

Explicit Conditions for Local Thermal Equilibrium in Porous Media Heat Conduction

PETER VADASZ

Department of Mechanical Engineering, Northern Arizona University, P. O. Box 15600, Flagstaff, AZ 86011-5600, U.S.A. e-mail: peter.vadasz@nau.edu

(Received: 7 June 2004; accepted in final form: 4 August 2004)

Abstract. Based on the traditional formulation of heat transfer in porous media it is demonstrated that Local Thermal Equilibrium (Lotheq) applies generally for any boundary conditions that are a combination of constant temperature and insulation. The resulting consequences are being analysed and discussed. Among these consequences it is shown that the linear relationship between the average temperature difference of the two phases and the heat transferred over the fluid-solid interface is inappropriate for use in connection with conditions of Lack of Local Thermal Equilibrium (La Lotheq).

Key words: local thermal equilibrium, heat conduction, dual-phase-lagging, thermal waves, hyperbolic heat conduction.

Nomenclature

Latin Symbols

\hat{e}_x	unit vector in the x direction.
\hat{e}_y	unit vector in the y direction.
\hat{e}_z	unit vector in the z direction.
$ Fo $	Fourier number, equals $\alpha_c \tau_q / L^2$.
$ N_\delta $	dual-phase-lagging mixed term dimensionless group, equals $\delta_c / L^2 = \alpha_c \tau_T / L^2$.
$ N_\beta $	dual-phase-lagging bi-harmonic term dimensionless group, equals $\beta_c / \alpha_c L^2$.
$ h $	integral heat transfer coefficient for the heat conduction at the solid–fluid interface (dimensional).
$ k_s $	effective thermal conductivity of the solid phase, equals $(1 - \varphi) \tilde{k}_s$ (dimensional).
$ \tilde{k}_s $	thermal conductivity of the solid phase, (dimensional).
$ k_f $	effective thermal conductivity of the fluid phase, equals $\varphi \tilde{k}_s$ (dimensional).
$ \tilde{k}_f $	thermal conductivity of the fluid phase, (dimensional).
$ L $	the length of the porous slab (dimensional).
$ \mathbf{q}_* $	heat flux vector (dimensional).
$ t_* $	time (dimensional).
$ T $	temperature, (dimensional).
$ T_C $	coldest wall temperature (dimensional).
$ T_H $	hottest wall temperature (dimensional).
$ x_* $	horizontal co-ordinate (dimensional).
$ \mathbf{x}_* $	spatial variables, equals (x_*, y_*, z_*) .

Greek Symbols

α_e	effective thermal diffusivity, (dimensional).
β_e	effective property coefficient to the dual-phase-lagging bi-harmonic term, (dimensional).
γ_s	solid phase effective heat capacity, equals $(1 - \varphi)\rho_s c_s$ (dimensional).
γ_f	fluid phase effective heat capacity, equals $\varphi\rho_f c_{p,f}$ (dimensional).
δ_e	effective property coefficient to the dual-phase-lagging mixed term, (dimensional).
θ	dimensionless temperature, equals $(T_* - T_C)/(T_H - T_C)$.
φ	porosity.
ρ_s	solid phase density.
ρ_f	fluid phase density.
τ_q	time lag associated with the heat flux, (dimensional).
τ_T	time lag associated with the temperature gradient, equals δ_e/α_e , (dimensional).

Subscripts

*	corresponding to dimensional values.
s	related to the solid phase.
f	related to the fluid phase.

1. Introduction

The problem of heat transfer in a fluid saturated porous domain subject to Lack of Local Thermal Equilibrium (La Lotheq) has been the subject of wide interest due to its theoretical importance as well as its practical applications. The industrial applications of porous materials show a dramatic growth, which, for example, includes the recent interest in metal foams as well as the investigation of bio-tissues. As a result of the growing interest in the latter, a more accurate understanding of the heat transfer phenomena in porous media continues to attract attention.

Applications of porous media heat transfer subject to La Lotheq were undertaken among others by Nield (1998, 2002), Minkowycz *et al.* (1999), Alazmi and Vafai (2002), Banu and Rees (2002), Baytas and Pop (2002), Kim and Jang (2002), Nield *et al.* (2002), and Rees (2002).

In particular, Nield (1998) shows that for uniform thermal conductivities the *steady state* conduction leads to Lotheq if the temperature or its normal derivative on the boundary are identical for both phases. In addition Nield (1998) expands the analysis to the forced convection problem between plane parallel walls. It turns out that the significance of the La Lotheq in porous media forced convection is controlled by a dimensionless group, the Nield number, given by $Ni = k_s/hl_c^2$, where k_s is the solid phase thermal conductivity, h is the fluid–solid heat transfer coefficient and l_c is a characteristic

length scale. The latter was further expressed by Nield (1998) explicitly in terms of the Darcy number, the Nusselt number related to the fluid–solid interface, porosity as well as the thermal conductivities and heat capacities of both phases.

Tzou (1995, 1997) refers to experimental results in porous media heat conduction identifying thermal oscillations and overshooting, and explains them by applying the Dual-Phase-Lagging (DuPhlag) model. In particular Minkowycz *et al.* (1999) link the La Lotheq model with the DuPhlag model in a similar manner to Tzou (1995, 1997).

