(r_s, x, t)$, while

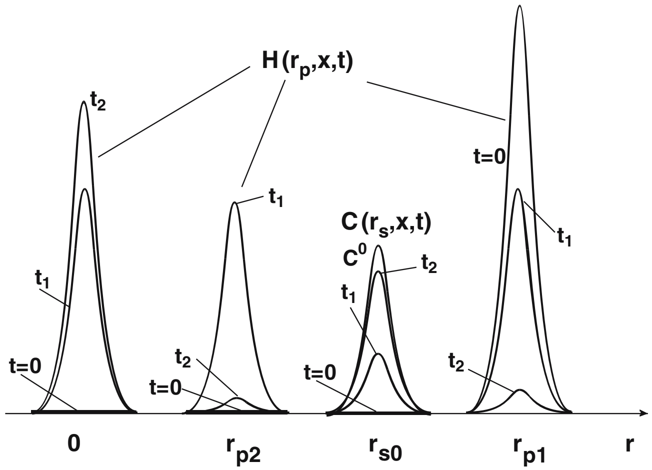

Fig. 2 Dynamics of pore and particle size distributions during deep bed filtration for the system with two pore and one particle sizes. We show the values o h corresponding to different pore radii, as well as the particle concentration corresponding to radius r_{s0}

in the rest of the integrals r_s is substituted by r_{s0}. Integration over r_p' with substitution of expression (34) for p results in the following equation:

$$\begin{aligned}\frac{\partial h_1}{\partial t}\delta(r_p - r_{p1}) + \frac{\partial h_2}{\partial t}\delta(r_p - r_{p2}) + \frac{\partial h_0}{\partial t}\delta(r_p) = ch_1 p_0 j_1 q_1 \delta(r_p - r_{p2})\\ +cj_2 h_2 q_2 \delta(r_p) - \left(cp_0 j_1 q_1 h_1 \delta(r_p - r_{p1}) + cj_2 q_2 h_2 \delta(r_p - r_{p2})\right). \qquad (36)\end{aligned}$$

Here and further on we use the designations

$$j_i = j(r_{s0}, r_{pi}); \quad q_i = q(r_{pi}), \quad \chi_i = \chi(r_{s0}, r_{pi}), \text{ etc.}$$

All the parameters j_i, χ_i are assumed to be constants. Since the particles do not enter the pores of the size r_{p2}, we have $\chi_2 = 0$ and, most likely, $j_2 = 1$.

On the contrary, the values of u_i in Eq. (36) and in the subsequent equations are not constants. At a given total flow rate U, they are to be found from Eq. (7), which, for the present situation, assumes the form of

$$u_1 = \frac{k_1 U}{k_1 h_1 + k_2 h_2}, \quad u_2 = \frac{k_2 U}{k_1 h_1 + k_2 h_2}. \qquad (37)$$

Equating the coefficients at different delta-functions in Eq. (36), we obtain the system of equations for h_0, h_1, h_2:

$$\frac{\partial h_1}{\partial t} = -p_0 c j_1 q_1 h_1; \qquad (38)$$

$$\frac{\partial h_2}{\partial t} = p_0 c j_1 q_1 h_1 - c j_2 q_2 h_2; \qquad (39)$$

$$\frac{\partial h_0}{\partial t} = c j_2 q_2 h_2. \qquad (40)$$

Under constant x and at a known concentration c, this system may be considered as a system of ordinary differential equations of the population balance type.

Let us now consider Eq. (18) for the particle concentration distribution. In this equation $C(r_s, x, t)$ may be substituted by $c(x, t)$ and r_s by r_{s0}, according to Eq. (33). The effective porosity $\phi(r_s, x, t)$ is calculated according to Eq. (9). Since the particles may only flow in the pores of the size r_{p1}, the only contribution to the effective porosity comes from these pores:

$$\phi(r_{s0}, x, t) = s_{10}h_1, \quad s_{10} = \chi(r_{s0}, r_{p1})s_1(r_{p1}).$$

The integrals in Eq. (18) are evaluated as follows: the flow integral is

$$\int_0^\infty \chi(r_{s0}, r_p) j(r_{s0}, r_p) q_1\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p = \chi_1 j_1 q_1 h_1,$$

and the capturing integral is

$$\int_0^\infty p\left(r_{s0}, r_p\right) j(r_{s0}, r_p) q_1\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p = p_0 j_1 q_1 h_1 + j_2 q_2 h_2.$$

Thus, the concentration Eq. (18) is reduced to the form of

$$s_{10}\frac{\partial}{\partial t}ch_1 + \chi_1 j_1 \frac{\partial}{\partial x}cq_1 h_1 = -\frac{1}{l}p_0 j_1 q_1 ch_1 - \frac{1}{l}j_2 q_2 ch_2. \tag{41}$$

Equations (37) to (41) form the closed system for the unknowns $c, h_1, h_2, h_0, q_1, q_2$.

