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Acoustics of Rotating Deformable Saturated Porous Media

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Abstract. We investigate wave propagation in elastic porous media which are saturated by incompressible viscous Newtonian fluids when the porous media are in rotation with respect to a Galilean frame. The model is obtained by upscaling the flow at the pore scale. We use the method of multiple scale expansions which gives rigorously the macroscopic behaviour without any prerequisite on the form of the macroscopic equations. For Kibel numbers $\mathcal{O}(1)$, the acoustic filtration law resembles a Darcy's law, but with a conductivity which depends on the wave frequency and on the angular velocity. The bulk momentum balance shows new inertial terms which account for the convective and Coriolis accelerations. Three dispersive waves are pointed out. An investigation in the inertial flow regime shows that the two pseudo-dilatational waves have a cut-off frequency.

Key words: acoustics, rotating porous media, filtration law, wave propogation in porous media, homogenisation.

Nomenclature

Roman Letters

$\mathbf{a}, \mathbf{a}^e$	elastic tensors.
$\mathbf{A}$	hermitian matrix.
$\mathbf{B}_{\alpha\beta}, \alpha, \beta = 1, 2$	dissipation/inertia tensors.
Ek	Ekman number.
$\mathbf{H}$	resistivity tensor.
k	wavenumber.
$\mathbf{K}$	conductivity tensor.
Ki	Kibel number.
l	characteristic size of the pores.
L	macroscopic characteristic length.
p	pressure.
P, Q, R	dimensionless numbers.
Rt	transient Reynolds number.
$\mathbf{u}_s$	wave solid displacement.
$\mathbf{v}_f$	wave fluid velocity.

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V	periodic cell.
V_p	pore volume in the periodic cell.
V_s	solid part of the periodic cell.
W_1, W_2	waves in the empty rotating porous matrix.
W_1', W_2', W_3'	waves in the saturated rotating porous matrix.
$\mathbf{x}$	dimensionless macroscopic space variable.
$\mathbf{X}$	dimensional space variable.
$\mathbf{y}$	dimensionless microscopic space variable.
Greek Letters	
ε	small parameter of separation of scale.
Γ	pore surface.
μ	viscosity.
ω	wave frequency.
$\mathbf{\Omega}$	angular velocity vector.
ϕ	porosity.
ρ_f, ρ_s	fluid and solid densities, respectively.

1. Introduction

The aim of this paper is to investigate wave propagation in non-Galilean deformable porous matrices saturated by an incompressible viscous Newtonian fluid. Acoustics in non-Galilean porous matrix is concerned with numerous practical applications going from geological applications to industry. At our knowledge, this problem has not received much attention until now. Existing works about rotating porous media are concerned by the permanent flow law in rigid matrices, Vadasz (1993, 1997), Auriault *et al.* (2000, 2002), Geindreau *et al.* (2004). In these works, Darcy's law is shown to be affected by the angular rotation Ω: the conductivity becomes dependent on Ω.

The present work can be seen as an extension of Auriault (2004), where wave propagation in rotating isotropic elastic solids was investigated. Two dispersive waves W_1 and W_2, each of them being a combination of the classical dilatational and shear waves, were pointed out. Waves W_1 and W_2 tend to the dilatational and shear waves, respectively, as Ω tends to zero. Wave W_1 shows a cut-off frequency $\omega = \Omega$.

To obtain the wave propagation model, we use an upscaling technique, i.e. the method of multiple scale expansions to determine the macroscopic flow from its description at the pore scale. Heterogeneous system as for example porous media may be modelled by an equivalent macroscopic continuous system if the condition of separation of scales is verified (Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980)

$$\varepsilon = \frac{l}{L} \ll 1, \tag{1}$$

where l and L are the characteristic lengths of the heterogeneities and of the macroscopic sample or excitation, respectively. In the present case L

can be assimilated to the wavelength. The macroscopic equivalent model is obtained from the description at the heterogeneity scale by (Auriault, 1991): (i) assuming the medium to be periodic, without loss of generality; (ii) writing the local description in a dimensionless form; (iii) evaluating the dimensionless numbers with respect to the scale ratio ε; (iv) looking for the unknown fields in the form of asymptotic expansions in powers of ε; (v) solving the successive boundary-value problems that are obtained after introducing these expansions in the local dimensionless description. The macroscopic equivalent model is obtained from compatibility conditions which are the necessary conditions for the existence of solutions to the boundary-value problems. The main advantages of the method rely upon the possibility of: (a) avoiding prerequisites at the macroscopic scale; (b) modelling finite size macroscopic samples; (c) modelling macroscopically non-homogeneous media or phenomena; (d) modelling problems with several separations of scales; (e) modelling several simultaneous phenomena; (f) determining whether the system "medium+phenomena" is homogenisable or not; (g) providing the domains of validity of the macroscopic models.

In Section 2, we investigate the wave propagation model in an empty elastic porous medium. The influence of the angular rotation Ω is measured by the Kibel number $Ki = \omega/\Omega$. The study is conducted with $Ki = \mathcal{O}(1)$. The macroscopic model is as in Auriault (2004), with the elastic tensor being now the effective elastic tensor. Wave W_1 shows a cut-off frequency $\omega = \Omega$. The acoustic flow law in a rigid rotating porous medium is addressed in Section 3. The obtained model is an extension of both the Biot's acoustic flow law (Biot, 1956a,b) and the permanent flow law in a rotating porous medium, Auriault *et al.* (2000, 2002): the conductivity depends on ω and Ω. The model for saturated elastic porous media is presented in Section 4. The structure of the model is similar to Biot's model, (Biot, 1956a,b), with new inertial terms due to $\Omega \neq 0$ in the bulk momentum balance and an Ω-dependent conductivity. Finally, Section 5 is devoted to the study of wave propagation at high frequency, i.e. in the inertial regime of the flow law. In this case an analytical investigation is possible. We show that three dispersive waves W_1', W_2' and W_3' are present, which are combinations of Biot's waves P_1, P_2 and S. Waves W_1', W_2' and W_3' tend to waves P_1, P_2 and S, respectively, as Ω goes to zero. Waves W_1' and W_2' show a cut-off frequency $\omega = \Omega$.

2. Acoustics of an Empty Rotating Porous Medium

The considered porous medium is spatially periodic and consists of repeated unit cells (parallelepipeds), see Figure 1. There are two characteristic length scales in this problem: the characteristic microscopic length scale l of the

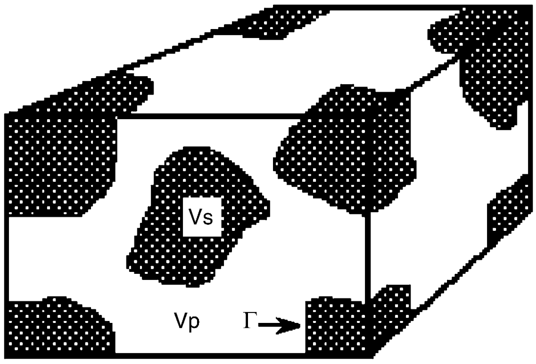

Figure 1. A period of the porous medium.

pores and of the unit cell, and the macroscopic length scale L that may be represented by the wavelength. Moreover we assume that the two length scales l and L are well separated

$$\frac{l}{L} = \varepsilon \ll 1 \tag{2}$$

The unit cell is denoted by V and is bounded by ∂V , the pore and the solid parts of the unit cell is denoted by V_p and V_s, respectively, and the fluid–solid interface inside the unit cell is Γ. The porous medium is rotating with respect to an inertial frame. To fix the ideas the porous medium is placed in the basket of a centrifuge of radius r which rotates at a constant angular velocity $\boldsymbol{\Omega} = \Omega \mathbf{e}_\Omega$, $\Omega =$ constant, see Figure 2.