The present paper aims at demonstrating that, for a fluid saturated porous medium subject to heat conduction (transient as well as steady state) and any combination of imposed temperatures and insulation on the boundary, the specific traditional form of representing the heat transferred over the fluid–solid interface (at the macro-level) via a linear relationship between the latter and the temperature difference of the two phases, that is, $Q_{sf} = h(T_s - T_f)$, is inappropriate for use in connection with conditions of La Lotheq because its application leads back to Lotheq conditions and to a very particular case of identical effective thermal diffusivities for both phases.

2. Governing Equations and General Solutions

Let us consider the heat conduction in a fluid saturated porous medium domain D , having a boundary Γ of a general shape that is exposed partially to different constant temperatures (Γ_1 and Γ_2) and partially to insulation conditions (Γ_3) (Figure 1). Heat conduction in porous media is governed at the macro-level by the following equations that represent averages over each phase within a Representative Elementary Volume (REV)

$$\gamma_s \frac{\partial T_s}{\partial t_*} = k_s \nabla_*^2 T_s - Q_{sf} \quad (1)$$

$$\gamma_f \frac{\partial T_f}{\partial t_*} = k_f \nabla_*^2 T_f + Q_{sf} \quad (2)$$

where Q_{sf} represents the heat transferred over the fluid–solid interface within the REV, and where $\gamma_s = (1 - \varphi)\rho_s c_s$ and $\gamma_f = \varphi\rho_f c_{p,f}$ are the solid phase and fluid phase effective heat capacities, respectively, φ is the porosity, $k_s = (1 - \varphi)\tilde{k}_s$ and $k_f = \varphi\tilde{k}_f$ are the effective thermal conductivities of the solid and fluid phases, respectively. When the assumption of Lotheq applies, it implies that $T_s \approx T_f$ and Equations (1) and (2) can be added, cancelling the Q_{sf} term, to produce one equation for heat conduction in porous media in the form

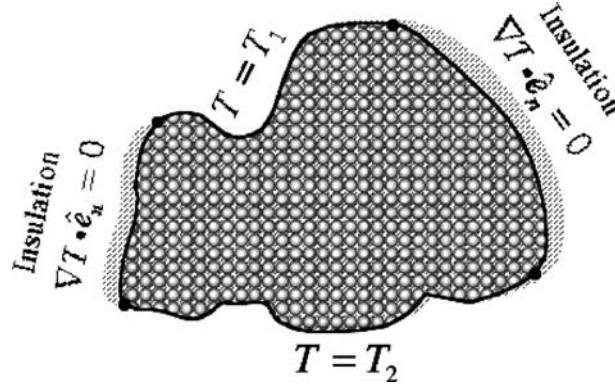


Figure 1. A fluid saturated porous domain subject to any combination of different constant temperatures and insulation on its boundaries.

$$\gamma_e \frac{\partial T}{\partial t_*} = k_e \nabla_*^2 T \quad (3)$$

where $\gamma_e = \gamma_s + \gamma_f$ and $k_e = k_s + k_f$. Otherwise, if the Lotheq assumption does not apply, the traditional formulation of the heat transferred over the fluid–solid interface (averaged over the REV) uses a linear relationship between Q_{sf} and the average temperature difference between the phases in the form

$$Q_{sf} = h(T_s - T_f) \quad (4)$$

where $h > 0$ is a macro-level integral heat transfer coefficient for the heat conduction at the fluid–solid interface (averaged over the REV) that is assumed independent of the phases temperatures. Substituting Equation (4) into Equations (1) and (2) yields the traditional form of these equations (Nield, 1998; Nield and Bejan, 1999)

$$\gamma_s \frac{\partial T_s}{\partial t_*} = k_s \nabla_*^2 T_s - h(T_s - T_f) \quad (5)$$

$$\gamma_f \frac{\partial T_f}{\partial t_*} = k_f \nabla_*^2 T_f + h(T_s - T_f) \quad (6)$$

Equations (5) and (6) are linearly coupled. The result of resolving the coupling between these equations as shown in the Appendix is represented by two independent equations for each phase in the form

$$\tau_q \frac{\partial^2 T_i}{\partial t_*^2} + \frac{\partial T_i}{\partial t_*} = \alpha_e \nabla_*^2 T_i + \delta_e \nabla_*^2 \left(\frac{\partial T_i}{\partial t_*} \right) - \beta_e \nabla_*^4 T_i \quad \forall i = s, f \quad (7)$$

where the index i can be s, representing the solid phase, or f, standing for the fluid phase and where the following notation was used

$$\tau_q = \frac{\gamma_s \gamma_f}{h(\gamma_s + \gamma_f)}; \quad \alpha_e = \frac{(k_s + k_f)}{(\gamma_s + \gamma_f)}; \quad \delta_e = \frac{(\gamma_s k_f + \gamma_f k_s)}{h(\gamma_s + \gamma_f)}; \quad \beta_e = \frac{k_s k_f}{h(\gamma_s + \gamma_f)} \quad (8)$$

