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Aggregating individual credences into collective binary beliefs: an impossibility result

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Abstract
This paper addresses how multiple individual credences on logically related issues should be aggregated into collective binary beliefs. We call this binarizing belief aggregation. It is vulnerable to dilemmas such as the discursive dilemma or the lottery paradox: proposition-wise independent aggregation can generate inconsistent or not deductively closed collective judgments. Addressing this challenge using the familiar axiomatic approach, we introduce general conditions on a binarizing belief aggregation rule, including rationality conditions on individual inputs and collective outputs, and determine which rules (if any) satisfy different combinations of these conditions. Furthermore, we analyze similarities and differences between our proofs and other related proofs in the literature and conclude that the problem of binarizing belief aggregation is a free-standing aggregation problem not reducible to judgment aggregation or probabilistic opinion pooling.

Keywords Belief aggregation · Judgment aggregation · Probabilistic opinion pooling · Belief binarization · Impossibility result · Social epistemology

1 Introduction

Binarizing belief aggregation

This paper addresses how multiple individual credences on logically related issues should be aggregated into collective binary beliefs. We call this *binarizing belief aggregation*. This addresses two different types of belief: binary belief, which allows only two options regarding a proposition, say A (she believes that A or she does not believe that A), and credence, which usually allows infinitely many options (she believes to a degree of x that A, where x is a numerical value). Binary beliefs and credences have their own merits and disadvantages. Credences are informative and

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sophisticated, whereas binary beliefs are uninformative in some complex contexts (e.g., decision contexts in uncertain environments). Meanwhile, binary beliefs are computationally efficient and decisive, whereas credences are often computationally demanding even for ideal reasoners.[1] Because one type does not dominate the other, we have a good reason to embrace both credences and binary beliefs in belief aggregation contexts. Considering two types of belief, we have the four possible combinations of belief aggregation problems presented in Fig. 1.[2]

The existing research regarding belief aggregation has primarily focused on the cases where individual and collective beliefs are of the same type, such as aggregating probabilistic beliefs in *probabilistic opinion pooling* (Genest and Zidek, 1986; Dietrich and List, 2016), or aggregating binary beliefs in the *judgment aggregation* literature (List and Puppe, 2009; Grossi and Pigozzi, 2014). However, this study focuses on aggregating individual credences into collective binary beliefs, leaving the fourth aggregation problem of aggregating individual binary beliefs into collective credences for further research.

The need for binarizing belief aggregation arises in various contexts. Regarding input data, credences would be generally preferred over binary beliefs. This is because we can treat input data as evidence for the resulting collective beliefs and we expect that sophisticated and informational input data are more likely to track the truth. Credences usually contain more information than binary beliefs, so binary beliefs may already suffer from information loss. However, for output data, binary beliefs could be appropriate in many cases.[3] For example, political parties describing society's problems through their programs would face criticism if these programs made or implied contradictory claims. In these processes, collective binary beliefs and rationality norms, such as consistency and logical closure, are commonly observed. This is not surprising as binary beliefs are efficient in conveying information and communicating with other agents. Furthermore, the type of belief in the final judgment is pre-determined in many institutions, whether written or customary, and often binary. For example, the jury's verdict in a criminal case is supposedly expressed as "guilty" or "not guilty." Combining these considerations regarding input and output suggests that there are some contexts where binarizing belief aggregation is particularly suitable.

[1] For more pros and cons of credences and binary beliefs, see Dietrich (2022).

[2] In addition to the four types of belief aggregation problems, Bradley and Wagner (2012) suggest a framework for modeling intermediate belief states (finitely many-valued doxastic states) between probabilistic and binary beliefs. More complex belief aggregation methods might include different dynamic processes: sequential evidence learning by Blackwell and Dubins (1962), deliberation processes (*the consensus formation model* by DeGroot (1974)), and higher-order evidence learning (the *peer disagreement* literature and *supra Bayesianism* by Morris (1974)). Note also that the input does not need to be individual beliefs to form collective beliefs. If we use belief elicitation mechanisms, such as the *prediction market* (Wolfers and Zitzewitz, 2004), the input can be individual actions, which supposedly reveal individual beliefs.

[3] We do not argue that only binary beliefs qualify as a group agent's belief type: for a group's decision-making under risk or uncertainty, it might be more appropriate that the group's beliefs take the form of credence; for mere summaries of individual beliefs, credences would be a suitable output data type.

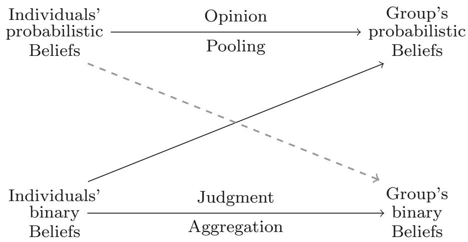

Fig. 1 Four types of belief aggregation problems

In addition to binarizing belief aggregation, another research field called belief binarization deals with different types of beliefs. This subject matter explores rational bridge principles between credences and binary beliefs,[4] We can employ the principles for a group agent as well as an individual agent. While belief binarization takes a single agent's credences as input, binarizing belief aggregation takes multiple agents' credences as input.

Figure 2 illustrates the relationship between binarizing belief aggregation and other research fields.[5] It should be noted that binary beliefs are not necessarily 0/1-valued probabilistic beliefs. Even if binary beliefs satisfy the "standard" rationality norm—logical closure and consistency—in formal epistemology (Alchourrón et al., 1985; Leitgeb, 2017), they do not necessarily correspond to some probabilistic beliefs unless they are complete. In this sense, binarizing belief aggregation cannot be subsumed under probabilistic opinion pooling, but belongs to a generalized notion of belief aggregation that can aggregate arbitrary [0, 1]-valued functions. Figure 2 also suggests how existing resources can be used to devise binarizing belief aggregation methods: any combination of an opinion pooling function and collective belief binarization yields a binarizing belief aggregation method; combining individual belief binarization and a judgment aggregation rule can do so.