2.1. LOCAL DESCRIPTION

For the sake of simplicity we consider that the material constituting the porous matrix is isotropic elastic. Relatively to the moving porous matrix frame, the momentum balance for the porous medium is

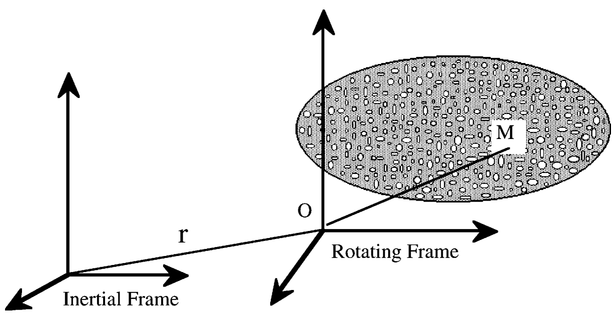

Figure 2. The rotating porous medium.

$$\mu_s \Delta_X \mathbf{U}_s + (\lambda_s + \mu_s)\nabla_X(\nabla_X \cdot \mathbf{U}_s) = \rho_s(\gamma_r + \gamma_e + \gamma_c) \quad \text{in } V_s, \tag{3}$$

where $\mathbf{U}_s$ is the displacement vector relative to the rotating frame, ρ_s is the density and λ_s and μ_s are the Lamé coefficients. The physical space variable is denoted $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3})$. Accelerations γ_r, γ_e and γ_c are the acceleration relative to the matrix frame, the convective and the Coriolis accelerations, respectively:

$$\gamma_r = \frac{\partial^2 \mathbf{U}_s}{\partial t^2}, \tag{4}$$

$$\gamma_e = \gamma(O) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{OM}), \tag{5}$$

$$\gamma_c = 2\mathbf{\Omega} \times \frac{\partial \mathbf{U}_s}{\partial t}, \tag{6}$$

where O is a fixed point in the investigated period and M is a current point in V_s. Equation (3) is completed by the stress free condition on Γ

$$\boldsymbol{\sigma}_s \cdot \mathbf{N} = (\lambda_s \nabla_X \cdot \mathbf{U}_s + 2\mu_s \mathbf{e}_X(\mathbf{U}_s)) \cdot \mathbf{N} = 0 \tag{7}$$

Due to the linearity of Equations (3) and (7), the displacement $\mathbf{U}_s$ can be decomposed into a permanent displacement $\mathbf{u}_p$ caused by the convective and the Coriolis accelerations γ_e and γ_c and the wave displacement $\mathbf{u}_s$

$$\mathbf{U}_s = \mathbf{u}_p + \mathbf{u}_s, \tag{8}$$

where $\mathbf{u}_s$ verifies

$$\mu_s \Delta_X \mathbf{u}_s + (\lambda_s + \mu_s)\nabla_x(\nabla_X \cdot \mathbf{u}_s)$$

$$= \rho_s \left(\frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}_s) + 2\mathbf{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t} \right) \quad \text{in } V_s, \tag{9}$$

$$= (\lambda_s \nabla_X \cdot \mathbf{u}_s + 2\mu_s \mathbf{e}_X(\mathbf{u}_s)) \cdot \mathbf{N} = 0 \quad \text{on } \Gamma. \tag{10}$$

2.2. DIMENSIONLESS LOCAL DESCRIPTION

The description (9–10) is driven by two dimensionless numbers. We use l as the characteristic length. These numbers are

$$Q_l = \frac{|\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2}|}{|\mu_s \Delta_X \mathbf{u}_s + (\lambda_s + \mu_s)\nabla_X(\nabla_X \cdot \mathbf{u}_s)|}$$

$$= \mathcal{O}\left(\frac{\rho_s \omega^2 l^2}{\lambda_s + 2\mu_s} \right) = \mathcal{O}\left(\frac{l^2}{L^2} \right) = \mathcal{O}\left(\varepsilon^2\right), \tag{11}$$

and the Kibel number Ki which appears in two ratios

$$\frac{|\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2}|}{|2\rho_s \mathbf{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t}|} = \mathcal{O}\left(\frac{\omega}{2\Omega} \right) = \mathcal{O}\left(\frac{Ki}{2} \right), \tag{12}$$

and

$$\frac{|2\rho_s \mathbf{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t}|}{|\rho_s \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}_s)|} = \mathcal{O}\left(\frac{2\omega}{\Omega}\right) = \mathcal{O}(2Ki), \tag{13}$$

where ω is the wave pulsation. To account for Coriolis effects, we will assume $Ki = \mathcal{O}(1)$. Smaller value $Ki = (\mathcal{O}(\varepsilon^p), p > 0$ are not investigated in the paper. For higher values $Ki = (\mathcal{O}(\varepsilon^p), p < 0$, Coriolis effects are negligible, and we recover the classical equation for elastic wave propagation in a Galilean frame. Finally, the dimensionless local description is in the following form, where all quantities are now dimensionless

$$\mu_s \Delta_y \mathbf{u}_s + (\lambda_s + \mu_s) \nabla_y (\nabla_y \cdot \mathbf{u}_s) = \varepsilon^2 \rho_s \left(\frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}_s) + 2\mathbf{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t} \right) \quad \text{in } V_s, \tag{14}$$

$$= (\lambda_s \nabla_y \cdot \mathbf{u}_s + 2\mu_s \mathbf{e}_y(\mathbf{u}_s)) \cdot \mathbf{N} = 0 \quad \text{on } \Gamma, \tag{15}$$

where $\mathbf{y} = \mathbf{X}/l$.

2.3. MACROSCOPIC MODELLING

The two characteristic lengths introduce two dimensionless space variables, i.e. the macroscopic dimensionless space variable $\mathbf{x} = \mathbf{X}/L$ and the local dimensionless space variable $\mathbf{y} = \mathbf{X}/l$, where $\mathbf{X}$ is the physical space variable. The displacement $\mathbf{u}_s$ is looked for in the form of the asymptotic expansion

$$\mathbf{u}_s = \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon^2 \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}, t) + \cdots, \tag{16}$$

where the different terms in the asymptotic expansion are V-periodic in $\mathbf{y}$. Introducing this expansion into Equations (14 and 15) and extracting like power terms in ε yield different boundary value problems to be investigated. By comparing the set (14 and 15) with the similar problem in non-rotating porous media, see e.g. (Auriault, 1997), it results that the difference concerns the three last terms in the right hand member of Equation (14). Since these terms are $\mathcal{O}(\varepsilon^2)$, it is easy to check by following the route in (Auriault, 1997) that the first term $\mathbf{u}^{(0)}$ verifies the dimensionless macroscopic relation

$$\nabla_x \cdot (\mathbf{a}^e \mathbf{e}_x(\mathbf{u}^{(0)})) = \rho_s^e \left(\frac{\partial^2 \mathbf{u}^{(0)}}{\partial t^2} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}^{(0)}) + 2\mathbf{\Omega} \times \frac{\partial \mathbf{u}^{(0)}}{\partial t} \right), \tag{17}$$

where $\mathbf{u}^{(0)}$ is independent of the local space variable $\mathbf{y}$, which means that the displacement is a rigid displacement at the pore scale and at the first order of approximation. Tensor $\mathbf{a}^e$ is the effective elastic tensor determined as in the static case, $\rho_s^e = <\rho_s>$ is the effective density, i.e. the volume average of the density ρ_s, and $\mathbf{e}_x$ is now the macroscopic deformation tensor. Finally the dimensionless macroscopic wave behaviour is in the form

$$\nabla_x \cdot (\mathbf{a}^e \mathbf{e}_x(\mathbf{u}_s)) = \rho_s^e \left(\frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}_s) + 2\mathbf{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t} \right) + \mathcal{O}(\varepsilon). \tag{18}$$

The above macroscopic model (18) is similar to the wave model for non-porous rotating elastic media, (Auriault, 2004). Let us recall the main features about wave propagation in rotating isotropic elastic media.

