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Stability Analysis of a Porous Layer Heated from Below and Subjected to Low Frequency Vibration: Frozen Time Analysis

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Abstract. We investigate the convection amplitude in an infinite porous layer subjected to a vibration body force that is collinear with the gravitational acceleration and heated from below. The analysis focuses on the specific case of low frequency vibration where the frozen time approximation is used. The results reveal that for moderate Vadasz numbers, increasing the magnitude of the acceleration stabilizes the convection. The results of the large Vadasz number analysis reveals that the acceleration plays a passive role in the stability of convection and the classical stability criteria for Rayleigh–Benard convection applies.

Key words: vibration, convection, porous media, Frozen time, Vadasz number.

Nomenclature

b_*	vibration amplitude.
Da	Darcy number, equals $k_0 * /H_*^2$.
$\hat{\mathbf{e}}_x$	unit vector in the x-direction.
$\hat{\mathbf{e}}_y$	unit vector in the y-direction.
$\hat{\mathbf{e}}_z$	unit vector in the z-direction.
Fr	Froude number, equals $\lambda_*^2/(g_* H_*^3)$.
g_*	gravitational acceleration.
H_*	the height of the porous layer.
k_*	permeability of the porous matrix.
L	the front aspect ratio of the porous layer, equals L_*/H_*.
L_*	the length of the porous layer.
p	dimensionless reduced pressure.
Pr	Prandtl number, equals v_*/λ_*.
R	porous media gravitational Rayleigh number, equals, $\beta_* \Delta T_C g_* k_C * H_*/v_* \lambda_*$.
s	convection wavenumber.
t_*	time.
T	dimensionless temperature, equals $(T_* - T_C)/(T_H - T_C)$.
T_C	coldest wall temperature.
T_H	hottest wall temperature.
u	horizontal x component of the filtration velocity.
v	horizontal y component of the filtration velocity.
$\mathbf{V}$	dimensionless filtration velocity vector, equals $u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y + w\hat{\mathbf{e}}_z$.

Va	Vadasz number, equals $\phi \sim Pr/Da$.
w	vertical z component of the filtration velocity.
W	the top aspect ratio of the porous layer, equals W_*/H_*.
W_*	the width of the layer.
x	horizontal length coordinate.
y	horizontal width coordinate.
z	vertical coordinate.

Greek Letters

ΔT_C	characteristic temperature difference.
Ω	scaled vibration frequency, equals, $\omega_* H_*^2/\lambda_*$.
α	a parameter related to the wave number, equals s^2/π^2.
β_*	thermal expansion coefficient.
γ	equals Va/π^2.
δ	equals, $\kappa Fr \Omega^2$.
κ	equals b_*/H_*.
λ_*	effective thermal diffusivity.
μ_*	fluid dynamic viscosity.
v_*	fluid kinematic viscosity.
ρ	fluid density.
σ	convection frequency.
ω_*	vibration frequency.
ϕ	porosity

Subscripts

*	dimensional values.
c	characteristic.
cr	critical values.
C	related to coldest wall.
H	related to hottest wall.
o	related to unmodulated quantities.

Over

$\sim$	scaled quantities.

1. Introduction

Over the past decade, interest in fundamental studies of thermal convection in porous media has significantly increased due to its presence in diverse engineering applications. Buoyancy induced flows in porous media play an important role in engineering applications, for example the presence mushy layers (a reactive porous medium) in binary alloy solidification. During solidification one finds the presence of three distinct layers *viz*. the solid layer, the melt layer and the mushy layer, which is sandwiched between the solid and melt regions. The mushy layer may be thought of as a two-phase zone/reactive porous medium that serves to smear the concentration gradient between the solid and melt regions. Experimental studies by Sample and Hellawell (1984) and Sarazin and Hellawell (1998) reveal important information on mechanism for channel formation. It is also demonstrated that the dynamics occurring in the mushy layer are critical to the quality of the final product. Amberg and Homsy (1993), Anderson and Worster (1995) and Worster (1992) provide an excellent numerical analysis of the effects of gravity on solidification, whilst Govender and Vadasz (2002a, b) and Govender (2003) extend their work by proposing rotation as a means of stabilising convection.