Thus far, addressing individual credences and collective binary beliefs has been rare in social epistemology and formal epistemology. Ivanovska and Slavkovik (2019, 2022) suggested a more general—embracing imprecise probabilities—framework than ours and focused on defining aggregators that capture different contexts and have different properties, rather than proving impossibility results. Thorn (2018) investigated joint aggregation of individual belief states, each comprising a quantitative and a qualitative belief, into a collective belief state. Our

[4] In most of the literature regarding belief binarization, belief is associated with high credence. One of them is the well-known *Lockean Thesis* which states that an agent should believe a proposition if and only if its probability exceeds a given threshold. However, the *lottery paradox* shows that the Lockean thesis might result in contradictory beliefs. Many attempts have been made to resolve this paradox (Kyburg, 1961; Leitgeb, 2017; Lin and Kelly, 2012).

[5] The interpretation of binarizing belief aggregation is flexible. The theory can be applied to belief binarization of an imprecise probability that is represented by a (finite) set of probabilistic beliefs. It can also be used for a judgment aggregation problem where a group is divided into several subgroups and each subgroup's credence is calculated somehow from the binary beliefs of subgroup members. For example, consider the case where a political party requests separate opinions of two subgroups (e.g., machine learning programmers and labor economists) on the effect of AI on the labor market, and the opinions of each subgroup are collected through an anonymous and proposition-wise independent procedure. This method is appropriate when we want to respect the subgroups' opinions, rather than individual opinions.

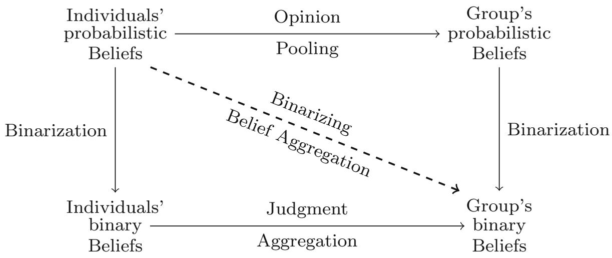

Fig. 2 Binarizing belief aggregation and other research fields

problem is different because we do not deal with joint aggregation that presupposes individual belief binarization. We can find literature where belief binarization methods are applied to judgment aggregation problems (e.g., Chandler (2013) and Cariani (2016)). In contrast, our theory applies belief binarization methods—e.g., combined with an opinion pooling method—to binarizing belief aggregation.

Impossibility result

For the start of research on binarizing belief aggregation, this study aims to explore impossibility theorems. Consider the following example. A political party wants to establish its position on the basic income policy. Hence, it asks three experts for opinions on the following logically related propositions: (A)"AI will outperform humans in all areas by 2050."; (B)"AI will replace humans in the labor market."; ($A \rightarrow B$)"If AI outperforms humans in all areas by 2050, then it will replace humans in the labor market." Three experts' opinions are given in Table 1.[6] Suppose that the party's beliefs in the three propositions and their negations are formed using the following method: a proposition is collectively believed if and only if the average credence is no less than 0.6. We observe that the resulting collective beliefs are neither deductively closed nor consistent. This example shows that binarizing belief aggregation faces a problem such as the *discursive dilemma* in judgment aggregation and the *lottery paradox* in belief binarization.[7]

The main question of this study will be how to formulate and generalize this problem. Thus, reviewing impossibilities in judgment aggregation, opinion pooling, and belief binarization would provide some hints. After generalizing the problem of proposition-wise majority voting in the discursive dilemma, the arrow-style axiomatic method has been applied to issues that are logically represented and interconnected (List and Pettit, 2002). Utilizing this method, much research in judgment aggregation has explicated the tension between proposition-wise independence and rationality norms respecting the logical relation between propositions. For example, independent aggregation generating complete and consistent collective judgments is, under certain minimal conditions, forced to be a dictatorship, if the issues have certain logical interconnection (Dokow and Holzman, 2010). While completeness and consistency are often assumed in judgment aggregation, a few

[6] Note that each agent's opinion in Table 1 is probabilistically coherent, if we regard each formula as the set of possible worlds satisfying the formula.

[7] A typical discursive dilemma in judgment aggregation can also be viewed as a dilemma in binarizing belief aggregation. We thank the first reviewer for this indication.

Table 1 Example of binarizing belief aggregation

Issues	A	$A \to B$	B
Agent 1	0.9	0.7	0.6
Agent 2	0.8	0.4	0.2
Agent 3	0.4	0.7	0.1
Collective Belief	Belief	Belief	Disbelief

studies weakened completeness to deductive closure, where dictatorships were replaced with oligarchies (Gärdenfors, 2006; Dietrich and List, 2008). In opinion pooling and belief binarization, the agenda has usually been assumed to have the structure of an algebra because the two fields usually deal with probability measures. Under this assumption, independent opinion pooling with a certain minor condition (certainty preservation) was shown to be restricted to linear pooling (McConway, 1981). Belief binarization can be thought of as anonymous proposition-wise independent judgment aggregation. Accordingly, some impossibilities demonstrated the tension between independence and rationality norms such as deductive closure (Dietrich and List, 2018, 2021).

These impossibilities shed some light on developing our theory. We address the following two questions. First, what kinds of impossibility results can be formulated? One may conjecture that, if we combine strong properties—completeness of collective beliefs and anonymity—with the strong agenda condition of being an algebra, then an impossibility of independent binarizing belief aggregation could be easily obtained. We go further and make stronger claims: (i) we relax completeness that is often required in judgment aggregation and assume deductive closure as in belief binarization, and (ii) we drop anonymity hidden in the belief binarization problem, which enables us to respect different degrees of expertise as in judgment aggregation or opinion pooling. Second, how are our theorems and proofs compared with impossibility results in judgment aggregation, opinion pooling, and belief binarization? We analyze the similarities and differences between our results and others and argue that our result is free-standing and novel.