2.4. WAVES IN ROTATING ISOTROPIC ELASTIC MEDIA

Consider an isotropic medium and waves at constant frequency ω. The vibrational displacement $\mathbf{u}_s$ verifies at the first order of approximation the wave equation

$$\mu_s^e \Delta_x \mathbf{u}_s + (\lambda_s^e + \mu_s^e)\mathbf{grad}_x \mathrm{div}_x \mathbf{u}_s = \rho_s^e \left[-\omega^2 \mathbf{u}_s + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}_s) + 2\mathbf{\Omega} \times i\omega \mathbf{u}_s \right], \tag{19}$$

where λ_s^e and μ_s^e are the effective Lamé coefficients and $i^2 = -1$. Equation (19) is an approximation: for simplicity the term $\mathcal{O}(\varepsilon)$ in (18) has been suppressed. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the rotating orthonormal basis with $\mathbf{\Omega} = \Omega \mathbf{e}_3$. A perturbation $\mathbf{u}_s$ that is colinear to $\mathbf{\Omega}$ is not affected by Coriolis acceleration. Therefore, we limit the analysis to wave displacements in the plane $(\mathbf{e}_1, \mathbf{e}_2)$. By applying successively the divergence and the curl operators to Equation (19), and after introducing $e = \mathrm{div}_x \mathbf{u}_s$ and $\mathbf{w} = \nabla_x \times \mathbf{u}_s = w_3 \mathbf{e}_3$, we obtain two coupled wave equations for e and w_3:

$$\left(\lambda_s^e + 2\mu_s^e\right) \Delta_x e = -\rho_s^e \left[\left(\omega^2 + \Omega^2\right) e + 2i\omega\Omega w_3 \right], \tag{20}$$

$$\mu_s^e \Delta_x w_3 = -\rho_s^e \left[\left(\omega^2 + \Omega^2\right) w_3 - 2i\omega\Omega e \right]. \tag{21}$$

Therefore, in contrast to wave propagation in an inertial medium, dilatational and shear waves do not propagate separately. The coupling appears in the right hand member which introduces a tensorial density $\rho_s^e \mathbf{A}$ with

$$\mathbf{A} = \begin{pmatrix} 1 + (\Omega/\omega)^2 & 2i\frac{\Omega}{\omega} & 0 \\ -2i\frac{\Omega}{\omega} & 1 + (\Omega/\omega)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

Matrix $\mathbf{A}$ is hermitian, ${}^t\mathbf{A} = \tilde{\mathbf{A}}$ (its transpose equals its complex conjugate). Therefore, it has two real eigenvalues $\sigma_1 = (1 - \Omega/\omega)^2$ and $\sigma_2 = (1 + \Omega/\omega)^2$. Remark that σ_1 cancels out for $\Omega = \omega$. When $\mathbf{\Omega}$ cancels out, we recover the uncoupled wave equations for propagation in an inertial isotropic elastic medium. Consider waves that propagate in the direction $\mathbf{e}_1$ of the form

$$e = A_1 \exp[ikx_1], \quad w_3 = A_2 \exp[ikx_1]. \tag{23}$$

Introducing these expressions into relations (20) and (21) gives two equations for the amplitudes A_1 and A_2:

$$(\lambda_s^e + 2\mu_s^e)k^2 A_1 = \rho_s^e(\omega^2 + \Omega^2)A_1 + 2i\rho_s^e \Omega\omega A_2, \tag{24}$$

$$\mu_s^e k^2 A_2 = \rho_s^e(\omega^2 + \Omega^2)A_2 - 2i\rho_s^e \Omega\omega A_1. \tag{25}$$

The existence of non trivial solutions for A_1 and A_2 yields the dispersion equation

$$\mu_s^e(\lambda_s^e + 2\mu_s^e)k^4 - \rho_s^e(\omega^2 + \Omega^2)(\lambda_s^e + 3\mu_s^e)k^2 + \rho_s^{e2}(\omega^2 - \Omega^2)^2 = 0. \tag{26}$$

Equation (26) admits two body waves W_1 and W_2 of wavenumbers k_1 and k_2, respectively. We have

$$\left(k_{\substack{1\\2}}\right)^2 = \frac{\rho_s^e\omega^2(\lambda_s^e + 3\mu_s^e)}{2\mu_s^e(\lambda_s^e + 2\mu_s^e)}\left(1 + \frac{\Omega^2}{\omega^2}\right)\left(1 \mp \sqrt{1 - \frac{4\mu_s^e(\lambda_s^e + 2\mu_s^e)}{(\lambda_s^e + 3\mu_s^e)^2}(\frac{\omega^2 - \Omega^2}{\omega^2 + \Omega^2})^2}\right). \tag{27}$$

Waves W_1 and W_2 are dispersive waves. As Ω goes to zero, we recover the classical non-dispersive elastic waves, the dilatational wave and the shear wave of speed c_d and c_s, respectively:

$$\lim_{\Omega \to 0} k_1^2 = k_d^2 = \frac{\rho_s^e\omega^2}{\lambda_s^e + 2\mu_s^e}, \quad c_d = \sqrt{\frac{\lambda_s^e + 2\mu_s^e}{\rho_s^e}}, \tag{28}$$

$$\lim_{\Omega \to 0} k_2^2 = k_s^2 = \frac{\rho_s^e\omega^2}{\mu_s^e}, \qquad c_s = \sqrt{\frac{\mu_s^e}{\rho_s^e}}. \tag{29}$$

The group velocity is defined by $c_1^{\mathrm{gr}} = \partial\omega/\partial k$. Typical dimensionless group velocities $c_1^{\mathrm{gr}*} = c_1^{\mathrm{gr}}/c_d$ and $c_2^{\mathrm{gr}*} = c_2^{\mathrm{gr}}/c_s$ of waves W_1 and W_2, respectively, are shown versus the Kibel number Ki in Figures 3 and 4 for different values of the Poisson's ratio η. Wave W_1 shows a cut-off frequency $\omega = \Omega$ for $\omega < \Omega$, the group velocity is negative and wave W_1 do not propagate. The corresponding wavenumber k_1 cancels out for $\omega = \Omega$. As $\omega \to \Omega$, the phase velocity c_1^{ph} of wave W_1 goes to ∞.

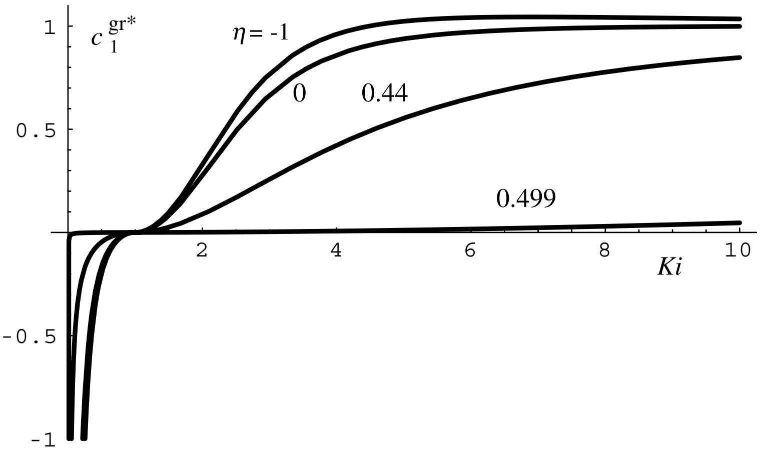

Figure 3. Dimensionless group velocity of wave W_1 versus the Kibel number Ki for different values of the Poisson's ratio η.

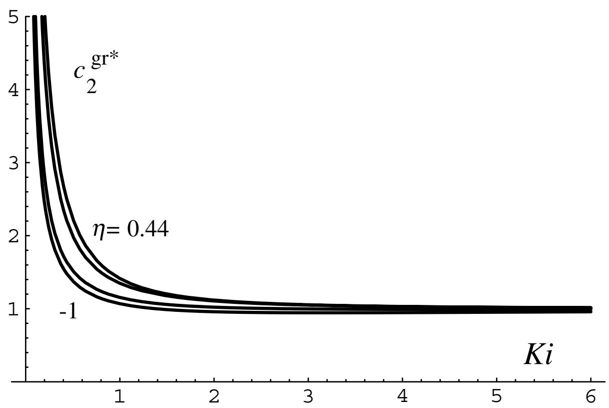

Figure 4. Dimensionless group velocity of wave W_2 versus the Kibel number Ki for different values of the Poisson's ratio η.

3. Generalised Acoustical Seepage Law in Rotating Rigid Porous Media

Consider now the small perturbation of an incompressible liquid flowing through a rigid porous medium which is placed in the basket of a centrifuge at constant angular velocity Ω. The porous medium is shown in Figures 1 and 2. The problem shows again two well separated characteristic length scales: the characteristic microscopic length scale l of the pores and of the unit cell, and the macroscopic length scale L that may be represented by the macroscopic pressure drop scale, with $l/L = \varepsilon \ll 1$.