Equation (7) is a linear equation that applies to each phase, while its parameters are effective coefficients common for both phases. By imposing a combination of Dirichlet (constant temperatures) and insulation boundary conditions (a particular case of Neumann boundary conditions in the form of a vanishing normal temperature gradient) expressed by

$$\begin{aligned} T_s &= T_f = T_1 && \text{on } \Gamma_1 \\ T_s &= T_f = T_2 && \text{on } \Gamma_2 \\ \nabla_* T_f \cdot \hat{e}_n &= \nabla_* T_s \cdot \hat{e}_n = 0 && \text{on } \Gamma_3 \end{aligned} \quad (9)$$

where \hat{e}_n is a unit vector normal to the boundary Γ , and assuming identical initial conditions for both phases in the form

$$(T_s)_{t_*=0} = (T_f)_{t_*=0} = T_o \quad \text{at } t_* = 0 \quad (10)$$

leads to the following boundary and initial conditions that are applicable to Equation (7)

$$\begin{aligned} T_i &= T_1 && \text{and} \quad \partial^2 T_i / \partial n^2 = 0 \quad \forall i = s, f \text{ on } \Gamma_1 \\ T_i &= T_2 && \text{and} \quad \partial^2 T_i / \partial n^2 = 0 \quad \forall i = s, f \text{ on } \Gamma_2 \end{aligned} \quad (11)$$

$$\begin{aligned} \partial T_i / \partial n &= 0 && \text{and} \quad \partial^3 T_i / \partial n^3 = 0 \quad \forall i = s, f \text{ on } \Gamma_3 \\ (T_i)_{t_*=0} &= T_o && \text{and} \quad (\partial T_i / \partial t_*)_{t_*=0} = \dot{T}_o \quad \forall i = s, f \text{ at } t_* = 0 \end{aligned} \quad (12)$$

Equation (7) that is identical for both phases, shares common parameters for both phases, solved subject to identical boundary and initial conditions for each phase, produces therefore a solution that is identical for both phases, that is,

$$T_s(t_*, \mathbf{x}_*) = T_f(t_*, \mathbf{x}_*) \quad \forall (t_* \geq 0, \mathbf{x}_* \in D) \quad (13)$$

where $\mathbf{x}_* = (x_*, y_*, z_*)$ represents the spatial variables. Equation (13) is identified as the requirement for Lotheq in porous media conduction causing the heat transfer at the fluid–solid interface $h(T_s - T_f)$ to vanish. It was obtained accurately from the original system of Equations (5) and (6)

subject to the specified boundary conditions, and other than that, no other imposed restrictions. It is therefore quite general. This result is quite astonishing and intriguing because it suggests that *Lotheq exists naturally in any porous domain subject to heat conduction and a combination of constant temperature and insulation boundary conditions*. However, this latter conclusion is not absolute but rather qualified and needs further investigation. Substituting Equation (13) into Equations (5) and (6) yields

$$\frac{\partial T_s}{\partial t} = \alpha_s \nabla^2 T_s \quad (14)$$

$$\frac{\partial T_f}{\partial t} = \alpha_f \nabla^2 T_f \quad (15)$$

where $\alpha_s = k_s/\gamma_s$ and $\alpha_f = k_f/\gamma_f$. The solution to Equations (14) and (15) subject to the same boundary and initial conditions as indicated in Equations (9) and (10) has to be identical to the corresponding solutions to Equation (7) subject to the equivalent boundary and initial conditions (11) and (12), respectively. This might be possible given the fact that Equation (7) has effective parameters that are different from the phase parameters that appear in Equations (11) and (12). The latter will be tested in Section 3 for a specific application of the general results that are presented here. Nevertheless, a further and more far reaching conclusion can be made by realising from (13) that Equations (14) and (15) are expected to produce an identical solution $T_s(t, \mathbf{x}) = T_f(t, \mathbf{x}) (t \geq 0, \mathbf{x} \in D)$ despite the fact that in general their respective thermal diffusivities may vary substantially. The latter cannot be accomplished unless $\alpha_s = \alpha_f$, leading to the inevitable conclusion that consistency requires the effective thermal diffusivities of both phases to be identical. The latter condition was not explicitly imposed *a priori*, nor implied in any of the subsequent derivations. Nevertheless, it was obtained as a result that is linked to the consequences of Equation (13). Along the way the only assumption made is represented by Equation(4) indicating that a linear relationship exists between the temperature difference of the two phases and the macro-level heat transferred over the fluid–solid interface within the REV. It was this specific representation that allowed the specific uncoupling of the equations leading to the result presented in Equation (13). *It is inevitable to conclude that application of Equation (4) implicitly assumes that $\alpha_s = \alpha_f$, and is inappropriate to use it in connection with conditions of La Lotheq because its application leads directly to Equation (13) that represents the Lotheq conditions.*