To answer these questions, the remainder of this paper is organized as follows. Section 2 illustrates our setting and formally defines binarizing belief aggregation. Section 3 formulates some (possible) axiomatic requirements on binarizing belief aggregation. Section 4 proves three impossibility theorems: the triviality result with deductive closure and anonymity, the oligarchy result without anonymity, and the non-existence theorem with completeness. Section 5 compares our main results with some related results in other research areas. Finally, in the conclusion section, we refer to some follow-up studies.

2 Binarizing belief aggregation

We now introduce our terminology and a formal definition of binarizing belief aggregation. Let W be a non-empty set of possible worlds and $\mathcal{P}(W)$ denote the set of all subsets of W. An *agenda* $\mathcal{A}$ on W is a non-empty subset of $\mathcal{P}(W)$ that is closed under complement (i.e., for all $A \in \mathcal{A}$, $A^c \in \mathcal{A}$, where A^c stands for the complement of A with respect to W). We call an element of $\mathcal{A}$ an *event*, which counts as a *proposition*. We denote the set of n individuals by $N := \{1, ..., n\}$ where $n \geq 1$. An agent's credences on $\mathcal{A}$ are represented by a credence function P from $\mathcal{A}$ to [0, 1]. We write $\vec{P} := (P_1, ..., P_n) = (P_i)_{i \in N}$ for a profile of n individuals' credences on $\mathcal{A}$. We call a function F taking $\vec{P}$ in a given domain into $F(\vec{P}) : \mathcal{A} \to [0, 1]$ an *aggregator*, where $F(\vec{P})$ represents collective credences on $\mathcal{A}$. For any individual or collective credence function, if its codomain is restricted to $\{0, 1\}$, then the function is called a *binary belief function* on $\mathcal{A}$. For a binary belief function Bel and a proposition $A \in \mathcal{A}$, $Bel(A) = 1$ means that A is believed, and $Bel(A) = 0$ means that A is not believed. In contrast, $P : \mathcal{A} \to [0, 1]$ is called *probabilistically coherent beliefs* on $\mathcal{A}$, or simply, *probabilistic beliefs* on $\mathcal{A}$, if P is extendable to a finitely additive probability function on the algebra $\mathfrak{a}(\mathcal{A})$ generated by $\mathcal{A}$ (i.e., the smallest algebra that includes $\mathcal{A}$).[8] Note that we do not require that a probabilistic belief function should be extendable to a σ-additive probability measure. This is because we will not need σ-additivity to prove any of our results. Binarizing belief aggregation deals with individuals' probabilistic beliefs and the group's binary beliefs. It is formally defined as follows:

Definition 1 (Binarizing Aggregator) Let $\mathcal{A}$ be an agenda. A binarizing aggregator (BA) F is a function that takes each profile $\vec{P}$ of n probabilistic belief functions on $\mathcal{A}$ in a given domain[9] and returns a binary belief function $F(\vec{P}) : \mathcal{A} \to \{0, 1\}$.

Notice that this notion of binarizing aggregator is more general than usual opinion pooling functions with the codomain restricted to 0/1-valued probability functions in the following two senses: (i) the agenda of a BA does not need to be an algebra, but just to be closed under complement, and (ii) the outputs of a BA does not need to be subject to a rationality condition (e.g., probabilistic coherence) but can be arbitrary 0/1-valued functions, which enables us to discuss various levels of rationality. Moreover, it is worth noting that we could have defined the notion of BA with more general inputs without imposing the rationality condition of probabilistic coherence —i.e., inputs of profiles of n arbitrary functions from $\mathcal{A}$ to [0, 1]—and then we could have incorporated the rationality condition later when we discuss individual rationality. However, we build individual rationality into the above definition for simplicity since we assume, throughout this paper, individual beliefs to be probabilistic beliefs, and we will compare our theory with opinion pooling theory, where probabilistic beliefs are usually addressed.

While studying binarizing belief aggregation, we also compare it with opinion pooling, judgment aggregation, and belief binarization. For a start, let us compare the above definition with the following formal definitions: a *judgment aggregator (JA)* is a function that takes each profile $\vec{P}$ of n binary belief functions on $\mathcal{A}$ in a given domain and returns a binary belief function; an *opinion pooling function (OP)* is a

[8] In the opinion pooling context, finitely additive probability measures are used and discussed in Herzberg (2015) and Nielsen (2019).

[9] A domain of F can be any non-empty set of profiles, which is not necessarily the set of all profiles.

function that takes as input each profile $\vec{P}$ of n probabilistic belief functions on $\mathcal{A}$ in a given domain and outputs a probabilistic belief function.[10][11]; finally, a *belief binarization function (BIN)* is a function that assigns to a probabilistic belief function on $\mathcal{A}$ a binary belief function. Note that a BIN can be construed as a special case of binarizing aggregator with the number n of individuals being one.

3 Rationality norms on binarizing aggregators

This section introduces axiomatic requirements for binarizing belief aggregation and explores their relations. The requirements can be divided into two groups. The first group concerns inputs and outputs and involves rationality norms imposed on individual and collective beliefs, whereas the second group is related to postulates governing aggregation rules themselves.

Individual and Collective Rationality

We focus on obtaining "rational" collective beliefs given "rational" individual beliefs. The kinds of rationality requirements may differ depending on whether they are imposed on probabilistic beliefs or binary beliefs: a probabilistic belief should be extendable to a finitely additive probability function, which is already assumed by the definition of BA; for binary beliefs, rationality requirements pertain to the notions of *consistency, deductive closure*, and *completeness*. To explicate these notions in our setting, where the agenda is not necessarily an algebra, we first define an entailment relation.