3.1. LOCAL DESCRIPTION

We assume the perturbation small enough so as to neglect non-linear terms. Relatively to the moving porous matrix frame, the momentum balance for the incompressible viscous Newtonian liquid is

$$\mu_f \Delta_X \mathbf{V}_f - \nabla_X p_f = \rho_f(\gamma_r + \gamma_e + \gamma_c) \quad \text{in } V_p, \tag{30}$$

where $\mathbf{V}_f$ is the fluid velocity vector relative to the matrix frame, p_f is the pressure, ρ_f is the density and μ_f is the viscosity. Gravitational acceleration is included in the pressure term. Accelerations γ_r, γ_e and γ_c are the acceleration relative to the matrix frame, the convective and the Coriolis accelerations, respectively:

$$\gamma_r = \frac{\partial \mathbf{V}_f}{\partial t}, \tag{31}$$

$$\gamma_e = \gamma(O) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{OM}), \tag{32}$$

$$\gamma_c = 2\mathbf{\Omega} \times \mathbf{V}_f. \tag{33}$$

Equation (30) is completed by the incompressibility condition and the adherence condition on Γ

$$\nabla_X \cdot \mathbf{V}_f = 0 \quad \text{in } V_p, \tag{34}$$

$$\mathbf{V}_f = 0 \quad \text{on } \Gamma. \tag{35}$$

Due to the linearity of Equations (30–35), the velocity $\mathbf{V}_f$ and the pressure p_f can be decomposed into a permanent velocity $\mathbf{v}_p$ and a permanent pressure p_p caused by the convective and the Coriolis accelerations γ_e and γ_c and the acoustic wave velocity $\mathbf{v}_f$ and pressure p

$$\mathbf{V}_f = \mathbf{v}_p + \mathbf{v}_f, \quad p_f = p_p + p. \tag{36}$$

The quasi-static flow v_p is described at the macroscopic level by a generalized Darcy's law which was investigated in (Auriault *et al.*, 2000, 2002). The acoustic velocity $\mathbf{v}_f$ and pressure p verify

$$\mu_f \Delta_X \mathbf{v}_f - \nabla_X p = \rho_f \left(\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}_f) + 2\mathbf{\Omega} \times \mathbf{v}_f \right) \quad \text{in } V_p, \tag{37}$$

$$\nabla_X \cdot \mathbf{v}_f = 0 \quad \text{in } V_p, \tag{38}$$

$$\mathbf{v}_f = 0 \quad \text{on } \Gamma, \tag{39}$$

where $\mathbf{u}_f$ is the displacement of the fluid wave perturbation.

3.2. DIMENSIONLESS LOCAL DESCRIPTION

As in part 1, we use the local length scale of a pore l as the characteristic length scale for the variations of the differential operators. The perturbation is caused by a macroscopic perturbation $\mathcal{O}(p/L)$ of the gradient of pressure which is equilibrated by the viscous forces $\mathcal{O}(\mu_f v_f/l^2)$ and the transient inertia forces $\mathcal{O}(\rho_f \omega v_f)$

$$\frac{p}{L} = \mathcal{O}\left(\frac{\mu_f v_f}{l^2}\right) = \mathcal{O}(\rho_f \omega v_f), \tag{40}$$

where ω is the wave pulsation. Therefore, we have the two following estimations of dimensionless numbers

$$P_l = \frac{|\nabla_X p|}{|\mu_f \Delta_X \mathbf{v}_f|} = \mathcal{O}\left(\frac{pl}{\mu_f v_f}\right) = \mathcal{O}(\varepsilon^{-1}), \tag{41}$$

and the local transient Reynold's number is evaluated as

$$Rt_l = \frac{|\rho_f \frac{\partial \mathbf{v}_f}{\partial t}|}{|\mu_f \Delta_X \mathbf{v}_f|} = \mathcal{O}\left(\frac{\rho_f \omega l^2}{\mu_f}\right) = \mathcal{O}(1). \tag{42}$$

For consistency, we adopt the same estimation of the Kibel number as in part 2

$$Ki = \frac{\omega}{\Omega} = \mathcal{O}(1). \tag{43}$$

Notice that the related local Ekman number Ek_l is also $\mathcal{O}(1)$

$$Ek_l = \frac{|\mu_f \Delta_X \mathbf{v}_f|}{|2\rho_f \boldsymbol{\Omega} \times \mathbf{v}_f|} = \mathcal{O}\left(\frac{\mu_f}{2\rho_f \Omega l^2}\right) = \mathcal{O}\left(\frac{Ki}{2Rt_l}\right) = \mathcal{O}(1). \tag{44}$$

Finally, the dimensionless local description, where for the sake of simplicity notations are kept unchanged, is in the form

$$\mu_f \Delta_y \mathbf{v}_f - \varepsilon^{-1} \nabla_y p = \rho_f \left(\frac{\partial \mathbf{v}_f}{\partial t} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u}_f) + 2\boldsymbol{\Omega} \times \mathbf{v}_f\right) \quad \text{in } V_p, \tag{45}$$

$$\nabla_y \cdot \mathbf{v}_f = 0 \quad \text{in } V_p, \tag{46}$$

$$\mathbf{v}_f = 0 \quad \text{on } \Gamma, \tag{47}$$

3.3. MACROSCOPIC MODELLING

The next step is to introduce multiple scale coordinates $\mathbf{x}$ and $\mathbf{y}$, and to look for the velocity $\mathbf{v}_f$ and the pressure p in the form of asymptotic expansions of powers of ε

$$\mathbf{v}_f = \mathbf{v}^{(0)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon \mathbf{v}^{(1)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon^2 \mathbf{v}^{(2)}(\mathbf{x}, \mathbf{y}, t) + \cdots, \tag{48}$$

$$p = p^{(0)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon p^{(1)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon^{(2)} p^2(\mathbf{x}, \mathbf{y}, t) + \cdots, \tag{49}$$

where the different terms in the asymptotic expansion are V-periodic in $\mathbf{y}$. Substituting these expansions in the set (45)–(47) gives, by identification of the like powers of ε, successive boundary value problems to be investigated. The lowest order approximation of the pressure verifies

$$\nabla_y p^{(0)} = 0, \quad p^{(0)} = p^{(0)}(\mathbf{x}, t). \tag{50}$$

Consider waves at constant pulsation ω. The first order approximation of the velocity $\mathbf{v}^{(0)}$ and the second order approximation of the pressure $p^{(1)}$ are determined by the following set

$$\mu_f \Delta_y \mathbf{v}^{(0)} - \nabla_x p^{(0)} - \nabla_y p^{(1)} = \rho_f \left(i\omega \mathbf{v}^{(0)} + \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \frac{\mathbf{v}^{(0)}}{i\omega} \right) + 2\mathbf{\Omega} \times \mathbf{v}^{(0)} \right) \quad \text{in } V_p, \tag{51}$$

$$\nabla_y \cdot \mathbf{v}^{(0)} = 0 \quad \text{in } V_p, \tag{52}$$

$$\mathbf{v}^{(0)} = 0 \quad \text{on } \Gamma. \tag{53}$$

The set (51)–(53) can be investigated as in Levy (1979), where $\Omega = 0$. We first remark that Equation (51) shows a tensorial density $\rho_f \mathbf{A}$ where the hermitian matrix $\mathbf{A}$ is defined in (22). Let us consider the Hilbert space $\mathcal{V}$ of all V-periodic irrotational complex vectors satisfying the boundary condition (53), with the scalar product (this is a consequence of the hermitian character of $\mathbf{A}$)

$$(\mathbf{u}, \mathbf{v})_{\mathcal{V}} = \int_{V_p} \left(\mu_f \frac{\partial v_k}{\partial y_j} + \frac{\partial \tilde{v}_k}{\partial y_j} + \rho_f u_j A_{jk} \tilde{v}_k \right) \mathrm{d}V_p, \quad \|\mathbf{u}\|^2_{\mathcal{V}} = (\mathbf{u}, \mathbf{u})_{\mathcal{V}}, \tag{54}$$

where $\tilde{v}_k$ is the complex conjugate of v_k. Multiplying (51) by test function $\tilde{\mathbf{u}}$ of $\mathcal{V}$, and integrating by parts on V_p yields the weak formulation

$$\forall \mathbf{u} \in \mathcal{V}, \quad \int_{V_p} \left(\mu_f \frac{\partial v_k^{(0)}}{\partial y_j} \frac{\partial \tilde{v}_k}{\partial y_j} + i\omega \rho_f \tilde{u}_k A_{kj} v_j^{(0)} \right) \mathrm{d}V_p = \int_{V_p} \frac{\partial p^{(0)}}{\partial x_k} \tilde{u}_k \mathrm{d}V_p. \tag{55}$$

The existence and uniqueness of $\mathbf{v}^{(0)}$ follows from the Lax-Milgram lemma. It appears that the velocity $\mathbf{v}^{(0)}$ is a linear vectorial function of the macroscopic gradient of pressure $\nabla_x p^{(0)}$

$$\mathbf{v}^{(0)} = -\mathbf{k}(\mathbf{y}, \omega, \mathbf{\Omega}) \nabla_x p^{(0)}, \tag{56}$$

where the tensor field $\mathbf{k}$ depends on ω and $\mathbf{\Omega}$.