Lately Govender (2004a, b) provided stability analyses for convection in gravity modulated porous layers heated from below and above. In addition Govender (2004c) provided a weak non-linear analysis of convection in a gravity modulated porous layer for the large amplitude scaling. In that paper he makes mention of the fact that the work may be extended to binary alloy solidification, a task that is currently underway. The objective of the current work is to analyse the stability of convection in a passive porous medium subjected to low frequency vibration for both the moderate and large Vadasz number scaling.

2. Problem Formulation

A sketch of the vibrating porous layer and the important boundary conditions are shown in Figure 1. Govender (2004a) presented a detailed formulation of the continuity, energy, and Darcy equations (extended to include vibration) for porous media. As the derivation will not be repeated here, readers are referred there for a detailed analysis. The dimensionless governing equations are presented as follows:

$$\nabla \cdot \mathbf{V} = 0. \tag{1}$$

$$\left(\frac{1}{Va}\frac{\partial}{\partial t} + 1\right)\mathbf{V} = -\nabla p - R[1 + \delta \ \cos(\Omega t)]T\hat{e}_z, \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \nabla^2 T. \tag{3}$$

The symbols $\mathbf{V}$, T and p represent the dimensionless filtration velocity vector, temperature and reduced pressure, respectively, and $\hat{e}_z$ is a unit vector in the z-direction. In Equation (2), Ω is the scaled frequency, defined as

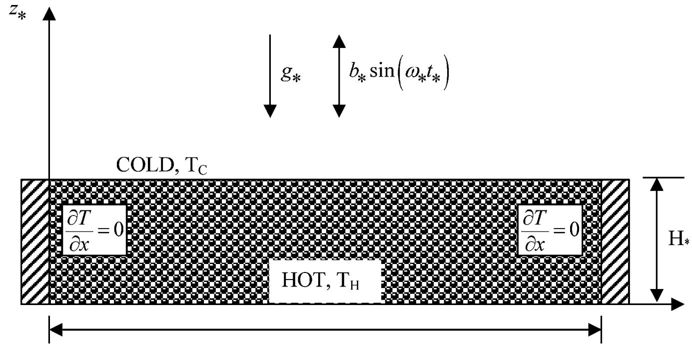

Figure 1. Differentially heated porous layer subjected to vibration.

$\Omega = \omega_* H_*^2/\lambda_*$, whilst the non-dimensional amplitude δ is defined as $\delta = \kappa Fr \Omega^2$, where $\kappa = b_*/H_*$ and Fr is the modified Froude number defined as $Fr = \lambda_*^2/(g_* H_*^3)$. The parameter Va is the Vadasz number, as pointed out by Straughan (2000), and includes the Prandtl and Darcy numbers as well as the porosity of the porous domain and is defined as $Va = \phi Pr/Da$, where $Pr = \nu_*/\lambda_*$ is the Prandtl number, $Da = k_{c*}/H_*^2$ is the Darcy number, ϕ is the porosity and ν_* stands for the kinematic viscosity of the fluid. It is only through this combined dimensionless group that the Prandtl number affects the flow in the porous media, see Vadasz (1998) for a full discussion on the numerical values that Pr can assume in a typical porous medium. In Equation.(2) one also observes the Rayleigh number, R; defined as $R = \beta_* \Delta T_C g_* H_* / \nu_* \lambda_*$. As all boundaries are rigid, the solution must follow the impermeability conditions there, i.e. $\mathbf{V} \cdot \hat{e}_n = 0$ on the boundaries, where $\hat{e}_n$ is a unit vector normal to the boundary. The temperature boundary conditions are: $T = 1$ at $z = 0$, $T = 0$ at $z = 1$ and $\nabla T \cdot \hat{e}_n = 0$ at $x = 0$ and $x = L$, representing the insulation condition on these walls. The partial differential equations (1)–(3) forms a non-linear coupled system which together with the corresponding boundary conditions accepts a basic motionless solution with a parabolic pressure distribution. The solutions for the basic temperature and flow field is given as, $T_B = 1 - z$ and $\mathbf{V}_B = 0$. To provide a non-trivial solution to the system it is convenient to apply the curl operator $(\nabla \times)$ twice on Equation (2) to obtain