Definition 2 (Entailment) Let $\mathcal{Y} \subseteq \mathcal{P}(W)$ and $A \subseteq W$. We say that $\mathcal{Y}$ entails A (in symbols $\mathcal{Y} \vDash A$) if there exists a finite subset $\mathcal{X} \subseteq \mathcal{Y}$ such that $\bigcap \mathcal{X} \subseteq A$.[12]

In this definition, the finiteness clause is included because weaker notions of deductive closure and consistency will suffice for our results, as will be explained in detail in what follows. Based on the notion of entailment, we define the following rationality norms.

Definition 3 (Rationality of Binary Beliefs) Let $Bel : \mathcal{A} \to \{0, 1\}$ be a binary belief function. We will use Bel not only for the function but also for the set of all believed propositions, i.e., $\{A \in \mathcal{A} | \; Bel(A) = 1\}$.

[10] In most of the opinion pooling literature, the underlying agenda is a σ-algebra, and σ-additive probability measures are addressed (McConway, 1981). However, the results relevant to our research—the characterization of linear pooling—can be obtained under the weaker assumption of finite-additivity on an algebra as well. Thus, we do not assume the agenda $\mathcal{A}$ to be a σ-algebra and probability functions to be σ-additive when we regard opinion pooling. Moreover, we do not demand our agenda $\mathcal{A}$ to form an algebra and thus we only require inputs and outputs to be extendable to a finite-additive probability function on $\mathfrak{a}(\mathcal{A})$.

[11] Note that Dietrich and List (2017a, 2017b) explored generalized opinion pooling where the agenda $\mathcal{A}$ does not need to be an algebra and is just required to be closed under complement as our agenda. However, unlike our definition of probabilistic beliefs, they require inputs and outputs of opinion pooling to be extendable to σ-additive probability measures.

[12] In this definition, we do not exclude $\mathcal{Y} = \emptyset$ and adopt the convention that $\bigcap \emptyset = W$, which implies $\emptyset \vDash W$.

(1) Bel is complete if $A \in Bel$ or $A^c \in Bel$ for all $A \in \mathcal{A}$.
(2) Bel is deductively closed if for all $A \in \mathcal{A}$ such that $Bel \vDash A$, $A \in Bel$.
(3) Bel is consistent if $Bel \nvDash \emptyset$.

Completeness says that, for every event, at least one of the event and its negation should be believed. This norm is quite demanding in the sense that suspending judgment on a proposition—believing neither the proposition nor its negation—is not allowed. Next, deductive closure means that every proposition that includes the intersection of *finitely* many believed propositions should be believed (see the finiteness clause in Definition 2).[13] This can be compared with the condition that every proposition that includes the intersection of *countably* many believed propositions should be believed and the condition that every proposition that includes $\bigcap Bel$ should be believed. If the agenda is finite, then those two conditions and our notion of deductive closure are all equivalent.[14] Our notion of deductive closure is weaker than those conditions.[15] For our results in this paper, our weaker notion of deductive closure will suffice. Lastly, consistency means that the set of believed propositions should not entail a contradiction, i.e., every intersection of *finitely* many believed propositions should be non-empty. This notion is weaker than the requirement that every intersection of *countably* many believed propositions should be empty, which we call *σ-consistency* and the requirement that $\bigcap Bel \neq \emptyset$, as we can observe similarly to the above.[16] We will need only the weaker notion of consistency for our results. Another alternative definition of consistency could be considered: a contradiction should not be believed—i.e., $Bel(\emptyset) = 0$. Under the assumption of $\emptyset \in \mathcal{A}$, our consistency of Bel implies $Bel(\emptyset) = 0$, and the converse holds if Bel is deductively closed. Since we do not always require that $\emptyset \in \mathcal{A}$, we will use the definition of part (3).

[13] Provided that the agenda is an algebra, our notions of deductive closure and consistency are the same as the notions of individual or social logical closure and consistency in Gärdenfors (2006).

[14] There are also cases with the agenda being infinite where no distinction is made between our notion of deductive closure and the other two notions: Let W be the set of all $0/1-$valuations on classical propositional language L with countably infinite atomic propositional letters and $\mathcal{A} = \{[\psi] \subseteq W | \ \psi \text{ is a formula in } L\}$ where $[\psi]$ is the set of all valuations assigning 1 to ψ. In this space, for any $Bel \subseteq \mathcal{A}$, any superset of $\bigcap Bel$ in the agenda $\mathcal{A}$ is a superset of the intersection of finitely many elements in Bel due to compactness of propositional logic. For example, consider $Bel = \{[\psi] \subseteq W | \ v(\psi) = 1\}$ for some $v \in W$. Then, $\bigcap Bel = \{v\}$ is not in $\mathcal{A}$ and hence not in Bel since a singleton valuation $\{v\}$ cannot be expressed by a formula. However, any superset of $\{v\}$ in the agenda $\mathcal{A}$ can be expressed by a finite conjunction of formulas and thus, Bel is not only deductively closed by our definition but also satisfies the other two stronger notions.

[15] Let us explain this with the following two examples. (1) Let W be the set $\mathbb{N}$ of natural numbers and the agenda $\mathcal{A} = \mathcal{P}(\mathbb{N})$. Consider the belief set $Bel = \{A \in \mathcal{P}(\mathbb{N}) | \ A^c \text{ is finite }\}$, called Fréchet filter on $\mathbb{N}$. (2) Let W be the set $\mathbb{R}$ of real numbers and the agenda $\mathcal{A}$ be the Borel algebra $\mathcal{B}$. Consider $Bel = \{A \in \mathcal{B} | \ (0, \epsilon] \subseteq A \text{ for some } \epsilon > 0\}$. In both examples, we have $\bigcap Bel = \emptyset \notin Bel$ and there is an empty intersection of a countable subset of Bel (e.g., $\bigcap \{[n, \infty) \in \mathcal{P}(\mathbb{N}) | \ n \in \mathbb{N}\} = \emptyset$ and $\bigcap \{(0, \frac{1}{2^n}] \in \mathcal{B} | \ n \in \mathbb{N}\} = \emptyset$). However, Bel is deductively closed according to our definition.