Consider now the volume balance (46). It gives at the second order

$$\nabla_y \cdot \mathbf{v}^{(1)} + \nabla_x \cdot \mathbf{v}^{(0)} \quad \text{in } V_p. \tag{57}$$

By integrating over V_p, we obtain

$$\nabla_x \cdot \langle \mathbf{v}^{(0)} \rangle = 0, \quad \langle \mathbf{v}^{(0)} \rangle = -\mathbf{K}(\omega, \mathbf{\Omega}) \nabla_x p^0,$$

$$\langle \mathbf{v}^{(0)} \rangle = \frac{1}{V} \int_{V_p} \mathbf{v}^{(0)} \mathrm{d}V_p, \quad \mathbf{K} = \frac{1}{V} \int_{V_p} \mathbf{k} \mathrm{d}V_p. \tag{58}$$

Finally, the dimensionless macroscopic behaviour is at the first order of approximation in the form

$$\nabla_x \cdot \langle \mathbf{v}_f \rangle = \mathcal{O}(\varepsilon), \quad \langle \mathbf{v}_f \rangle = -\mathbf{K}(\omega, \mathbf{\Omega}) \nabla_x p + \mathcal{O}(\varepsilon). \tag{59}$$

Remarks are as follows

- The filtration tensor $\mathbf{K}$ depends on both the wave pulsation ω and the rotational velocity $\mathbf{\Omega}$. It is generally not symmetric.
- Model (59) is an approximation since $\mathbf{v}_f = \mathbf{v}^{(0)} + \mathcal{O}(\varepsilon)$.
- Model (59) is valid in the range $\varepsilon \ll Ek_l \ll \varepsilon^{-1}, \varepsilon \ll Ki \ll \varepsilon^{-1}$ which can be quite large if the separation of scales is good.
- Consider a porous medium which is macroscopically isotropic, of conductivity $K(\omega, \mathbf{0})$ in absence of rotation. When submitted to a rotation of axis X_3, tensor $\mathbf{K}$ takes the following form in the frame (X_1, X_2, X_3).

$$\mathbf{K} = \begin{pmatrix} K_{11}(\omega, \mathbf{\Omega}) & -K_{21}(\omega, \mathbf{\Omega}) & 0 \\ K_{21}(\omega, \mathbf{\Omega}) & K_{11}(\omega, \mathbf{\Omega}) & 0 \\ 0 & 0 & K(\omega, \mathbf{0}) \end{pmatrix}. \tag{60}$$

Equation (59_2) is a momentum balance which can be rewritten in the form

$$\nabla_x p = -\mathbf{H}(\omega, \mathbf{\Omega}) \langle \mathbf{v}_f \rangle + \mathcal{O}(\varepsilon), \tag{61}$$

where the complex valued tensor $\mathbf{H} = \mathbf{H}_1 + i\mathbf{H}_2 = \mathbf{K}^{-1}$. Its real part $\mathbf{H}_1$ is a dissipation density whereas its imaginary part $\mathbf{H}_2$ is an inertial density. Two flow regimes can be distinguished. For $\omega < \omega_c = \mu_f / \rho_f l^2$, viscous effects dominate and for $\omega > \omega_c$, inertial effects are preponderent. An approximation of $\mathbf{H}$ for the inertial regime can be easily obtained by neglecting the viscous term in (51)

$$-\nabla_x p^{(0)} - \nabla_y p^{(1)} = \rho_f \left(i\omega \mathbf{v}^{(0)} + \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \frac{\mathbf{v}^{(0)}}{i\omega} \right) + 2\mathbf{\Omega} \times \mathbf{v}^{(0)} \right). \tag{62}$$

After volume averaging this equation becomes

$$-\nabla_x \phi p^{(0)} = \rho_f \left(i\omega \langle \mathbf{v}^{(0)} \rangle + \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \frac{\langle \mathbf{v}^{(0)} \rangle}{i\omega} \right) + 2\Omega \times \langle \mathbf{v}^{(0)} \rangle \right), \tag{63}$$

which yields

$$\mathbf{H} = \frac{i\omega\rho_f}{\phi} \mathbf{A}, \tag{64}$$

where $\mathbf{A}$ is defined in (22).

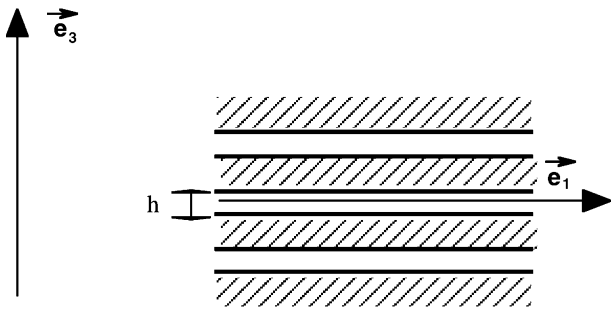

Figure 5. Flat pore.

3.4. PLANE FISSURES

Consider a porous system which is composed of plane fissures of thickness h and parallel to plane (X_1, X_2) and the rotation $\mathbf{\Omega e}_3$, see Figure 5. Consider a macroscopic gradient of pressure $\nabla_x p^{(0)} = G_1 \mathbf{e}_1$ in the $\mathbf{e}_1$ direction. Due to the particular boundary value problem at stake, velocity $\mathbf{v}^{(0)} = (v_1(y_3), v_2(y_3), 0)$ verifies

$$\mu_f \frac{d^2 v_1}{dy_3^2} - G_1 = i\omega\rho_f \left(\left(1 + \left(\frac{\Omega}{\omega} \right)^2 \right) v_1 + 2i \frac{\Omega}{\omega} v_2 \right), \tag{65}$$

$$\mu_f \frac{d^2 v_2}{dy_3^2} = i\omega\rho_f \left(-2i \frac{\Omega}{\omega} v_1 + \left(1 + \left(\frac{\Omega}{\omega} \right)^2 \right) v_2 \right), \tag{66}$$

$$\mathbf{v}^{(0)} \left(\pm \frac{h}{2} \right) = 0 \tag{67}$$

When $\Omega = 0$, the solution is (Auriault *et al.*, 1985)

$$v_1 = -k(\omega) G_1, \quad v_2 = 0, \tag{68}$$

which yields the conductivity tensor

$$\mathbf{K}(\Omega = 0) = \begin{pmatrix} K(\omega) & 0 & 0 \\ 0 & K(\omega) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{69}$$

$$K(\omega) = \frac{\phi}{i\omega\rho_f} \left(1 - \frac{2\tanh(h(i\omega/\nu)^{1/2}/2)}{h(i\omega/\nu)^{1/2}} \right), \tag{70}$$

where ϕ is the porosity and $\nu = \mu_f/\rho_f$. When $\Omega \neq 0$, it is convenient to rewrite system (65)–(66) in the principal axis of matrix **A**

$$\mathbf{e}'_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + i\mathbf{e}_2), \quad \mathbf{e}'_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - i\mathbf{e}_2), \quad \mathbf{e}'_3 = \mathbf{e}_3, \tag{71}$$

$$v'_1 = \frac{1}{\sqrt{2}}(v_1 - iv_2), \quad v'_2 = \frac{1}{\sqrt{2}}(v_1 + iv_2), \quad v'_3 = 0. \tag{72}$$