The initial-boundary value problem, describing injection of a single particle size suspension into a porous medium which initially consists only of the pores of the size r_{p1} is:

$$t = 0 : c = h_2 = h_0 = 0, h_1 = h_1^0; \tag{42}$$

$$x = 0 : c = c^0. \tag{43}$$

Thus, the mobile species (particle) concentration is set in the initial and boundary conditions. The immobile species (pore) concentrations are set just in initial conditions. The boundary $x = 0$ coincides with the characteristic line of the hyperbolic system (37) to (41), and the initial-boundary value problem (42), (43) is of the Goursat type (Tikhonov and Samarskii 1990).

The system formulated above allows generalization onto the case where a pore is completely plugged by N particles, i.e., after the sequential plugging of a pore by n particles the size of the pore becomes equal to zero. In this case, $N + 1$ kinetic equations appear for the pore concentrations $h_1, h_2, \ldots, h_N$ and h_0. Introduction of the more complex capturing mechanisms is also possible.

5.2 Solution of the problem

The mathematical problem stated above possesses an exact (although cumbersome) analytical solution. The formal details of the solutions are presented in Appendix B. The procedure is rather general, and multiple other examples of the "discrete capturing" may be resolved in a similar way. In this section we present qualitative analysis of the results obtained. The results are illustrated in Figs. 2 and 3.

Figure 2 illustrates the evolution of probability distribution functions for pores and particles according to analytical solution presented in Appendix B. The large pore concentration decreases due to particle deposition in large pores—particles consume large pores and form small pores. The particle concentration increases during the propagation of suspended particle wave. The zero-radius-pore concentration increases during plugging of small pores. The small pore concentration first increases as a result of the large pore plugging, reaches maximum and afterwards it decreases due to small pore plugging.

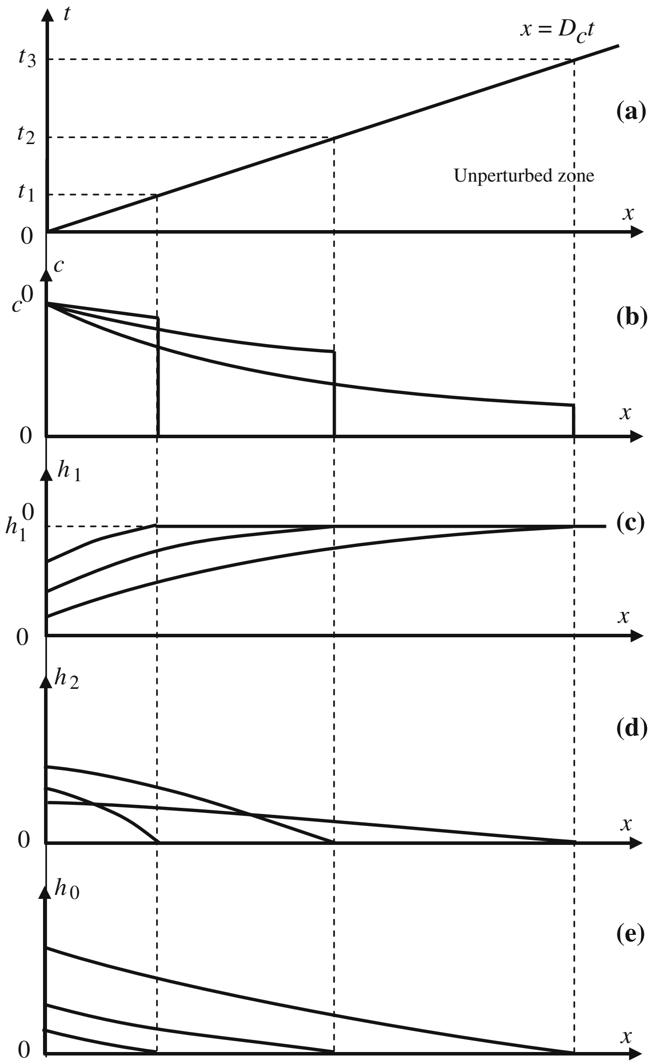

Fig. 3 (**a**) Propagation of suspension concentration front in (x, t) plane; (**b**) – (**e**) Distributions of the particle concentration and of the concentrations of pores of different sizes at the three different time moments

Figure 3 presents the dynamics of the suspended concentration front and all concentration profiles. All the concentrations are functions of x and t. They are presented in the three different time moments.