In addition, the following result applies for α_e , if $\alpha_s = \alpha_f = \alpha$ is substituted in Equation (8)

$$\alpha_e = \alpha_s = \alpha_f = \alpha \quad (16)$$

3. Application to Heat Conduction in a Porous Slab

The problem of one-dimensional heat conduction in a fluid saturated porous slab as presented in Figure 2 is considered in order to establish whether Equation (7) can produce solutions that are identical to the solutions to Equations (14) and (15). The condition that $\alpha_s = \alpha_f = \alpha_e = \alpha$, which was obtained in the previous section, is explicitly enforced when using Equations (7), (14), and (15). For the convenience of the solution, the equations are presented in the following dimensionless form by using L to scale the independent length variable x_* , that is, $x = x_*/L$, L^2/α_e to scale the time, that is, $t = t_*\alpha_e/L^2$, and introducing the dimensionless temperature, $\theta = \theta_s = \theta_f$, defined in the form $\theta = (T - T_C)/(T_H - T_C)$ leading to

$$Fo \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = \nabla^2 \theta + N_\delta \nabla^2 \left(\frac{\partial \theta}{\partial t} \right) - N_\beta \nabla^4 \theta \tag{17}$$

where the Fourier number, Fo , and two additional dimensionless groups arise

$$Fo = \frac{\alpha_e \tau_q}{L^2}; N_\delta = \frac{\delta_e}{L^2} = \frac{\alpha_e \tau_T}{L^2}; N_\beta = \frac{\beta_e}{\alpha_e L^2} \tag{18}$$

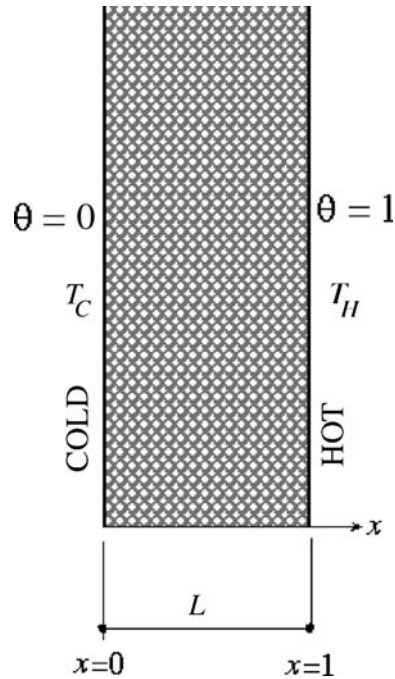


Figure 2. A fluid saturated porous slab subject to different constant temperatures on its walls.

The dimensionless form of Equation (14) or (15) is

$$\frac{\partial \theta}{\partial t} = \nabla^2 \theta \quad (19)$$

for both the solid and the fluid phases because of assuming $\alpha_s = \alpha_f = \alpha_e = \alpha$ as indicated above. For the one dimensional slab considered here and presented in Figure 2, Equations (17) and (19) become

$$Fo \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + N_\delta \frac{\partial^3 \theta}{\partial t \partial x^2} - N_\beta \frac{\partial^4 \theta}{\partial x^4} \quad (20)$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \quad (21)$$

and their corresponding boundary and initial conditions are

$$x = 0 : \theta = 0 \text{ and } \partial^2 \theta / \partial x^2 = 0; \quad x = 1 : \theta = 1 \text{ and } \partial^2 \theta / \partial x^2 = 0 \quad (22)$$

$$t = 0 : \theta = \theta_o = \text{const. and } \dot{\theta} = \dot{\theta}_o = 0 \quad (23)$$

The second boundary conditions of $\partial^2 \theta / \partial x^2 = 0$ at $x = 0$ and $x = 1$ was obtained as a result of realising that imposing the constant temperature on the walls applies equally well to the solid as well as fluid phases. Therefore at $x_* = 0$ one obtains $T_s = T_f = T_C = \text{const.}$ and at $x_* = L$ it produces $T_s = T_f = T_H = \text{const.}$ As a result, on these boundaries $T_s = T_f$ and $\partial T_s / \partial t_* = \partial T_f / \partial t_* = 0$ which upon substitution into the one dimensional version of Equations (5) and (6) yields $\partial^2 T_s / \partial x_*^2 = \partial^2 T_f / \partial x_*^2 = 0$ at $x_* = 0$ and at $x_* = L$. The dimensionless version of the latter is $\partial^2 \theta / \partial x^2 = 0$ at $x = 0$ and $x = 1$ as indicated in Equation (22). Similarly, for constant initial conditions that are identical for both phases, that is, $(T_s)_{t_*=0} = (T_f)_{t_*=0} = T_o = \text{const.}$, the different terms in Equations (5) and (6) are as follows: $(\partial^2 T_s / \partial x_*^2)_{t_*=0} = (\partial^2 T_f / \partial x_*^2)_{t_*=0} = 0$, $h(T_s - T_f)_{t_*=0} = 0$ leading to $(\partial T_s / \partial t_*)_{t_*=0} = (\partial T_f / \partial t_*)_{t_*=0} = 0$. The dimensionless version of the latter is $\partial \theta / \partial t = \dot{\theta} = \dot{\theta}_o = 0$ at $t = 0$ as indicated in Equation (23). Substituting the condition $\alpha_s = \alpha_f = \alpha_e = \alpha$ into the definitions (8) yield

$$\delta_e = \frac{2\alpha\gamma_s\gamma_f}{h(\gamma_s + \gamma_f)} = 2\alpha\tau_q; \quad \beta_e = \frac{\gamma_s\gamma_f\alpha^2}{h(\gamma_s + \gamma_f)} = \alpha^2\tau_q \quad (24)$$

leading upon substitution into Equation (18)

$$Fo = \frac{\alpha\tau_q}{L^2}; \quad N_\delta = \frac{\delta_e}{L^2} = \frac{2\alpha\tau_q}{L^2} = 2Fo; \quad N_\beta = \frac{\alpha^2\tau_q}{\alpha L^2} = \frac{\alpha\tau_q}{L^2} = Fo \quad (25)$$

This result of $N_\delta = 2Fo$ and $N_\beta = Fo$ is an immediate consequence of imposing the condition $\alpha_s = \alpha_f = \alpha_e = \alpha$, and is actually the dimensionless equivalent of the latter.