$$\left(\frac{1}{Va}\frac{\partial}{\partial t} + 1\right)\nabla^2 \mathbf{V} + R[1 + \delta \sin(\Omega t)]\left[\frac{\partial^2 T}{\partial x\, \partial z}\hat{e}_x + \frac{\partial^2 T}{\partial y\, \partial z}\hat{e}_y - \nabla_H^2 T \hat{e}_z\right] = 0, \tag{4}$$

for a solenoidal velocity field, Equation (1). The horizontal Laplacian operator in Equation (4) is defined as $\nabla_H^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

3. Linear Stability Analysis

The system accepts a basic motionless solution of the form $\mathbf{V}_{\mathrm{B}} = 0$ and $T_{\mathrm{B}} = 1 - z$. Assuming small perturbations around the basic solution in the form $\mathbf{V} = \mathbf{V}_{\mathrm{B}} + \mathbf{V}'$ and $T = T_{\mathrm{B}} + T'$, and linearising Equations (1)–(4) yields the following linear system:

$$\left(\frac{1}{Va}\frac{\partial}{\partial t} + 1\right)\nabla^2\mathbf{V}' + R[1 + \delta\cos(\Omega t)]\left[\frac{\partial^2 T'}{\partial x\,\partial z}\hat{e}_x + \frac{\partial^2 T'}{\partial y\,\partial z}\hat{e}_y - \nabla_{\mathrm{H}}^2 T'\hat{e}_z\right] = 0, \tag{5}$$

$$\left[\frac{\partial}{\partial t} - \nabla^2\right]T' - w' = 0, \tag{6}$$

where w' is the perturbation to the vertical component of the filtration velocity. The boundary conditions in the z-direction required for solving Equations (5) and (6) are $w' = T' = 0$ at $z = 0$ and $z = 1$. In the x-direction $\partial T/\partial x = 0$ at $x = 0$ and $x = L$. The coupling between Equations (5) and (6) can be removed by considering the vertical component of Equation (5) and eliminating w' to provide one equation for the temperature perturbation in the form

$$\left(\frac{1}{Va}\frac{\partial}{\partial t} + 1\right)\nabla^2\left[\frac{\partial}{\partial t} - \nabla^2\right]T' - R_m(t)\nabla_{\mathrm{H}}^2 T' = 0, \tag{7}$$

where $R_m(t) = R[1 + \delta\cos(\Omega t)]$. Assuming an expansion into normal modes in the x- and y-directions, and a time-dependent amplitude $\theta(t)$ of the form

$$T' = \theta(t)\exp[i(s_x y + s_y z)]\sin(\pi z) + c.c., \tag{8}$$

where $c.c.$ stands for the complex conjugate terms and $s^2 = s_x^2 + s_y^2$. Substituting Equation (8) into the Equation (7) provides an ordinary differential equation for the amplitude $\theta(t)$:

$$\frac{1}{\gamma}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \pi^2\left(\frac{\alpha + 1}{\gamma} + 1\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} - F(\alpha)[\tilde{R}_m(t) - \tilde{R}_{\mathrm{o}}]\theta = 0, \tag{9}$$