The suspended concentration front moves at the constant speed D_c. The initial suspended and initial pore concentrations hold in the zone ahead of the concentration front. The profiles show that the suspended concentration monotonically decreases from the injected value c^0 at the inlet to the value $c(D_c t, t)$ behind the front. The suspended concentration behind the front exponentially decreases with time from c^0 at the beginning of injection up to zero when time tends to infinity. The suspended concentration is discontinuous on the front. On the contrary, all the pore concentrations are continuously distributed. The concentrations of the small and of the zero-sized pores decrease along the linear coordinate from maximum values at the inlet to zeros on the concentration front. The large pore concentration decreases from its initial value on the front to minimum value at the inlet. All the concentrations turn to their initial values along each characteristic when x (or t) tends to infinity.

Figure 3b–e show the particle and the pore concentration profiles at different moments. The suspended concentration $c(x,t)$ and the large pore concentration $h_1(x,t)$ decrease with time, since the particles and the largest pores are consumed by the capture-plugging process. The concentration $h_0(x,t)$ of the zero-sized pores increases with time, because the corresponding pores appear during plugging. The concentration $h_2(x,t)$ of the pores of the intermediate radius shows a non-monotonous behavior. Initially, the prevailing process is the particle capture by pores of the radius r_{p1} and their transformation into pores of the radius r_{p2}. At this stage the value of $h_2(x,t)$ increases, starting from zero. At a later stage, the particle capture by pores of the radius r_{p2} prevails over the process of formation of these pores, and the value of $h_2(x,t)$ decreases back to zero.

The inlet large pore concentration is always positive and tends to zero when time tends to infinity. The small pore concentration at the inlet initially increases, but afterwards also tends to zero. The zero-sized pore concentration tends to the initial value h_1^0. Thus, the state of the system where all large and small pores are plugged by the particles is reached asymptotically, when time tends to infinity.

The porosity and permeability of the medium decrease with time, and this decrease develops later for the points located further from the injection side.

If a pore can be plugged by N particles, the general structure of the flow zone (Fig. 3) holds, although the amounts of pores of intermediate radii may exhibit non-trivial behavior. The suspension concentration front speed is the same, and all concentrations ahead of this front coincide with those in initial conditions.

5.3 Connection to the classical deep bed filtration model

Let us show that the suspension concentration $c(x,t)$ and the retained concentration $\sigma(x,t)$ obey the deep bed filtration model that accounts for accessibility and flux reduction:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\phi_0\gamma\left(\sigma\right)c\right)+U\frac{\partial}{\partial x}\left(\alpha\left(\sigma\right)c\right)=-\frac{\partial\sigma}{\partial t},\\ &\frac{\partial\sigma}{\partial t}=\lambda\left(\sigma\right)\left(1-\alpha\right)cU. \end{aligned} \tag{44}$$

Here γ is the porosity fraction accessible to particles, i.e., the accessibility factor; α is the total flux fraction transporting particles via the accessible pore volume, i.e., the flux reduction factor, and λ is the filtration coefficient. The value of α is similar to that appearing in Eqs. (21) to (25). The value of γ differs from the value of Γ in Eq. (24) in that the last value is the reduction factor for the current porosity, while γ is related to the initial porosity. The capture rate is proportional to the flux through inaccessible pores. It is important to highlight that $c(x,t)$ is the suspended particle concentration in the accessible pore volume, i.e., it is equal to the number of particles per unit of the accessible pore volume.

If the particles are exceedingly smaller than the pores, then both factors are equal to unity, and the model (44) reduces to the so-called classical deep bed filtration model (Iwasaki 1937; Herzig et al. 1970). What is required to prove, is that, for a model studied, the thus defined values of γ, α and λ are functions of σ. This proof is given below.