4. Analytical Solution

The solution to Equation (20) is separated into steady state θ_{ss} and transient θ_{tr} parts in the form $\theta = \theta_{ss} + \theta_{tr}$. The steady state is the linear solution $\theta_{ss} = x$ which satisfies the boundary conditions Equation (22). The transient solution θ_{tr} has to fulfil the equation

$$Fo \frac{\partial^2 \theta_{tr}}{\partial t^2} + \frac{\partial \theta_{tr}}{\partial t} = \frac{\partial^2 \theta_{tr}}{\partial x^2} + N_\delta \frac{\partial^3 \theta_{tr}}{\partial t \partial x^2} - N_\beta \frac{\partial^4 \theta_{tr}}{\partial x^4} \tag{26}$$

and the following boundary and initial conditions

$$x = 0 : \theta_{tr} = 0 \text{ and } \partial^2 \theta_{tr} / \partial x^2 = 0; \quad x = 1 : \theta_{tr} = 0 \text{ and } \partial^2 \theta_{tr} / \partial x^2 = 0 \tag{27}$$

$$t = 0 : \theta_{tr} = (\theta_o - x) \text{ and } \dot{\theta}_{tr} = \dot{\theta}_o = 0 \tag{28}$$

The solution is obtained by separation of variables in the form of two equations for $\theta_{tr} = \phi_n(t)u_n(x)$ presented in the form

$$\frac{d^2 \phi_n}{dt^2} + c_{f,n} \frac{d\phi_n}{dt} + \omega_n^2 \phi_n = 0 \tag{29}$$

$$\frac{d^2 u_n}{dx^2} + \kappa_n u_n = 0 \tag{30}$$

The solution of Equation (30) subject to the homogeneous boundary conditions $u_n = 0$ at $x = 0$ and at $x = 1$ is $u_n = a_1 \sin(n\pi x)$ and the resulting eigenvalues are $\kappa_n = n\pi \quad \forall n = 1, 2, 3, \dots$. As a result, the coefficients $c_{f,n}$ and ω_n^2 in Equation (29) are being defined in the form

$$c_{f,n} = Fo^{-1}(1 + N_\delta n^2 \pi^2); \omega_n^2 = Fo^{-1} n^2 \pi^2 (1 + n^2 \pi^2 N_\beta) \tag{31}$$

By imposing the condition $N_\delta = 2Fo$ and $N_\beta = Fo$ which is an immediate consequence of requiring $\alpha_s = \alpha_f = \alpha_e = \alpha$ yields the following definitions

$$c_{f,n} = Fo^{-1}(1 + 2Fon^2 \pi^2) = Fo^{-1} + 2n^2 \pi^2; \tag{32}$$

$$\omega_n^2 = Fo^{-1} n^2 \pi^2 (1 + n^2 \pi^2 Fo) = Fo^{-1} n^2 \pi^2 + n^4 \pi^4$$

Equation (29) represents a linear damped oscillator. Its eigenvalues are

$$\lambda_{1n} = -\frac{c_{f,n}}{2} \left[1 + \sqrt{1 - 4 \frac{\omega_n^2}{c_{f,n}^2}} \right] \tag{33}$$

$$\lambda_{2n} = -\frac{c_{f,n}}{2} \left[1 - \sqrt{1 - 4 \frac{\omega_n^2}{c_{f,n}^2}} \right] \tag{34}$$

For conditions consistent with Equation (31) it is simple to show that $4\omega_n^2/c_{f,n}^2 < 1 \quad \forall n = 1, 2, 3 \dots$ leading to real eigenvalues for all values of n . The latter result means that the solution is always overdamped, that is, it has the form

$$\theta = x + \sum_{n=1}^{\infty} (A_{1n}e^{\lambda_{1n}t} + A_{2n}e^{\lambda_{2n}t}) \sin(n\pi x) \quad (35)$$

By using the initial conditions specified in Equation (28) with $\theta_o = 0$, representing a step temperature change of the hot wall at $x = 1$ (to $T = T_H$ or $\theta = 1$), from an initial uniform temperature in the slab $T = T_C$ (or, $\theta = 0$) and $\dot{\theta}_o = 0$ one obtains the following expressions for the coefficients A_{1n} and A_{2n}

$$A_{1n} = \frac{2(-1)^n \lambda_{2n}}{(\lambda_{2n} - \lambda_{1n})n\pi} \quad (36)$$

$$A_{2n} = \frac{-2(-1)^n \lambda_{1n}}{(\lambda_{2n} - \lambda_{1n})n\pi} \quad (37)$$