[16] Consider the two examples in Footnote 15. In both examples, there is an empty intersection of countably many believed propositions, and $\bigcap Bel = \emptyset$. However, they are consistent according to our definition.

This paper will address two kinds of rationality norms on binary beliefs: (i) consistency and deductive closure, and (ii) consistency and completeness. Let us compare them. First, it is well-known that deductive closure follows from consistency and completeness. Second, unlike binary beliefs with consistency and deductive closure, binary beliefs with completeness and consistency count as 0/1-valued probabilistic beliefs and vice versa, as the following lemma states. Note that they might not be extendable to σ-additive probability functions on the σ-algebra generated by the agenda. To obtain the latter property, we need to strengthen consistency of binary beliefs to σ-consistency.

Lemma 1 *Let W be a non-empty set, $\mathcal{A} \subseteq \mathcal{P}(W)$ be an agenda, and $Bel : \mathcal{A} \rightarrow \{0, 1\}$ be a binary belief function. Then, (i) Bel is complete and consistent iff Bel is extendable to a finitely additive 0/1-valued probability function on the algebra $\mathfrak{a}(\mathcal{A})$ generated by $\mathcal{A}$. (ii) Bel is complete and σ-consistent iff Bel is extendable to a σ-additive 0/1-valued probability function on the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$.*

Now, we are ready to state some postulates of BAs concerning individual and collective rationality. First, the requirement of individual rationality is related to the domain of an aggregator. The requirement is alluded to by what is called universal domain, which states that the domain should include all profiles of "rational" individual beliefs.

Definition 4 (Universal Domain) A BA F satisfies universal domain (UD) if the domain of F is the set of all profiles of n probabilistic belief functions on $\mathcal{A}$.

Let us define the universal domain of an OP and a JA to compare our results about a BA with the corresponding ones regarding an OP or a JA later. We define the universal domain of an OP as the same as the universal domain of a BA in the above definition; a JA satisfies universal domain if and only if its domain is the set of all profiles of n binary belief functions on $\mathcal{A}$ that are consistent and complete. The reason why we require completeness here, rather than deductive closure, is that most of the literature on judgment aggregation assumes completeness of individual beliefs, and we would like to compare our results with the ones in judgment aggregation. Moreover, consistent and complete binary beliefs can be viewed as 0/1-valued probabilistic beliefs, as shown in Lemma 1, which can be inputs of a BA. Accordingly, the domain of a JA can be thought of as a subset of the domain of a BA, and a JA can be regarded as a restriction of a BA.

Next, we will define constraints imposed on the outputs of an aggregator, i.e., collective beliefs. While UD regulates the domain of an aggregator, collective rationality governs the codomain.

Definition 5 (Collective Rationality) A BA F satisfies collective completeness (CCP)/ collective consistency (CCS)/ collective deductive closure (CDC) if $F(\vec{P})$ is complete/ consistent/ deductively closed for all $\vec{P}$ in the domain of F, respectively.

In opinion pooling, collective rationality corresponds to the requirement that $F(\vec{P})$ is a probabilistic belief function. In judgment aggregation, we have the same definition as above with the domain of F understood as the domain of a JA. Some research investigates judgment aggregation under the assumption of CCP and CCS (Dokow and Holzman, 2010; Nehring and Puppe, 2010), and some under the assumption of CCS and CDC (Gärdenfors, 2006; Dietrich and List, 2008). In our study, we will address both assumptions.

Unanimity, Anonymity, and Independence

Now, we consider the second group of postulates of BAs. Irrespective of whether F is a BA, an OP, or a JA, these postulates can be defined in the same ways, except that the domain of F is interpreted differently. Let $\vec{P}(A)$ denote the vector $(P_1(A), \ldots, P_n(A))(= (P_i(A))_{i \in N})$ for any $A \in \mathcal{A}$. We define a kind of unanimity required whenever everyone has a probabilistic belief of 1 or everyone has a probabilistic belief of 0.

Definition 6 (Unanimity) Let F be an aggregator.

(1) F satisfies certainty preservation (CP) if for all $A \in \mathcal{A}$ and all $\vec{P}$ in the domain of F, if $\vec{P}(A) = \vec{1}$ $(:= (1, \ldots, 1) \in [0,1]^n)$, then $F(\vec{P})(A) = 1$.
(2) F satisfies zero preservation (ZP) if for all $A \in \mathcal{A}$ and all $\vec{P}$ in the domain of F, if $\vec{P}(A) = \vec{0}$ $(:= (0, \ldots, 0) \in [0,1]^n)$, then $F(\vec{P})(A) = 0$.

CP posits that, if everyone in a group is certain of a proposition being true, then the group should also believe it. Meanwhile, ZP states that, if everyone in a group is certain of a proposition being false, then the group should not believe it (the group does not necessarily have to disbelieve it though). In opinion pooling, CP and ZP are equivalent.[17] In binarizing belief aggregation as well as judgment aggregation, we have the following lemma.

Lemma 2 *In binarizing belief aggregation and judgment aggregation, the following holds*:

(1) Given CCS, CP implies ZP.
(2) Given CCP, ZP implies CP.
(3) Let $W, \emptyset \in \mathcal{A}$. Then, $F(\vec{P})(W) = 1$ by CDC or CP, and $F(\vec{P})(\emptyset) = 0$ by CCS or ZP.
(4) Let $\emptyset \in \mathcal{A}$ and F satisfy CDC and CP. Then, F satisfies ZP if F satisfies CCS.