Comparison to Eq. (41) shows that, in order to obey Eqs. (44), the accessible porosity fraction should be defined as

$$\gamma = s_{10}h_1/\phi_0. \tag{45}$$

The total particle flux via accessible pore volume is

$$\chi_1 j_1 q_1 h_1.$$

Therefore, with the help of Eq. (37), the flux reduction factor should be expressed as

$$\alpha = \frac{\chi_1 j_1 k_1 h_1}{k_1 h_1 + k_2 h_2}. \tag{46}$$

Finally, comparison of Eqs. (44) and (41), with application of Eq. (46), results in the following expression for the filtration coefficient:

$$\lambda = \frac{p_0 j_1 k_1 h_1 + j_2 k_2 c h_2}{l[(1 - \chi_1 j_1) k_1 h_1 + k_2 h_2]}. \tag{47}$$

Equations (45) to (47) show that, in order to prove Eq. (44), it is necessary to prove that h_1 and h_2 are functions of σ. This proof is described in Appendix C.

In the case where a pore can be completely plugged by n particles, the n pore size population appears in the model described above. It may be shown that the suspended particle and the total deposited concentrations are also described by the same phenomenological deep bed filtration model (44), although with different dependencies γ, α and λ.

The same result may be obtained for more complex mechanisms of plugging than the mechanism considered above (an example is considered in the next subsection). Thus, the result expressed by system (44) is rather general, being valid for any suspended flow of particles of a single size. However, the function $\lambda(\sigma)$ may be history-dependent and boundary problem-dependent, since, for example, initial presence of the pores of the intermediate size will lead to a different dependence $\lambda(\sigma)$.

The suspended concentration at the core effluent is usually measured in the laboratory deep bed filtration corefloods (Higgo et al. 1993; Kretzschmar et al. 1995; and others). The filtration coefficient $\lambda\,(\sigma)$ is determined from the breakthrough curve $c\,(0, t)$ by solution of the inverse problem. After tuning of the model on the breakthrough curve, prediction of the deposition profile $\sigma\,(x, t)$ is obtained from the solution of the direct problem for the system (44).

The method for measurement of the deposition profiles was developed by Al-Abduwani et al. (2004). The results make it possible to validate a deep bed filtration model by comparison between the two deposition profiles: the profile obtained by prediction from the model, preliminarily tuned on the breakthrough curve, and the profile obtained from direct laboratory measurements. Agreement between the two profiles would indicate adequacy of the model, while their disagreement would indicate the presence of an additional physical mechanism, which is not taken into account by the model.

The analysis above indicates that involvement of different mechanisms of the particle capture is insufficient for explanation of the disagreement between the two profiles. Indeed, as it was stated above, the total retained and suspended concentrations for the n-particle plugging system fulfill the same system (44) as in the case of one-particle plugging. Introduction of additional pore populations of intermediate sizes results in distribution of the total deposition over different pores, but the total deposited concentration remains the same. If the dependencies $\alpha(\sigma)$, $\gamma(\sigma)$ are known beforehand, and $\lambda(\sigma)$ is fitted from the breakthrough curve, the predicted deposition profile may still match badly the measured profile (as was observed by Al-Abduwani et al. 2004). Introduction of the exceedingly more complex mechanisms of particle capturing cannot resolve this problem. The only possible resolution seems to be introduction of particles of different sizes. The availability of particles of different sizes may cause the chromatographic effect, resulting in alteration of the deposition profile. Study of this deposition profile with varying parameters of the model may result in agreement between the modeling and the experimental data.

5.4 Generalization of the example

More general physics for the particle capture has been incorporated into the pore network models (Rege and Fogler 1987, 1988; Siqueira et al. 2003). It was assumed that the size exclusion capture may happen at each pore entrance. Alternatively, the particle could penetrate into a pore and become captured there. The probability of particle capture inside the pore is determined by electrical, gravity and sorption mechanisms of the particle retention. Thus, the initial pore may be closed by capturing due to the size exclusion mechanism, or its radius may be reduced due to capturing inside the pore. Let us show how this system can be described in the framework of the proposed population balance model. We describe only the simplest possible approach, although a more general model may also be derived and studied analytically.