That gives

$$\mu_f \frac{\mathrm{d}^2 v'_1}{\mathrm{d}y_3^2} - \frac{G_1}{\sqrt{2}} = i\omega\rho_f \left(1 - \frac{\Omega}{\omega}\right)^2 v'_1, \tag{73}$$

$$\mu_f \frac{\mathrm{d}^2 v'_2}{\mathrm{d}y_3^2} - \frac{G_1}{\sqrt{2}} = i\omega\rho_f \left(1 + \frac{\Omega}{\omega}\right)^2 v'_2, \tag{74}$$

$$\mathbf{v}'\left(\pm\frac{h}{2}\right) = 0. \tag{75}$$

Equations for v'_1 and v'_2 are uncoupled. By analogy with (68), we obtain

$$\langle v'_1 \rangle = -\frac{K\left(\omega\left(1 - \frac{\Omega}{\omega}\right)^2\right)}{\sqrt{2}} G_1, \quad \langle v'_2 \rangle = -\frac{K\left(\omega\left(1 + \frac{\Omega}{\omega}\right)^2\right)}{\sqrt{2}} G_1. \tag{76}$$

Finally, by returning to axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, we obtain

$$K_{11}(\omega, \Omega) = \frac{1}{2}\left(K\left(\omega\left(1 - \frac{\Omega}{\omega}\right)^2\right) + K\left(\omega\left(1 + \frac{\Omega}{\omega}\right)^2\right)\right), \tag{77}$$

$$K_{21}(\omega, \Omega) = \frac{i}{2}\left(K\left(\omega\left(1 - \frac{\Omega}{\omega}\right)^2\right) - K\left(\omega\left(1 + \frac{\Omega}{\omega}\right)^2\right)\right). \tag{78}$$

Due to the isotropy in plane (X_1, X_2), the permeability tensor is the form

$$\mathbf{K} = \begin{pmatrix} K_{11}(\omega, \boldsymbol{\Omega}) & -K_{21}(\omega, \boldsymbol{\Omega}) & 0 \\ K_{21}(\omega, \boldsymbol{\Omega}) & K_{11}(\omega, \boldsymbol{\Omega}) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{79}$$

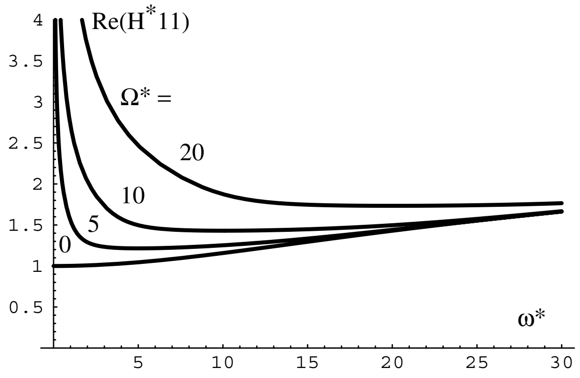

Figure 6. Dimensionless real part of H_{11} versus dimensionless frequency ω^* for different rotations Ω.

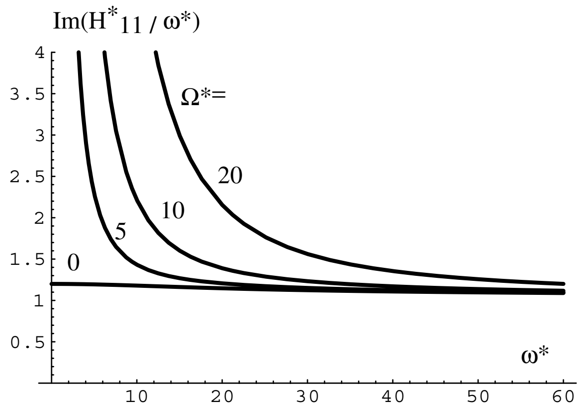

Figure 7. Dimensionless imaginary part of H_{11}/ω versus dimensionless frequency ω^* for different rotations Ω^*.

Real and imaginary parts of the components of $\mathbf{H}(\omega, \Omega) = \mathbf{K}^{-1}$ are given in Figures 6–9. The dimensionless quantities in use in these Figures are

$$\Omega^* = \frac{h^2 \rho_f}{4\mu_f}\Omega, \quad \omega^* = \frac{h^2 \rho_f}{4\mu_f}\omega, \quad \mathrm{Re}(H_{11}^*) = K(0)\mathrm{Re}(H_{11}), \tag{80}$$

$$\mathrm{Re}(H_{12}^*) = K(0)\mathrm{Re}(H_{12}), \quad K(0) = \frac{\phi h^2}{12\mu}, \tag{81}$$

$$\mathrm{Im}\left(\frac{H_{11}^*}{\omega^*}\right) = \frac{\phi}{\rho_f}\mathrm{Im}\left(\frac{H_{11}}{\omega}\right), \quad \mathrm{Im}\left(\frac{H_{12}^*}{\omega^*}\right) = \frac{\phi}{\rho_f}\mathrm{Im}\left(\frac{H_{12}}{\omega}\right). \tag{82}$$

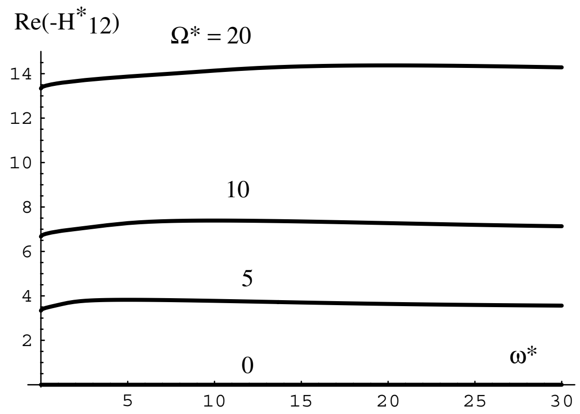

Figure 8. Dimensionless real part of $-H_{12}$ versus dimensionless frequency ω^* for different rotations Ω^*.

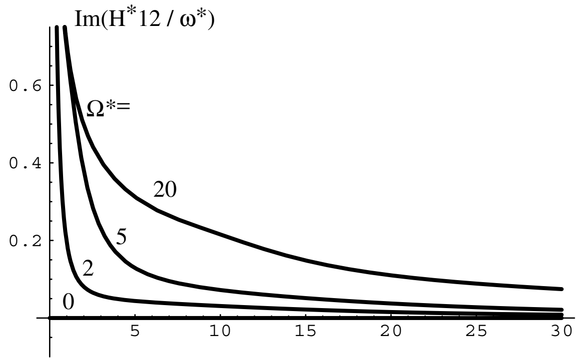

Figure 9. Dimensionless imaginary part of H_{12}/ω versus dimensionless frequency ω^* for different rotations Ω^*.

4. Acoustics of Saturated Rotating Elastic Porous Media

We now investigate the macroscopic behaviour of an elastic porous media saturated by an incompressible Newtonian liquid under an acoustic excitation, when the system is submitted to a constant angular velocity Ω.

4.1. Pore scale description

The pore scale description is given by (9) for the solid matrix and (37)–(38) for the liquid, with the continuity of the normal stress and of the displacement on the pore surface Γ

$$\mu_s \Delta_X \mathbf{u}_s + (\lambda_s + \mu_s)\boldsymbol{\nabla}_X(\boldsymbol{\nabla}_X \cdot \mathbf{u}_s)$$
$$= \rho_s \left(\frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u}_s) + 2\boldsymbol{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t} \right) \quad \text{in } V_s, \tag{83}$$

$$\mu_f \Delta_X \mathbf{v}_f - \boldsymbol{\nabla}_X p = \rho_f \left(\frac{\partial \mathbf{v}_f}{\partial t} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u}_f) + 2\boldsymbol{\Omega} \times \mathbf{v}_f \right) \quad \text{in } V_p, \tag{84}$$

$$\boldsymbol{\nabla}_X \cdot \mathbf{v}_f = 0 \quad \text{in } V_p, \tag{85}$$

$$(\lambda_s \boldsymbol{\nabla}_X \cdot \mathbf{u}_s + 2\mu_s \mathbf{e}_X(\mathbf{u}_s)) \cdot \mathbf{N} = (-p\mathbf{I} + 2\mu_f \mathbf{e}_X(\mathbf{v}_f)) \cdot \mathbf{N} \quad \text{on } \Gamma, \tag{86}$$