5. Results and Discussion

The solution (35) was evaluated for two different values of Fo number and is presented in Figures 3 and 4. The results corresponding to $Fo = 10$, $N_\delta = 2Fo = 20$ and $N_\beta = Fo = 10$ are presented in Figure 3 showing a typical Fourier diffusion solution. Similarly, the results corresponding to $Fo = 100$, $N_\delta = 2Fo = 200$ and $N_\beta = Fo = 100$ are presented in Figure 4 showing similar Fourier diffusion results. In order to compare the two results that belong to different values of the parameters the difference between the two was evaluated showing that its absolute value is less than 10^{-3} . It seems compelling to explain this similar behavior. To do so, we use Equation (21) and evaluate the following terms

$$Fo \frac{\partial^2 \theta}{\partial t^2} = Fo \frac{\partial}{\partial t} \left(\frac{\partial \theta}{\partial t} \right)_{\text{Eq. (21)}} = Fo \frac{\partial}{\partial t} \left(\frac{\partial^2 \theta}{\partial x^2} \right) \quad (38)$$

$$N_\beta \frac{\partial^4 \theta}{\partial x^4} = N_\beta \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \theta}{\partial x^2} \right) = N_\beta \frac{\partial^2}{\partial x^2} \left(\frac{\partial \theta}{\partial t} \right)_{\text{Eq. (21)}} = N_\beta \frac{\partial}{\partial t} \left(\frac{\partial^2 \theta}{\partial x^2} \right) \quad (39)$$

Substituting (38) and (39) into Equation (20) yields

$$Fo \frac{\partial}{\partial t} \left(\frac{\partial^2 \theta}{\partial x^2} \right) + N_\beta \frac{\partial}{\partial t} \left(\frac{\partial^2 \theta}{\partial x^2} \right) - N_\delta \frac{\partial}{\partial t} \left(\frac{\partial^2 \theta}{\partial x^2} \right) + \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} = 0 \quad (40)$$

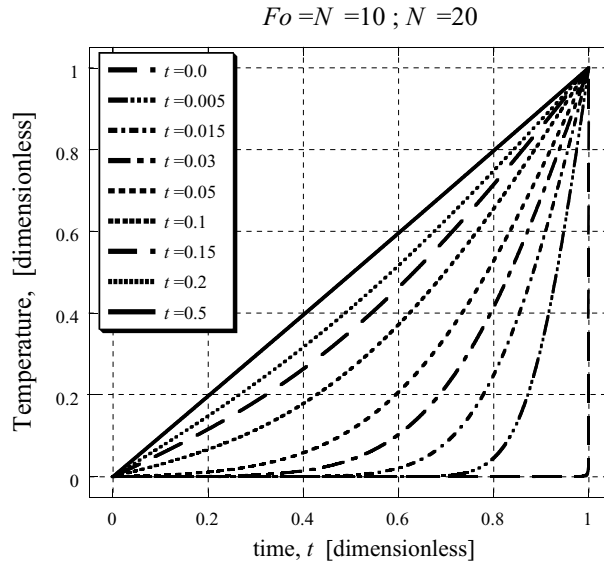


Figure 3. The temperature solution of Equation (26) in a porous slab as a function of $x = x_*/L$ for different values of time, and corresponding to $Fo = 10$, $N_\delta = 20$ and $N_\beta = 10$, and the following initial conditions $\theta_o = 0$ and $\dot{\theta}_o = 0$.

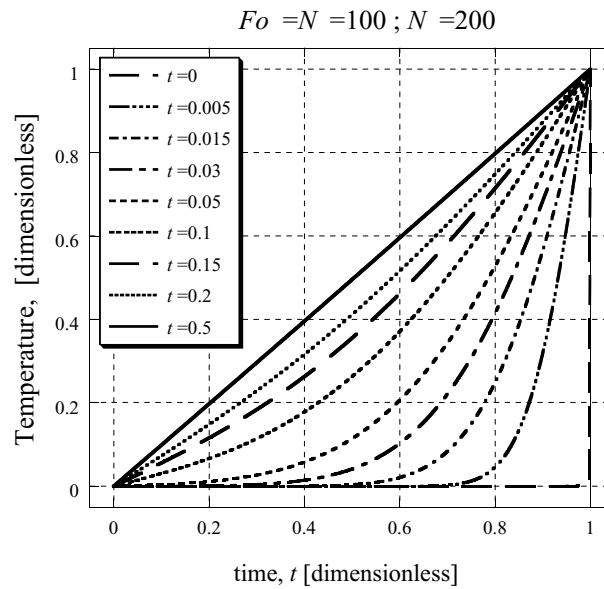


Figure 4. The temperature solution of Equation (26) in a porous slab as a function of $x = x_*/L$ for different values of time, and corresponding to $Fo = 100$, $N_\delta = 200$ and $N_\beta = 100$, and the following initial conditions $\theta_o = 0$ and $\dot{\theta}_o = 0$.

which leads to

$$(Fo + N_\beta - N_\delta) \frac{\partial}{\partial t} \left(\frac{\partial^2 \theta}{\partial x^2} \right) + \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} = 0 \quad (41)$$

The term in the parentheses $(Fo + N_\beta - N_\delta)$ can be evaluated subject to the condition that $N_\delta = 2Fo$ and $N_\beta = Fo$ which is an immediate consequence of imposing $\alpha_s = \alpha_f = \alpha_e = \alpha$. The result of enforcing the latter condition leads to $(Fo + N_\beta - N_\delta) = 0$ which, upon substitution into Equation (41), produces the Fourier diffusion equation $\partial\theta/\partial t - \partial^2\theta/\partial x^2 = 0$, explaining the results presented in Figures 3 and 4.