As previously, we assume that all the particles are of the only size r_{s0}. The required modification to the previously described discrete model is that the particle may reduce the size of the pore r_{p1} to r_{p2} with the probability p_1 (corresponding to capturing inside the pore) or totally close the pore with the probability of p_2 (corresponding to size exclusion capturing). As above, we assume that the particle is captured by a pore of the radius r_{p2} by the size exclusion mechanism, and the radius of such pore is reduced to zero:

$$p(r_{s0}, r_{p1} \to r'_p) = p_1\delta(r'_p - r_{p2}) + p_2\delta(r'_p); \quad p(r_{s0}, r_{p2} \to r'_p) = \delta(r'_p).$$

The system (38) to (40) of equations for kinetics of pore plugging is changed to

$$\begin{aligned}
\frac{\partial h_1}{\partial t} &= -p_0 c j_1 q_1 h_1, \quad p_0 = p_1 + p_2;\\
\frac{\partial h_2}{\partial t} &= p_1 c j_1 q_1 h_1 - c j_2 q_2 h_2;\\
\frac{\partial h_0}{\partial t} &= p_2 c j_1 q_1 h_1 + c j_2 q_2 h_2.
\end{aligned}$$

The rest of the governing equations remain the same.

The procedure for derivation of the governing equations and their solution is similar to that described above and in Appendix B. The necessary modifications to the final formulae for this case are given at the end of Appendix B.

The phenomenological system of Eq. (44) considered in the previous subsection may also be derived for this case. Equations (45) to (47) remain valid, although with a different function $f(h_1)$. Thus, introduction of the two simultaneous physical mechanisms of particle capture does not qualitatively change the mathematical description of the filtration process. The only way to change the description significantly seems to be introduction of the particle size distribution.

6 Conclusions

Derivation of the stochastic model for deep bed filtration with incomplete pore plugging and analytical modeling of two-pore size filtration, described in this paper, lead to the following conclusions:

(1) Filtration of particulate suspensions in porous media, with a possibility of incomplete plugging of pores by particles, is described in the framework of the statistic formalism. Size distributions of particles and pores are considered. A model for capturing of a particle by a pore is rather general, allowing for description of both complete and incomplete capturing caused by a variety of physical mechanisms.
(2) A closed system of kinetic equations for the size distributions of the suspended and captured particles, as well as for the pore size distribution, is derived. The system takes into account the process of capturing and the consequent porosity and permeability reduction. It accounts for the possibility of different particles to move at velocities which are different from the average velocity of the flow.
(3) Averaging of the system of kinetic equations leads to the system of hydrodynamic equations. The hydrodynamic system is closed and includes the equation for evolution of the permeability and porosity of the medium, describing the process of the formation damage.
(4) A problem of deep bed filtration of single-sized particles with complete plugging of pores by two particles allows for exact solution. The validity of a known phenomenological model for the deep bed filtration is established in the framework of this example.
(5) The suspended particle and total retained particle concentrations are described by the classical deep bed filtration model for the case of any number of particles that completely plug pores.
(6) The two-capture mechanism system of size exclusion at the pore entrance with particle sorption inside the pore is also described by the classical deep bed filtration model.

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Appendix A. Conservation of the total amount of free pores

In this appendix we derive the conservation law for the total amount of free pores. The value of $h\,(x, t)$, defined as

$$h\,(x,t) = \int_0^\infty H\left(r_\mathrm{p}, x, t\right) \mathrm{d}r_\mathrm{p}, \tag{48}$$

remains invariable with time

$$\frac{\partial h}{\partial t} = 0. \tag{49}$$

We do not use the fact of pore number conservation in the derivations above. We prove it only in order to demonstrate one property of our model. This property follows from an assumption that a particle entering a pore does not plug or split it, but just decreases its size. This assumption is true, for example, if the particle sizes are much smaller than the pore sizes. In other types of models (Sharma and Yortsos 1987a, b, c) it may be assumed, for example, that a particle may totally plug a pore, and that such a pore "disappears". Alternatively, it may be assumed that the particle splits a pore into two other pores. In both cases, the number of pores is not conserved (although in the first case the conservation of the pores may be kept by introduction of the zero-sized pores).