$$\mathbf{u}_f = \mathbf{u}_s \quad \text{on } \Gamma, \tag{87}$$

4.2. DIMENSIONLESS PORE SCALE DESCRITION

We use the estimations that were adopted in parts 2 and 3

$$Q_l = \frac{|\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2}|}{|\mu_s \Delta_X \mathbf{u}_s + (\lambda_s + \mu_s)\boldsymbol{\nabla}_X(\boldsymbol{\nabla}_X \cdot \mathbf{u}_s)|} = \mathcal{O}(\varepsilon^2), \tag{88}$$

$$\frac{|\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2}|}{|2\rho_s \boldsymbol{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t}|} = \mathcal{O}\left(\frac{|2\rho_s \boldsymbol{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t}|}{|\rho_s \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u}_s)|} \right) = \mathcal{O}(1), \tag{89}$$

$$P_l = \frac{|\boldsymbol{\nabla}_X p|}{|\mu_f \Delta_X \mathbf{v}_f|} = \mathcal{O}(\varepsilon^{-1}), \tag{90}$$

$$Rt_l = \frac{|\rho_f \frac{\partial \mathbf{v}_f}{\partial t}|}{|\mu_f \Delta_X \mathbf{v}_f|} = \mathcal{O}(1), \quad Ki = \frac{\omega}{\Omega} = \mathcal{O}(1). \tag{91}$$

We also assume that macroscopic stresses and displacements of the solid and the liquid are of similar orders of magnitude

$$\frac{|\boldsymbol{\sigma}_s| L}{|\boldsymbol{\sigma}_f| L} = \mathcal{O}(1), \quad \frac{|\mathbf{u}_s|}{|\mathbf{u}_f|} = \mathcal{O}(1). \tag{92}$$

In absence of medium rotation, $\Omega = 0$, estimations (89) and $(91)_2$ become unnecessary and we recover estimations that yield to the two-phase Biot's acoustics behaviour (Biot, 1956a,b, 1962). In presence of a medium rotation, the dimensionless pore scale description takes the form (notations are leaved unchanged)

$$\mu_s \Delta_y \mathbf{u}_s + (\lambda_s + \mu_s)\nabla y(\nabla_y \cdot \mathbf{u}_s) \\ = \varepsilon^2 \rho_s \left(\frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u}_s) + 2\boldsymbol{\Omega} \times \frac{\partial \mathbf{u}_s}{\partial t} \right) \quad \text{in } V_s, \tag{93}$$

$$\mu_f \Delta_y \mathbf{v}_f - \varepsilon^{-1} \nabla_y p = \rho_f \left(\frac{\partial \mathbf{v}_f}{\partial t} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u}_f) + 2\boldsymbol{\Omega} \times \mathbf{v}_f \right) \quad \text{in } V_p, \tag{94}$$

$$\nabla_y \cdot \mathbf{v}_f = 0 \quad \text{in } V_p, \tag{95}$$

$$\left(\lambda_s \nabla_y \cdot \mathbf{u}_s + 2\mu_s \mathbf{e}(\mathbf{u}_s)\right) \cdot \mathbf{N} = -(-p\mathbf{I} + 2\varepsilon \mu_f \mathbf{e}_y(\mathbf{v}_f)) \cdot \mathbf{N} \quad \text{on } \Gamma, \tag{96}$$

$$\mathbf{u}_f = \mathbf{u}_s \quad \text{on } \Gamma. \tag{97}$$

4.3. MACROSCOPIC MODELLING

The displacement $\mathbf{u}_s$, the velocity $\mathbf{v}_f$ and the pressure p are looked for in the form of the asymptotic expansion

$$\mathbf{u}_s = \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon^2 \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}, t) + \cdots, \tag{98}$$

$$\mathbf{v}_f = \mathbf{v}^{(0)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon \mathbf{v}^{(1)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon^2 \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}, t) + \cdots, \tag{99}$$

$$p = p^{(0)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon p^{(1)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon^{(2)} p^2(\mathbf{x}, \mathbf{y}, t) + \cdots, \tag{100}$$

where the different terms in the asymptotic expansion are V-periodic in $\mathbf{y}$. The process is quite similar to those described in different works (Levy, 1979; Auriault, 1980; Burridge and Keller, 1981; Auriault, 1997), where the two-phase Biot model is demonstrated by upscaling. It is easy to show, by following the homogenisation process in these papers (see particularly Auriault (1997)) and the results in parts 2 and 3, that the first order approximation of the dimensionless macroscopic monochromatic behaviour verifies the following set

$$\nabla_x \cdot (\mathbf{a}^e \mathbf{e}_x(\mathbf{u}_s) - \boldsymbol{\alpha} p) = -\omega^2 \mathbf{q} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{q}) + 2\boldsymbol{\Omega} \times i\omega \mathbf{q}) + \mathcal{O}(\varepsilon), \tag{101}$$

$$\mathbf{q} = \rho_s^e \mathbf{u}_s + \rho_f \langle \mathbf{u}_f \rangle, \tag{102}$$

$$\nabla_x \cdot (\langle \mathbf{v}_f \rangle - n i\omega \mathbf{u}_s) = -\boldsymbol{\alpha} i\omega \ \mathbf{e}(\mathbf{u}_s) - \beta i\omega \ p + \mathcal{O}(\varepsilon), \tag{103}$$

$$\langle \mathbf{v}_f \rangle - \phi i\omega \mathbf{u}_s = -\mathbf{K}(\omega, \boldsymbol{\Omega})(\nabla_x p - \rho_f \omega^2 \mathbf{u}_s - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{u}_s) - 2\boldsymbol{\Omega} \times i\omega \mathbf{u}_s)) + \mathcal{O}(\varepsilon). \tag{104}$$

The left hand side of momentum balance (101) is as in Biot's model and it introduces the Biot's second order tensor $\boldsymbol{\alpha}$ and the total stress $\boldsymbol{\sigma}^t = \mathbf{a}^e \mathbf{e}_x(\mathbf{u}_s) - \alpha \ p$. The right hand member is an inertial term similar to the inertial term in (18). It comprises inertia of both solid and liquid as in Biot's model, but with complementary terms due to the angular velocity $\boldsymbol{\Omega}$.

The volume balance (103) is not modified by the presence of $\boldsymbol{\Omega} \neq \mathbf{0}$. Finally, the generalized Darcy's law (104) is as in Biot's model, with added inertial terms related to $\boldsymbol{\Omega}$. Tensor $\mathbf{K}$ is the generalised conductivity tensor defined in part 3. Remark that the divergence operator in (103) kills the antisymmetric part of tensor $\mathbf{K}$ in homogeneous media. Of course, letting $\boldsymbol{\Omega} = 0$ yields the Biot's model. As it is shown below, we recover a Biot's model where the conductivity now depends on the angular velocity $\boldsymbol{\Omega}$ and where the fluid and the solid densities ρ_s^e and ρ_f are replaced by tensorial densities $\rho_s^e \mathbf{A}$ and $\rho_f \mathbf{A}$, respectively.

To obtain a more symmetric Biot type description, let us introduce partial stresses $-\phi p \mathbf{I}$ and $\boldsymbol{\sigma}_S$ of the fluid and the solid

$$\sigma_{ij}^t = a_{ijkl}^e e_{xkl}(\mathbf{u}_s) - \alpha_{ij} \ \ p = \sigma_{sij} - \phi p I_{ij}$$

and the averaged displacement $\bar{\mathbf{u}}_f$ of the fluid

$$\langle \mathbf{v}_f \rangle = \phi i \omega \bar{\mathbf{u}}_f.$$

Let us also introduce the inverse $\mathbf{H}$ of $\mathbf{K}$:

$$H_{ij} = H_{1ij} + i H_{2ij} = (K_{ij})^{-1}.$$

Equation (104) can now be put in the form (for simplicity, all the approximations $\mathcal{O}(\varepsilon)$ are omitted)

$$-\frac{\partial \phi p}{\partial x_i} = \phi^2 H_{ij} i \omega (\bar{u}_{fj} - u_{sj}) - \phi \omega^2 \rho_f u_{si}.$$