Next, let us introduce the anonymity norm of a BA, which requires that collective beliefs should not be inclined to some particular agents' opinions. Although this norm has been extensively studied in social choice theory and judgment aggregation, this kind of fairness norm might be questionable in epistemic collective decisions. Indeed, it could be better to prioritize some agents' opinions who are experts on the issue. However, there may be situations where it is unknown to whom the submitted opinions belong or which agents are experts on the issue, or where the group consists

[17] In opinion pooling, (i) $\vec{P}(A^c) = \vec{1}$ is equivalent to $\vec{P}(A) = \vec{0}$, and (ii) $F(\vec{P})(A) = 0$ is equivalent to $F(\vec{P})(A^c) = 1$. However, in binarizing belief aggregation, (ii) does not hold unless we assume that CCP and CCS. Under the assumption of CDC and CCS, $F(\vec{P})(A^c) = 1$ does not follow from $F(\vec{P})(A) = 0$.

of epistemic peers. In the next section, we will proceed with and without this norm to address various situations.

Definition 7 (Anonymity) Let F be an aggregator. F satisfies anonymity (AN) if $F((P_{\pi(i)})_{i \in N}) = F((P_i)_{i \in N})$ for all $\vec{P}$ in the domain of F and all permutations π on N.

The last property we now introduce is the most controversial one. Independence between propositions ensures that the resulting belief in a proposition depends only on individual probabilistic beliefs in that proposition; thus, an aggregation should be performed proposition-wise. It is of practical use as we do not need to consider all values of a profile if we want to focus on the result of one proposition. However, this norm can create tension with the requirement of consistency and deductive closure or the one of consistency and completeness, when propositions are logically related. This tension has been pointed out as one of the main culprits of impossibility results in the judgment aggregation literature. Let us define the independence norm and a stronger norm, the systematicity norm.

Definition 8 (Independence and Systematicity) Let F be an aggregator.

(1) F satisfies proposition-wise independence (IND) if for all $A \in \mathcal{A}$, there is a function $G_A : [0,1]^n \to \{0,1\}$ such that $F(\vec{P})(A) = G_A(\vec{P}(A))$ for all $\vec{P}$ in the domain of F.
(2) F satisfies systematicity (SYS) if there is a function $G : [0,1]^n \to \{0,1\}$ such that $F(\vec{P})(A) = G(\vec{P}(A))$ for all $\vec{P}$ in the domain of F and all $A \in \mathcal{A}$

Systematicity can be viewed as the independence norm plus the *neutrality* norm, where a BA F is neutral if and only if for all $A, B \in \mathcal{A}$, if $\vec{P}(A) = \vec{P}(B)$, then $F(\vec{P})(A) = F(\vec{P})(B)$ for all $\vec{P}$ in the domain of F. Neutrality means that each proposition is determined to be believed or not by the same rule. Combining neutrality with independence yields the norm of systematicity—i.e., the group's belief in each proposition should be determined by the same function of individual probabilistic beliefs in that proposition.

4 Impossibility results

This section presents the main results. We prove that any proposition-wise independent BA with collective deductive closure and anonymity satisfying universal domain and certainty and zero preservation yields a trivial aggregation function. Furthermore, we drop anonymity and show that it is not so helpful in avoiding degenerate procedures. Additionally, we demonstrate that if we add the assumption of collective consistency and collective completeness, then there will be no aggregation function that satisfies the above-mentioned postulates.

Here and in the following section, we assume the complexity of the logical interconnections in the agenda to be a *non-trivial* algebra, defined as follows.

Definition 9 (Non-trivial Algebra) An algebra $\mathcal{A}$ is non-trivial if it has at least three elements besides the empty set and W.

Thus, a trivial algebra on a set W has the form of $\{\emptyset, A, A^c, W\}$ for some $A \subseteq W$. In this case, the logical connection is so minimal that collective deductive closure is too lenient for an independent BA to yield impossibility results. Thus, we require that the agenda is not a trivial algebra in order to formulate impossibility results. Non-trivial algebras have quite complex logical connections. In Wang and Kim (2023), we have relaxed this assumption of being a non-trivial algebra and found minimal agenda conditions to obtain each result in this study. The reason why we provide our main results under a restricted assumption in this study is to compare our results with similar findings in opinion pooling and belief binarization, where finitely additive probability functions on an algebra are usually dealt with.[18] Moreover, we take the aggregation problems where individual beliefs are finitely additive probability functions on a given algebra to be the most typical ones in binarizing belief aggregation. Therefore, this paper provides direct proofs for this restricted but most typical case of binarizing belief aggregation.

Triviality Result

To prove our main results, we need a lemma stating that certainty and zero preserving (CP, ZP) independent (IND) binarizing belief aggregation satisfies systematicity (SYS) under the assumption of universal domain (UD) and collective deductive closure (CDC). We will compare this with the corresponding result in opinion pooling, which says that given UD of opinion pooling—since $\mathcal{A}$ is a non-trivial algebra, the universal domain is the set of all profiles of n finitely additive probability functions—, an OP satisfies CP and IND if and only if it satisfies SYS (McConway, 1981; Herzberg, 2015). In judgment aggregation, there is a lemma, called the contagion lemma, that can be stated in the same way as ours, except that UD of binarizing belief aggregation is replaced by UD of judgment aggregation (Dokow and Holzman, 2010).

Lemma 3 (*Contagion Lemma*) *Let $\mathcal{A}$ be a non-trivial algebra and F be a BA with UD. If F satisfies CDC*, *ZP*, *CP*, *and IND*, *then it satisfies SYS.*

This lemma shows that to obtain SYS from CP and IND, we need CDC and ZP, neither of which is required to get the corresponding result in opinion pooling. First, let us focus on CDC. Recall that the requirement of collective rationality on a credence function is that it is extendable to a finitely additive probability measure. In opinion pooling, this requirement is satisfied not by a separate collective rationality condition, but rather by the definition of the codomain of opinion pooling functions. Accordingly, one may expect that some collective rationality conditions might be needed in binarizing belief aggregation. This lemma shows that we need to add CDC. Second, ZP can be replaced with CCS by Lemma 2 (4). Note that to prove this lemma ZP (or equivalently CCS) is only used for the case where $\emptyset \in \mathcal{A}$. If our agenda is not an algebra and $\emptyset \notin \mathcal{A}$, then ZP is not required. Provided that $\emptyset \in \mathcal{A}$ as

[18] For an exception, see Footnote 11.

in this section, we need ZP, which does not follow from CP, unlike the case in opinion pooling.