In order to prove Eq. (49), we integrate Eq. (14) over r_p:

$$\frac{\partial h(x,t)}{\partial t} = \int_0^\infty \mathrm{d}r_\mathrm{s} \left[C(r_\mathrm{s},x,t) \begin{pmatrix} \int_0^\infty \mathrm{d}r_\mathrm{p} \int_{r_\mathrm{p}}^\infty \mathrm{d}r'_\mathrm{p} \left\{ \alpha\left(r_\mathrm{s}, r'_\mathrm{p} \to r_\mathrm{p}\right) q\left(r'_\mathrm{p},x,t\right) H\left(r'_\mathrm{p},x,t\right) \right\} - \\ \int_{r'_\mathrm{p}}^\infty \mathrm{d}r_\mathrm{p} \left\{ \alpha\left(r_\mathrm{s}, r_\mathrm{p}\right) q\left(r_\mathrm{p},x,t\right) H\left(r_\mathrm{p},x,t\right) \right\} \end{pmatrix} \right]$$

$$= \int_0^\infty \mathrm{d}r_\mathrm{s} \left[C(r_\mathrm{s},x,t) \begin{pmatrix} \int_{r_\mathrm{p}}^\infty \mathrm{d}r'_\mathrm{p} \left\{ \alpha\left(r_\mathrm{s}, r'_\mathrm{p}\right) q\left(r'_\mathrm{p},x,t\right) H\left(r'_\mathrm{p},x,t\right) \right\} - \\ \int_{r'_\mathrm{p}}^\infty \mathrm{d}r_\mathrm{p} \left\{ \alpha\left(r_\mathrm{s}, r_\mathrm{p}\right) q\left(r_\mathrm{p},x,t\right) H\left(r_\mathrm{p},x,t\right) \right\} \end{pmatrix} \right].$$

The two terms in the round parentheses are clearly equal, within redesignation of the integration variable. This proves Eq. (49).

Appendix B Solution of the system for model cases

The goal of this Appendix is to solve the Goursat problem (42), (43) for the system of Eqs. (37) to (41) derived in Section 5.1 and describing a model example of the one-size particle filtration problem. We bring also the results for the generalization of the example described in Section 5.4.

Let us first consider hyperbolic Eq. (41). Since the boundary condition is discontinuous with regard to the suspended concentration, the solution contains a concentration shock front travelling with a characteristic velocity D_c. The mass balance conditions at the front are:

$$s_{10}\,[ch_1]\,D_\mathrm{c} = \chi_1 j_1\,[cq_1 h_1]\,,$$
$$[h_1]\,D_\mathrm{c} = [h_2]\,D_\mathrm{c} = [h_0]\,D_\mathrm{c} = 0.$$

From the last conditions it follows that the pore concentrations h_i are continuous along the shock moving at non-zero speed. In view of Eq. (37), the velocities u_i are also continuous. The suspended concentration shock has a speed of

$$D_\mathrm{c} = \frac{\chi_1 j_1 q_1}{s_{10}}.$$

From the characteristic form of the system (38) to (40), (41) it follows that concentrations c, h_2 and h_0 ahead of the front $x = D_\mathrm{c}t$ are zeros, and that the large pore concentration h_1 is equal to its initial value h_1^0. Therefore, these conditions for the pore concentrations are also valid on the front. Then, from the last equation and Eq. (37) it follows that the shock velocity D_c is constant and is equal to

$$D_\mathrm{c} = \frac{\chi_1 j_1 U}{s_{10} h_1^0}.$$

In order to solve the formulated system of equations, let us first notice that the system (38) to (40) makes it possible to establish the relations between h_1, h_2, h_0. Dividing (39) by (38), we obtain

$$\frac{\mathrm{d}h_2}{\mathrm{d}h_1} = -1 + a\frac{h_2}{h_1}, \quad a = \frac{j_2 k_2}{p_0 j_1 k_1}.$$

This equation is solved by the substitution $y(h_1) = h_2/h_1$. The solution is

$$h_2 = f(h_1) = C_1 h_1^a + \frac{h_1}{a-1}, \quad C_1 = -\frac{\left(h_1^0\right)^{1-a}}{a-1}. \tag{50}$$

Here the constant of integration C_1 is found from the condition on the suspended concentration front:

$$x = D_c t : h_2 = 0, h_1 = h_1^0.$$

Summing Eqs. (38) to (40) and integrating, we obtain

$$h_1 + h_2 + h_0 = \text{const} = h_1^0. \tag{51}$$

This equation is in accordance with the result of Appendix A. The value of h_0 may be found from (51) as a function of h_1, h_2 and therefore, in view of Eq. (50), as a function of h_1 only.