where $\alpha = s^2/\pi^2, \gamma = Va/\pi^2, \tilde{R} = R/\pi^2, F(\alpha) = \pi^4\alpha/(\alpha + 1)$ and $\tilde{R}_{\mathrm{o}}$ is the unmodulated Rayleigh number defined as $\tilde{R}_{\mathrm{o}}(\alpha + 1)^2/\alpha$. The solution for the moderate Vadasz number solution involving moderate to large frequencies has been extensively investigated by Govender (2004a), so readers are referred to that paper for a full analysis. This paper will focus only on low frequency effects ($\Omega < \square 1$) and the growth rate factor with no modulation is then assumed to apply instantaneously as follows by taking $\theta(t) = \mathrm{e}^{\sigma t}$ in Equation (8). Subtsituting in Equation (9) and refining yields the following growth rate factor for moderate Vadasz numbers:

$$\sigma = \varphi\left[\left(\frac{\tilde{R}_m(t)-\eta}{\tilde{R}_o-\eta}\right)^{1/2} -1\right], \tag{16}$$

where $\tilde{R}_m(t) = R[1+\delta\cos(\Omega t)]$ is considered here to be a slowly varying, time-dependent Rayleigh number, $\varphi^2 = F(\alpha)\gamma(\tilde{R}_o-\eta)$ and the parameter $\eta = -\tilde{R}_o(\alpha+1-\gamma)^2/[4\gamma(\alpha+1)]$. For marginal stability we require that there be no net growth or decay over once cycle, i.e.

$$\int_0^{2\pi/\Omega} \sigma\,\mathrm{d}t = 0. \tag{17}$$

Allowing $\tau_0 = \Omega t$, yields the following criterion for marginal stability,

$$\int_0^{2\pi/\Omega} \sigma\ \mathrm{d}t = \int_0^{2\pi}(1+\kappa_o\sin\tau_0)^{1/2}\mathrm{d}\tau_0 = 2\pi\left(\frac{\tilde{R}_0-\eta}{\tilde{R}-\eta}\right)^{1/2}, \tag{18}$$

where $\kappa_0 = \delta\tilde{R}/(\tilde{R}-\eta) \ll 1$. Using the first three terms of the binomial expansion of $(1+\kappa_o\ \sin\tau_0)^{1/2}$ and performing the integral in Equation (18) yields

$$|\delta| = 4\frac{\tilde{R}-\eta}{\tilde{R}}\left[1-\left(\frac{\tilde{R}_0-\eta}{\tilde{R}-\eta}\right)^{1/2}\right]^{1/2} \qquad \text{for } \kappa_0 \ll 1. \tag{19}$$

It is observed from Equation (19) that for low frequency the stability is dependent on the acceleration $\delta = \kappa Fr\Omega^2$, and not independently by κFr and Ω as discovered by Govender (2004a). It is clearly seen in Figure 2 that for increasing values of acceleration δ, the convection is stabilized as observed by the increasing values of critical Rayleigh number. There is also a corresponding increase in the critical wavenumbers for increasing values of acceleration. It is also noted that below $\delta \approx 2$ there is a marginal increase in the critical Rayleigh number and wavenumber, whilst beyond $\delta \approx 2$ there is a marked increase in the critical Rayleigh number and wavenumber. For large Vadasz numbers, i.e. $\gamma \to \infty$, Equation (9) has the following form:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = F(\alpha)[\tilde{R}_m(t)-\tilde{R}_o]\theta. \tag{20}$$

For low frequency effects ($\Omega < \square 1$), the growth rate factor with no modulation is then assumed to apply instantaneously as follows by substituting $\theta(t) = \mathrm{e}^{\sigma t}$ in Equation (20), to yields the following definition for σ for large Vadasz numbers:

$$\sigma = F(\alpha)(\tilde{R}_m(t)-\tilde{R}_0). \tag{21}$$

Integrating Equation (21) over one cycle yields, $\tilde{R} = \tilde{R}_0$, which implies that for large Vadasz numbers the critical conditions for Rayleigh–Benard