6. Conclusions

The conditions for Lotheq in porous media conduction are shown to apply generally for any boundary conditions that are a combination of constant temperature and insulation, based on the traditional formulation of heat transfer in porous media. Among the resulting consequences of the latter it is shown that the linear relationship between the average temperature difference of the two phases and the heat transferred over the fluid-solid interface is inappropriate for use in connection with conditions of La Lotheq.

Appendix

The objective of this Appendix is to show the derivation of Equation (7) from Equations (5) and (6), in the form

$$\gamma_s \frac{\partial T_s}{\partial t_*} = k_s \nabla_*^2 T_s - h(T_s - T_f) \quad (A.1)$$

$$\gamma_f \frac{\partial T_f}{\partial t_*} = k_f \nabla_*^2 T_f + h(T_s - T_f) \quad (A.2)$$

The diffusion terms in these equations are a result of replacing the $-\nabla_* \cdot \mathbf{q}_{*s}$ and $-\nabla_* \cdot \mathbf{q}_{*f}$ terms by using Fourier's Law in the form $\mathbf{q}_{*s} = -k_s \nabla_* T_s$ and $\mathbf{q}_{*f} = -k_f \nabla_* T_f$ to yield the Laplacian terms. When the Lotheq assumption is not valid, conditions appropriate for the case when the temperature difference between the two phases is not small, the two Equations (A.1) and (A.2) are coupled and need to be solved simultaneously. There are two ways to resolve the coupling between these equations. The first way is in isolating T_f in Equation (A.1) to obtain

$$T_f = \frac{\gamma_s}{h} \frac{\partial T_s}{\partial t_*} - \frac{k_s}{h} \nabla_*^2 T_s + T_s \quad (A.3)$$

and substituting it into Equation (A.2), re-grouping the terms and dividing the resulting equation by $(\gamma_s + \gamma_f)$, to produce one equation for T_s in the form

$$\tau_q \frac{\partial^2 T_s}{\partial t_*^2} + \frac{\partial T_s}{\partial t_*} = \alpha_e \nabla_*^2 T_s + \delta_e \nabla_*^2 \left(\frac{\partial T_s}{\partial t_*} \right) - \beta_e \nabla_*^4 T_s \quad (\text{A.4})$$

where the following notation was used

$$\tau_q = \frac{\gamma_s \gamma_f}{h(\gamma_s + \gamma_f)}; \quad \alpha_e = \frac{(k_s + k_f)}{(\gamma_s + \gamma_f)}; \quad \delta_e = \frac{(\gamma_s k_f + \gamma_f k_s)}{h(\gamma_s + \gamma_f)}; \quad \beta_e = \frac{k_s k_f}{h(\gamma_s + \gamma_f)} \quad (\text{A.5})$$

Alternatively, by isolating T_s in Equation (A.2) and substituting it into Equation (A.1) one obtains an identical equation as (A.4) for T_f in the form

$$\tau_q \frac{\partial^2 T_f}{\partial t_*^2} + \frac{\partial T_f}{\partial t_*} = \alpha_e \nabla_*^2 T_f + \delta_e \nabla_*^2 \left(\frac{\partial T_f}{\partial t_*} \right) - \beta_e \nabla_*^4 T_f \quad (\text{A.6})$$

Note the negative sign in front of the bi-harmonic term as distinct from Tzou (1997), which shows incorrectly a positive sign in front of this term.

The second way in resolving the coupling between the two equations is by presenting them in the equivalent operator form

$$\begin{bmatrix} \left(\gamma_s \frac{\partial}{\partial t_*} - k_s \nabla_*^2 + h \right); & -h \\ -h; & \left(\gamma_f \frac{\partial}{\partial t_*} - k_f \nabla_*^2 + h \right) \end{bmatrix} \begin{bmatrix} T_s \\ T_f \end{bmatrix} = 0 \quad (\text{A.7})$$

Then, their uncoupled form is obtained by evaluating the operator determinant leading to

$$\left[\left(\gamma_s \frac{\partial}{\partial t_*} - k_s \nabla_*^2 + h \right) \left(\gamma_f \frac{\partial}{\partial t_*} - k_f \nabla_*^2 + h \right) - h^2 \right] T_i = 0 \quad \forall i = s, f \quad (\text{A.8})$$

where the index i can be s, representing the solid phase, or f, standing for the fluid phase. Equation (A.8) can be presented explicitly, after dividing it by $h(\gamma_s + \gamma_f)$ in the form

$$\tau_q \frac{\partial^2 T_i}{\partial t_*^2} + \frac{\partial T_i}{\partial t_*} = \alpha_e \nabla_*^2 T_i + \delta_e \nabla_*^2 \left(\frac{\partial T_i}{\partial t_*} \right) - \beta_e \nabla_*^4 T_i \quad \forall i = s, f \quad (\text{A.9})$$

where the index i can be s, representing the solid phase, or f, standing for the fluid phase This Equation (A.9) is identical to (A.4) and (A.6) obtained by the substitution method.