For the Goursat problem formulated above, the immobile pore concentrations at the boundary $x = 0$ may be found without solving the problem for $x > 0$ (Tikhonov and Samarskii 1990). Substitution of the boundary value c^0 and expression (37) for u_1 into Eq. (38) for immobile species kinetics results in the differential equation for $h_1(0, t)$. Its solution and utilization of the initial condition (42) result in the following implicit equation for $h_1(0, t)$:

$$\left(1 + \frac{k_2}{k_1(a-1)}\right) h_1(0,t) - \frac{k_2 \left(h_1^0\right)^{1-a}}{k_1 a(a-1)} h_1(0,t)^a = \left(1 + \frac{k_2}{k_1 a}\right) h_1^0 - U c^0 p_0 j_1 t. \tag{52}$$

The values of $h_2(0, t)$ and $h_0(0, t)$ may be found from Eqs. (50) and (51), correspondingly.

Let us now bring Eq. (41) to the form of the conservation law, applying Eqs. (38), (39). Multiplication of Eq. (41) by l, Eq. (38) by -2, Eq. (39) by -1 and addition of the three obtained equations results in the conservation law:

$$\frac{\partial}{\partial t}(s_{10} l c h_1 - 2h_1 - h_2) + \chi_1 j_1 l \frac{\partial}{\partial x} c q_1 h_1 = 0.$$

The value of ch_1 is now expressed from Eq. (38). Substitution into the last equation results in

$$\frac{\partial}{\partial t}\left(-\frac{s_{10} l}{p_0 j_{11} q_1}\frac{\partial h_1}{\partial t} - 2h_1 - h_2\right) - \frac{\partial}{\partial x}\left(\frac{\chi_1 l}{p_0}\frac{\partial h_1}{\partial t}\right) = 0.$$

Integrating the last equation by time from zero to t, we obtain:

$$\frac{s_{10} l}{p_0 j_{11} q_1}\frac{\partial h_1}{\partial t} + 2h_1 + h_2 + \frac{\chi_1 l}{p_0}\frac{\partial h_1}{\partial x} = C_2.$$

The constant C_2 in the last equation is found from the initial conditions (42), (43). Finally, in consideration of Eqs. (50), (51), Eq. (41) becomes

$$\frac{s_{10}(k_1 h_1 + k_2 f(h_1))}{\chi_1 j_1 k_1 U}\frac{\partial h_1}{\partial t} + \frac{\partial h_1}{\partial x} = -\frac{p_0}{\chi_1 l}\left[2\left(h_1 - h_1^0\right) + f(h_1)\right]. \tag{53}$$

The last is the linear hyperbolic equation with regard to h_1. It may be represented in the characteristic form. Along the characteristics

$$\frac{dt}{dx} = \frac{s_{10}(k_1 h_1 + k_2 f(h_1))}{\chi_1 j_1 k_1 U},$$
$$\frac{dh_1}{dx} = -\frac{p_0}{\chi_1 l}\left[2\left(h_1 - h_1^0\right) + f(h_1)\right].$$

This system of differential equations allows for exact, although implicit, solution in terms of the two integrals:

$$\int_{h_1(x,t)}^{h_1(0,t)} \frac{\mathrm{d}h_1}{2\left(h_1 - h_1^0\right) + f\left(h_1\right)} = \frac{p_0 x}{\chi_1 l}, \tag{54}$$

$$\int_{h_1(x,t)}^{h_1(0,t_0)} \frac{(k_1 h_1 + k_2 f(h_1))\mathrm{d}h_1}{2\left(h_1 - h_1^0\right) + f\left(h_1\right)} = \frac{p_0 j_1 k_1 U}{l s_{10}}(t - t_0). \tag{55}$$

For a given x and t, the three transcendental Eqs. (52), (54) and (55) form the system, from which three unknowns $h_1(x,t)$, t_0 and $h_1(0,t_0)$ are to be found.

The concentration c may now be expressed with the help of Eq. (38). Differentiating Eq. (54) by t, expressing $\partial h_1/\partial t$ and substituting into Eq. (38), we find:

$$\begin{aligned} c(x,t) &= -\frac{1}{p_0 j_{11} q_1 h_1(x,t)} \frac{\partial h_1(x,t)}{\partial t} \\ &= -\frac{k_1 h_1(x,t) + k_2 f(h_1(x,t))}{p_0 j_{11} k_1 U h_1(x,t)} \frac{\partial h_1(0,t)}{\partial t} \frac{2\left(h_1(x,t) - h_1^0\right) + f\left(h_1(x,t)\right)}{2\left(h_1(0,t) - h_1^0\right) + f\left(h_1(0,t)\right)}. \end{aligned} \tag{56}$$

This completes the description of the solution. Behind the suspension front, the solution is given by Eqs. (52), (54) and (55) for $h_1(x,t)$, by Eq. (56) for c, and by Eqs. (50) and (51) for h_2 and h_0, correspondingly. In the area ahead of the suspension front all the variables are equal to their initial values.