Now, we prove our main results formulated by the subsequent two theorems stating that the conditions for obtaining SYS in the previous lemma lead to degenerate procedures. It is worth comparing our theorems with the corresponding ones in judgment aggregation formulated in our terminology: if a JA F satisfies UD of JA, ZP, CP, IND, and CDC, then F is oligarchic, and if AN is added, then F is the unanimity rule (Dietrich and List 2008[19]). In opinion pooling, given UD of OP, an OP satisfies CP and IND if and only if it is a linear pooling function (McConway, 1981). In binarizing belief aggregation, the first main result is related to the unanimity rule defined as follows.

Definition 10 (Unanimity Rule) An aggregator F is the unanimity rule if

$$F(\vec{P})(A) = \begin{cases} 1 & \text{if } \vec{P}(A) = \vec{1} \\ 0 & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{A}$ and all profiles $\vec{P}$ in the universal domain.

Thus, the unanimity rule states that a proposition is believed if and only if every individual assigns the probability of 1 to the proposition. Therefore, a proposition should not be believed even if only one individual assigns a probability slightly below 1 (e.g., 0.9999). Thus, the unanimity rule is quite demanding in two senses: (i) it requires unanimous opinions to believe a proposition, and (ii) the unanimous opinions should not have a probability less than 1.[20] Consequentially, we consider the unanimity rule to be trivial and also call it the trivial rule. Here is our first main result, called the triviality result:

Theorem 4 (*Triviality Result*) *Let $\mathcal{A}$ be a non-trivial algebra. The only BA satisfying UD, ZP, CP, IND, CDC, and AN is the unanimity rule.*

This theorem shows that, under certain mild conditions (UD, ZP, and CP), it is impossible that an anonymous (AN) proposition-wise independent (IND) binarizing belief aggregation guarantees deductively closed collective beliefs (CDC), except for the unanimity rule. Note that we can replace ZP with CCS, by Lemma 2 (4) since we have CDC, CP, and $\emptyset \in \mathcal{A}$. Completeness is not required to obtain this result. CDC serves as a sufficient rationality condition on collective beliefs to create tension with IND. Interestingly, the conditions leading to the unanimity rule are the same as those in the corresponding result in judgment aggregation. One might question whether our result can be directly derived from that result, but this is not the case. In the next

[19] Their formal setting and their notion of deductive closure are different from ours. However, we can easily check that their results can be adapted to our setting. Note that they assumed, instead of ZP and CP, the weaker condition of unanimity preservation such that if $P_i = P$ for all i, then $F((P_i)_{i \in N}) = P$, which follows from ZP and CP, but not in general vice versa. However, it is easily shown that, if we have IND, the converse also holds. Therefore, their result is equivalent to the above statement.

[20] Note that the unanimity rule parallels the notion of the unanimity rule in judgment aggregation, where, however, the opinion short of 1 means the opinion of 0, as opposed to binarizing belief aggregation.

section, we will conduct a detailed comparison of our results with those in judgment aggregation and provide the reasons for the differences.

It is worth understanding the structure of our proof because this method will also be applied to prove the next theorem and discussed in several places throughout this paper. First, SYS follows from UD, CDC, ZP, CP, and IND by Lemma 3, so we have a function G such that $F(\vec{P})(A) = G(\vec{P}(A))$ and $G(\vec{1}) = 1$. Our proof unfolds in three steps. [Step 1] presents the following two key facts, utilizing the agenda condition of being a non-trivial algebra and the property of CDC (closure under superset for (Fact 1) and closure under intersection for (Fact 2)):

(Fact 1) if $\vec{a} \leq \vec{b}$ and if $G(\vec{a}) = 1$, then $G(\vec{b}) = 1$
(Fact 2) if $\vec{a} + \vec{b} - \vec{1} \geq \vec{0}$ and if $G(\vec{a}) = 1$ and $G(\vec{b}) = 1$, then $G(\vec{a} + \vec{b} - \vec{1}) = 1$

where $\leq$ is applied component-wise.[21] (Fact 1) means that if G assigns 1 to a vector, then G does so to every component-wise greater vector than that, which we call *upward closure of* $G^{-1}(1)$ (the preimage of 1 under G). In contrast, (Fact 2) claims that certain component-wise smaller vectors than a vector in $G^{-1}(1)$ are also contained in it, which we call *restricted downward closure of* $G^{-1}(1)$. Next, [Step 2] demonstrates that any non-trivial BA where $G(\vec{a}) = 1$ with some $a_i \neq 1$ yields

$$G(\vec{a}[a_i \mapsto 0, a_l \mapsto 1 \text{ for all } l \neq i]) = 1$$

indicating that substituting a_i and a_l with 0 and 1, respectively, does not alter the value of G, i.e., we have $(1, ..., 1, 0, 1, ..., 1) \in G^{-1}(1)$ where 0 is the i-th component. Indeed, through the upward closure of $G^{-1}(1)$, we can substitute a_l with 1 for all $l \neq i$; by iterating the process of upward and restricted downward closure of $G^{-1}(1)$, we can then substitute a_i with 0. Finally, using AN and (Fact 2), [Step 3] shows

$$G((0, ..., 0)) = 1$$

which contradicts ZP.

One may ask whether AN is the main culprit in the difficulty of a proposition-wise BA to get deductively closed collective beliefs. The following theorem shows that that is not the case, and dropping AN is insufficient to avoid the difficulty.

Without Anonymity: Oligarchy

Now, we drop the assumption of anonymity and show that it leads to other degenerate BAs as well. To this end, we define an oligarchy first. Recall that N denotes the set of all individuals.