The derivation of Equation (A.9) or its equivalent forms (A.4) and (A.6) was the main objective of this Appendix, and is presented as Equation (7) in the text.

In addition, an equivalent equation presented in terms of the heat flux instead of temperature is derived below by using Equations (1) and (2) that are presented in the following equivalent form

$$\gamma_s \frac{\partial T_s}{\partial t_*} + \nabla_* \cdot \mathbf{q}_{*s} + h(T_s - T_f) = 0 \quad (\text{A.10})$$

$$\gamma_f \frac{\partial T_f}{\partial t_*} + \nabla_* \cdot \mathbf{q}_{*f} - h(T_s - T_f) = 0 \quad (\text{A.11})$$

by using Fourier's Law $\mathbf{q}_{*s} = -k_s \nabla_* T_s$ and $\mathbf{q}_{*f} = -k_f \nabla_* T_f$. By applying the gradient operator (∇_*) on Equations (A.10) and (A.11) and the Fourier Law expressed in the form

$$\nabla_* T_s = -\mathbf{q}_{*s}/k_s; \quad \nabla_* T_f = -\mathbf{q}_{*f}/k_f \quad (\text{A.12})$$

to replace the $\nabla_* T_s$ and $\nabla_* T_f$ terms, produces the following system

$$\begin{bmatrix} \left(\frac{\gamma_s}{k_s} \frac{\partial}{\partial t_*} - \nabla_* (\nabla_* \cdot) + \frac{h}{k_s} \right); & \left(-\frac{h}{k_f} \right) \\ \left(-\frac{h}{k_s} \right); & \left(\frac{\gamma_f}{k_f} \frac{\partial}{\partial t_*} - \nabla_* (\nabla_* \cdot) + \frac{h}{k_f} \right) \end{bmatrix} \begin{bmatrix} \mathbf{q}_{*s} \\ \mathbf{q}_{*f} \end{bmatrix} = 0 \quad (\text{A.13})$$

Following a similar process as for the temperature one obtains the uncoupled equation for the heat flux in the form

$$\begin{aligned} \tau_q \frac{\partial^2 \mathbf{q}_{*i}}{\partial t_*^2} + \frac{\partial \mathbf{q}_{*i}}{\partial t_*} &= \alpha_e \nabla_* (\nabla_* \cdot \mathbf{q}_{*i}) + \delta_e \frac{\partial \nabla_* (\nabla_* \cdot \mathbf{q}_{*i})}{\partial t_*} \\ &\quad - \beta_e \nabla_* [\nabla_*^2 (\nabla_* \cdot \mathbf{q}_{*i})] \quad \forall i = s, f \end{aligned} \quad (\text{A.14})$$

where the definitions of the parameters are identical to Equation (A.5).

References

- Alazmi, B. and Vafai, K.: 2002, Constant wall heat flux boundary conditions in porous media under local thermal non-equilibrium conditions, *Int. J. Heat Mass Transfer* **45**, 3071–3087.
- Banu, N. and Rees, D. A. S.: 2002, Onset of Darcy–Benard convection using a thermal non-equilibrium model, *Int. J. Heat Mass Transfer* **45**, 2221–2238.
- Baytas, A. C. and Pop, I.: 2002, Free convection in a square porous cavity using a thermal nonequilibrium model, *Int. J. Thermal Sci.* **41**, 861–870.
- Kim, S. J. and Jang, S. P.: 2002, Effects of Darcy number, the Prandtl number, and the Reynolds number on local thermal non-equilibrium, *Int. J. Heat Mass Transfer* **45**, 3885–3896.
- Minkowycz, W. J., Haji-Shiekh, A. and Vafai, K.: 1999, On departure from local thermal equilibrium in porous media due to a rapidly changing heat source: the Sparrow number, *Int. J. Heat Mass Transfer* **42**, 3373–3385.
- Nield, D. A.: 1998, Effects of local thermal nonequilibrium in steady convective processes in a saturated porous, medium: forced convection in a channel, *J. Porous Media* **1**, 181–186.
- Nield, D. A.: 2002, A note on the modeling of local thermal non-equilibrium in a structured porous medium, *Int. J. Heat Mass Transfer* **45**, 4367–4368.

- Nield, D. A., and Bejan, A.: 1999, *Convection in Porous Media*, 2nd edn., Springer-Verlag, New York.
- Nield, D. A. Kuzentsov, A. V. and Xiong, M.: 2002, Effect of local thermal non-equilibrium on thermally developing forced convection in a porous medium”, *Int. J. Heat Mass Transfer* **45**, 4949–4955.
- Rees, D. A. S.: 2002, Vertical free convective boundary-layer flow in a porous medium using a thermal nonequilibrium model: elliptical effects, *Zeitschrift fur angewandte Mathematik und Physik ZAMP* **53**, 1–12
- Tzou, D. Y.: 1995, A unified field approach for heat conduction from macro-to-micro-scales, *ASME J. Heat Transfer* **117**, 8–16.
- Tzou, D. Y.: 1997, *Macro-to-Microscale Heat Transfer; The Lagging Behavior*, Taylor & Francis, Washington, DC.