Let us determine the suspended concentration along the front $x = D_\mathrm{c}t$. It is convenient to use directly Eq. (41). On the front we have $h_1 = h_1^0$, $h_2 = 0$. Thus, from Eq. (41) it follows that

$$c(t, D_\mathrm{c}t) = c^0 \exp\left(-t/T\right), \quad T = \frac{l s_{10} h_1^0}{p_0 j_1 U}$$

The capture of particles on the front is going on just by thick pores, whose concentration h_1^0 is constant. Hence, the rate of decrease of the suspended concentration on the front is determined by the probability p_0 of capture in large pores. The value of T introduced in the last equation is the characteristic time of the concentration decrease due to capturing in the large pores.

For the generalized case where a pore is completely plugged by n particles, the $N+1$ kinetic equations appearing for the pore concentrations $h_1, h_2, \ldots, h_N$ and h_0 allow for first integrals $h_i\left(h_1\right), i = 2, 3 \ldots N$. This also makes it possible to reduce the system to two equations: for the suspended concentration c, and for the kinetics of h_1-plugging. Consequently, an exact analytical solution for such a problem is also possible.

Another generalization, the problem described in Section 5.4, may be solved in a similar way. Instead for Eq. (50) it may be obtained that

$$h_2 = f(h_1) = -\frac{b\left(h_1^0\right)^{1-a}}{a-1} h_1^a + \frac{b h_1}{a-1} : b = \frac{p_1}{p_0} \tag{57}$$

Of the system of transcendental Eqs. (52), (55), (55), the first equation remains invariable. The last two equations are substituted by (accounting for Eq. (57)):

$$\int_{h_1}^{h_1(0,t)} \frac{\mathrm{d}h_1}{(b+1)\left(h_1 - h_1^0\right) + f\left(h_1\right)} = \frac{p_0 x}{\chi_1 l},$$
$$\int_{h_1}^{h_1(0,t_0)} \frac{\left(k_1 h_1 + k_2 f(h_1)\right)}{(b+1)\left(h_1 - h_1^0\right) + f\left(h_1\right)} = \frac{p_0 j_1 k_1 U}{l s_{10}}(t - t_0)$$

Finally, Eq. (56) is substituted by

$$c(x,t) = -\frac{k_1 h_1(x,t) + k_2 f(h_1(x,t))}{p_0 j_{11} k_1 U h_1(x,t)} \frac{\partial h_1(0,t)}{\partial t} \frac{(b+1)\left(h_1(x,t) - h_1^0\right) + f\left(h_1(x,t)\right)}{(b+1)\left(h_1(0,t) - h_1^0\right) + f\left(h_1(0,t)\right)}$$

Appendix C Proof of the validity of the classical filtration equations for the model of Section 5

The goal of this Appendix is to prove the validity of the classical filtration model given by Eq. (44) (Section 5.4) in the framework of the simplified deep bed filtration model with one particle sizes and two pore sizes described in Section 5.1. In the proof, we will rely upon the exact solution obtained in Appendix B. As shown in Section 5.4, it is sufficient to proof that the pore concentrations h_1, h_2 are the functions of the suspended particle concentration σ. Moreover, since, in view of Eq. (50), h_2 is function of h_1, it is sufficient to prove that $h_1 = g(\sigma)$.

Let us find the retained particle concentration σ in terms of h_i. A captured particle transforms a large pore into a small pore and transforms a small pore into a zero-sized pore. Large pores do not contain retained particles. Each small pore contains one particle, and each zero-sized pore contains two retained particles. Therefore,

$$\sigma = \frac{h_2 + 2h_0}{l}.$$

(The multiplier $1/l$ is necessary, since the values of h_i are determined per unit surface, while σ is determined as the concentration per unit volume, cf. Eq. (10).) Accounting for Eqs. (50), (51), we obtain

$$\sigma = \frac{2h_1^0 - 2h_1 - h_2}{l} = \frac{2h_1^0 - 2h_1 - f\left(h_1\right)}{l}.$$

The large pore concentration can be expressed from the last equation by an inverse function

$$h_1 = g\left(\sigma\right).$$

This finishes the proof of the validity of the system (51) in the framework of the developed model for deep bed filtration.

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