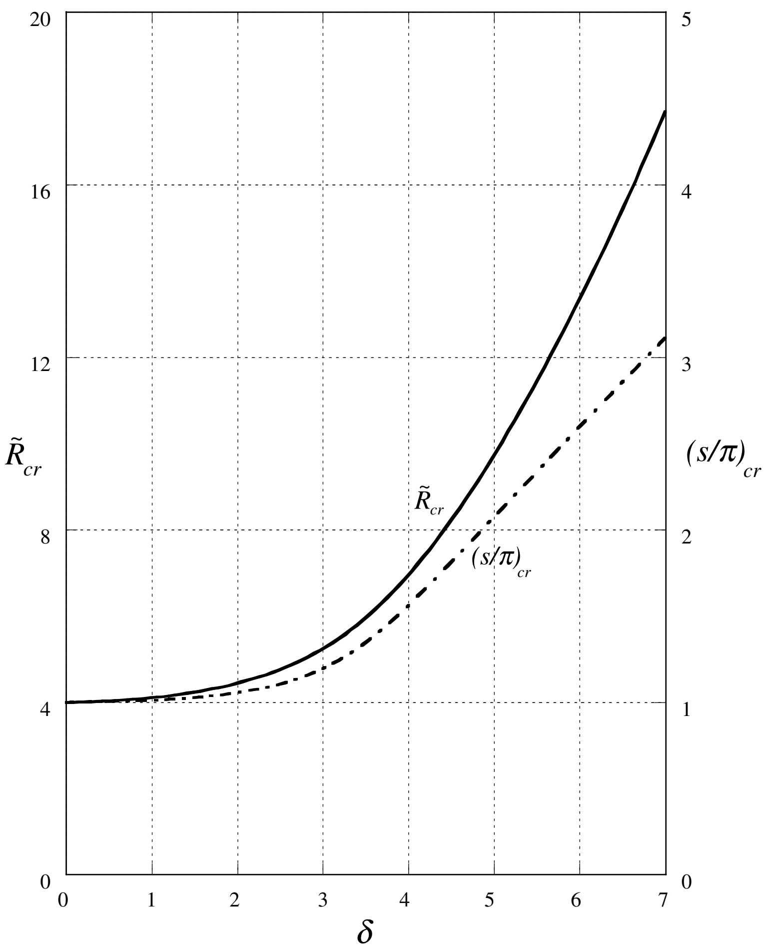

Figure 2. Critical Rayleigh number and wavenumber curves for various values of acceleration δ.

convection hold, i.e. $\tilde{R}_{\mathrm{cr}} = 4$ and $\alpha_{\mathrm{cr}} = 1$. In addition the acceleration δ is seen to play passive role in the stability of convection at low frequencies and large Vadasz numbers.

5. Conclusion

The current work investigates the effect of low frequency gravity modulation on the stability of convection in a differentially heated porous layer. The results show that for large Vadasz numbers the stability criteria for unmodulated Rayleigh–Benard convection applies, i.e. the stability criteria is independent of the acceleration δ. For moderate Vadasz numbers it is discovered that convection is stabilized for increasing values of the acceleration δ.

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Appendix. Derivation of Equation (18)

Allowing $\tau_0 = \Omega t$, yields the following criterion for marginal stability,

$$\int_0^{2\pi} \sigma \, \mathrm{d}\tau_0 = \int_0^{2\pi} \varphi \left[\left(\frac{\tilde{R}_m(t) - \eta}{\tilde{R}_0 - \eta} \right)^{1/2} - 1 \right] \mathrm{d}\tau_0 = 0. \tag{A.1}$$

Substituting the Rayleigh number definition $\tilde{R}_m(\tau_0) = \tilde{R}[1 + \delta \cos \tau_0]$ in Equation (A.1) and integrating yields,

$$\int_0^{2\pi} (1 + \kappa_o \ \sin \tau_0)^{1/2} \mathrm{d}\tau_0 = 2\pi \left(\frac{\tilde{R}_0 - \eta}{\tilde{R} - \eta} \right)^{1/2}, \tag{A.2}$$

where $\kappa_0 = \delta \tilde{R}/(\tilde{R} - \eta) << 1$. For small values of κ_0 we may use the binomial expansion to determine the acceleration δ in terms of the characteristic Rayleigh number and the parameter η.

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