By subtracting member to member the above equality from Equation (101) yields

$$\frac{\partial \sigma_{sij}}{\partial x_j} = -\phi^2 H_{1ij} i \omega (\bar{u}_{fj} - u_{sj}) - (\rho_s^e - \phi \rho_f) A_{ij} \omega^2 u_{sj} - \phi \rho_f A_{ij} \omega^2 \bar{u} f_j.$$

On the other hand, the mass balance (103) writes

$$p = -\alpha_{ij} \beta^{-1} e_{xij}(\mathbf{u}_s) - \phi \beta^{-1} \frac{\partial}{\partial x_i} (\bar{u} f_i - u_{si}).$$

Finally, with the Biot's notations (Biot, 1956a,b) and returning to time derivatives when possible for a better lecture, the monochromatic behaviour of the solid–fluid system is in the following form

$$\frac{\partial}{\partial x_j} \langle \sigma_{sij} \rangle = B_{11ij} \ddot{u}_{sj} + B_{12ij} \ddot{\bar{u}}_{fj}, \tag{105}$$

$$\frac{\partial}{\partial x_j} (-\phi p) = B_{21ij} \ddot{u}_{sj} + B_{22ij} \ddot{\bar{u}}_{fj}, \tag{106}$$

$$\langle\sigma_{sij}\rangle = \pi_{ijkh} e_{xkh}(\mathbf{u}_s) + Q_{ij}\theta, \quad -\phi p = Q_{jk} e_{xjk}(\mathbf{u}_s) + R\theta,$$

where $\boldsymbol{\pi}, \mathbf{Q}, R$ and θ are given by

$$\pi_{ijkh} = a^e_{ijkh} + \beta^{-1}(\alpha_{ij} - \phi I_{ij})(\alpha_{kh} - \phi I_{kh}),$$
$$Q_{ij} = \phi\beta^{-1}(\alpha_{ij} - \phi I_{ij}), \quad R = \phi^2\beta^{-1}, \quad \theta = \frac{\partial \bar{u}_{fi}}{\partial x_i}$$

and where the tensors $\mathbf{B}_{\alpha\beta}, \alpha, \beta = 1, 2$, are defined by

$$B_{11ij} = (\rho^e_s - \rho^e_f)A_{ij} - i\phi^2\omega^{-1}H_{ij} = \rho^e_s A_{ij} - B_{12ij},$$
$$B_{12ij} = B_{21ij} = \rho^e_f A_{ij} + i\phi^2\omega^{-1}H_{ij},$$
$$B_{22ij} = -i\phi^2\omega^{-1}H_{ij},$$

where $\rho^e_f = \phi\rho_f$.

5. High Frequency Wave Propagation in Rotating Saturated Elastic Porous Media

Investigating wave propagation is much more intricate than in the classical Biot's case (Biot, 1956a,b). Corresponding to the three uncoupled Biot waves (two dilatational waves P_1 and P_2 and one rotational wave S), we obtain three coupled dispersive dilatational–rotational waves. These three waves, W'_1, W'_2 and W'_3 , tend to waves P_1, P_2 and S, respectively, when Ω goes to 0. However, dispersion is introduced now by both the frequency dependence ($\omega \neq 0$) and the angular velocity ($\boldsymbol{\Omega} \neq 0$). To get an insight into these waves, consider the inertial regime where $\mathbf{H} \approx i\omega\rho_f\phi^{-1}\mathbf{A}$. Tensors $\mathbf{B}_{\alpha\beta}$ reduce to

$$B_{11ij} = \rho^e_s A_{ij}, \quad B_{12ij} = B_{21ij} = 0, \quad B_{22ij} = \rho^e_f A_{ij}.$$

As in parts 2 and 3, we consider an angular velocity $\boldsymbol{\Omega} = \Omega\mathbf{e}_3$ and displacements in the plane $(\mathbf{e}_1, \mathbf{e}_2)$. The porous medium is isotropic. We successively apply the divergence and the rotational operators to Equations (105) and (106) and we note $e_s = \mathrm{div}_x \mathbf{u}_s$, $e_f = \mathrm{div}_x \mathbf{u}_f$, $\mathbf{w}_s = \boldsymbol{\nabla}_x \times \mathbf{u}_s = w_s\mathbf{e}_3$ and $\mathbf{w}_f = \boldsymbol{\nabla}_x \times \mathbf{u}_f = w_f\mathbf{e}_3$. We obtain

$$(\lambda^e_s + 2\mu^e_s)\Delta_x e_s + Q\Delta_x e_f = -\rho^e_s[(\omega^2 + \Omega^2)e_s + 2i\omega\Omega w_s], \tag{107}$$

$$Q\Delta_x e_s + R\Delta_x e_f = -\rho^e_f[(\omega^2 + \Omega^2)e_f + 2i\omega\Omega w_f], \tag{108}$$

$$\mu^e_s\Delta_x w_s = -\rho^e_s[(\omega^2 + \Omega^2)w_s - 2i\omega\Omega e_s], \tag{109}$$

$$0 = -\rho^e_f[(\omega^2 + \Omega^2)w_f - 2i\omega\Omega e_f]. \tag{110}$$

After introducing the value of w_f from (110), Equation (108) becomes

$$Q\Delta_x e_s + R\Delta_x e_f = -\rho_f^e(\omega^2 - \Omega^2)e_f. \tag{111}$$

Equations (107), (109) and (111) show three coupled dilatational–rotational waves. We look for e_s, e_f and w_s in the form

$$e_s = A_1 \exp[ikx_1], \quad e_f = A_2 \exp[ikx_1], \quad w_s = A_3 \exp[ikx_1]. \tag{112}$$

Introducing these expressions into relations (107), (111) and (109) gives three equations for the amplitudes A_1, A_2 and A_3:

$$(\lambda_s^e + 2\mu_s^e)k^2 A_1 + Qk^2 A_2 = \rho_s^e(\omega^2 + \Omega^2)A_1 + 2i\rho_s^e\Omega\omega A_3, \tag{113}$$

$$Qk^2 A_1 + Rk^2 A_2 + Rk^2 A_2 = \rho_f^e(\omega^2 - \Omega^2)A_2, \tag{114}$$

$$\mu_s^e k^2 A_3 = \rho_s^e(\omega^2 + \Omega^2)A_3 - 2i\rho_s^e\Omega\omega A_1. \tag{115}$$

The existence of non trivial solutions for A_1, A_2 and A_3 yields the dispersion equation

$$a_1 k^6 + a_2 k^4 + a_3 k^2 + a_4 = 0,$$

with

$$a_1 = (\lambda_s^e + 2\mu_s^e)\mu^e R - Q^2\mu_s^e,$$
$$a_2 = (Q^2 - (\lambda_s^e + 3\mu_s^e)R)\rho_s^e(\omega^2 + \Omega^2) - (\lambda_s^e + 2\mu_s^e)\mu_s^e\rho_f^e(\omega^2 - \Omega^2),$$
$$a_3 = (\lambda_s^e + 3\mu_s^e)(\omega^4 - \Omega^4)\rho_s^e\rho_f^e + R\rho_s^{e2}(\omega^2 - \Omega^2)^3,$$
$$a_4 = -\rho_s^e\rho_f^{e2}(\omega^2 - \Omega^2)^3.$$

Coefficients a_3 and a_4 cancel out for $\omega = \Omega$. Then two among the three waves W_1', W_2' and W_3' show a cut-off frequency at $\omega = \Omega$. When $\omega = \Omega$ Equation (111) gives

$$\Delta_x e_f = -\frac{Q}{R}\Delta_x e_s.$$

Introducing this result into (107) yields (116), which together with (109) gives the system

$$\left(\lambda_s^e + 2\mu_s^e - \frac{Q^2}{R}\right)\Delta_x e_s = -\rho_s^e[\left(\omega^2 + \Omega^2\right)e_s + 2i\omega\Omega w_3], \tag{116}$$

$$\mu_s^e\Delta_x w_s = -\rho_s^e[\left(\omega^2 + \Omega^2\right)w_s - 2i\omega\Omega e_s]. \tag{117}$$

System (116) and (117) is similar to system (20) and (21) for the empty porous matrix, after replacing λ_s^e by $\lambda_s^e - Q^2/R$. That shows that wave W_3' has no cut-off frequency at $\omega = \Omega$. Finally waves W_1' and W_2' are the two waves with a cut-off frequency at $\omega = \Omega$: they do not propagate when $\omega < \Omega$.

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