Definition 11 (Oligarchy) An aggregator F is an oligarchy if there is a non-empty subset M of N such that

[21] (Fact 1) represents a kind of *monotonicity* defined by

(MON) if $\vec{P}(A) \leq \vec{P}'(A)$ and $F(\vec{P})(A) = 1$, then $F(\vec{P}')(A) = 1$.

$$F(\vec{P})(A) = \begin{cases} 1 & \text{if } P_i(A) = 1 \text{ for all } i \in M \\ 0 & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{A}$ and all profiles $\vec{P}$ in the universal domain. When $|M| = 1$, we call F a dictatorship.

So an oligarchy refers to a rule where the unanimous certain beliefs of a group of oligarchs are the necessary and sufficient condition for the collective belief in a proposition. This procedure is also problematic partly because individuals other than the oligarchs are excluded from the decision process. However, one could argue that relying on experts, who are considered oligarchs, would not be irrational to obtain true collective beliefs. Nevertheless, non-oligarchy is still a rational requirement even in epistemic contexts, as after excluding non-experts, the decision process among the oligarchs can be viewed as the trivial aggregation among the oligarchs. With this type of degenerate BA, we have another impossibility theorem stating that the same conditions as the triviality result, except for AN lead to oligarchies.

Theorem 5 (*Oligarchy Result*) *Let $\mathcal{A}$ be a non-trivial algebra. The only BAs satisfying UD, ZP, CP, IND, and CDC are the oligarchies.*

This theorem generalizes the triviality result and shows that dropping AN leads to the oligarchy result. In the proof, we adopted [Step 1]—thus, (Fact 1) and (Fact 2)—and [Step 2] in the proof of the triviality result since we have not used AN to prove them. The two proofs are similar in spirit. However, instead of deriving $G((0, ..., 0)) = 1$ from $G(\vec{a}) = 1$ where $a_i \neq 1$ for some $i \in N$, we derived $G((\mathbb{1}_M(i))_{i \in N}) = 1$, where $\mathbb{1}_M(i) = 1$ if $i \in M$, otherwise $\mathbb{1}_M(i) = 0$, from $G(\vec{b}) = 1$ where $b_j \neq 1$ for some $j \in N \setminus M$. In this process, we used [Step 2] and applied induction analogously to [Step 3]—except that induction is used on non-oligarchs, not on all individuals—to prove that, even if all non-oligarchs certainly believe that a proposition is false, the oligarchs' unanimous certain beliefs in that proposition yield the collective belief in it.

With CCP and CCS: Non-existence

Until now, we have assumed only CDC. However, if we impose a stronger rationality constraint on collective belief, we obtain another impossibility result, which we achieve as a corollary of the oligarchy result. In judgment aggregation, there is an impossibility result for a finite agenda, which can be rephrased in our terminology as follows: if F satisfies UD of judgment aggregation, CP, IND, CCP, and CCS, then F is a dictatorship (Dokow and Holzman, 2010), and if we add AN, then there is no such JA. We now present our result in binarizing belief aggregation.

Corollary 6 (*Non-existence Result*) *Let $\mathcal{A}$ be a non-trivial algebra. There is no BA satisfying UD, CP, IND, CCP, and CCS.*

Note that, to use the contagion lemma and the oligarchy result, we add CCS since we need to get CDC from CCP and CCS. Under the assumption of CCS and CCP, it holds that ZP if and only if CP (see Lemma 2). In judgment aggregation, from the same assumptions, we obtain dictatorships, where a dictator's binary beliefs are complete. By contrast, in binarizing belief aggregation, a dictator's binary beliefs determined somehow from her probabilistic beliefs—e.g., in the way that a proposition is believed if and only if a dictator gives it a probability of 1—hardly satisfy completeness. That is why we need neither non-dictatorship nor anonymity to obtain the above impossibility result in binarizing belief aggregation.

To summarize this section, we proved three *impossibility theorems*—provided that $\mathcal{A}$ is a non-trivial algebra, there is no BA satisfying any of the following:

(1) UD, ZP, CP and IND + CDC + Non-oligarchy;
(2) UD, ZP, CP and IND + CDC + AN + Non-triviality;
(3) UD, CP and IND + CCS and CCP.

Note that, from the impossibility of (1), we could obtain the ones of (2) and (3) because the only anonymous oligarchy is the trivial rule. Nevertheless, we first provided a direct proof of the triviality result separately and then modified the proof to obtain the oligarchy result. This is because the fact that the triviality result follows from the oligarchy result holds only under certain agenda conditions, such as the agenda being a non-trivial algebra.[22] In some other settings, where the oligarchy result does not hold,[23] we need to prove the triviality result separately using [STEP 3].

5 Comparison with judgment aggregation, opinion pooling, and belief binarization

Judgment Aggregation and Binarizing Belief Aggregation

Let us compare our results with certain impossibility results in judgment aggregation. The impossibility results with (1),(2), and (3) in Table 2 summarize our results in the last section. The ones with (1′) and (2′) are derived from Dietrich and List (2008) and reformulated in our terminology, and the ones with (3′) and (4′) are from Dokow and Holzman (2010) and Nehring and Puppe (2010). To compare all of them in a common underlying space, we assume, throughout this section, the agenda $\mathcal{A}$ to be a non-trivial algebra as in the last section. When we compare (3) with (3′) and (4′), we further assume $\mathcal{A}$ to be finite as in Dokow and Holzman (2010) and Nehring and Puppe (2010).

The universal domain of judgment aggregation can be viewed as a subset of the one of binarizing belief aggregation since consistent and complete binary beliefs can be seen as probabilistic beliefs, as proven in Lemma 1. Therefore, the restriction of a BA F with UD to the universal domain of judgment aggregation, denoted by $F\upharpoonleft$, can be regarded as a JA. Conversely, any extension of a JA F' to the universal domain of binarizing belief aggregation, denoted by $F'\urcorner$, can be seen as a BA. So it is of interest to know whether our results can be obtained "directly" from the corresponding

[22] In Wang and Kim (2023), we have proven that path-connectedness and even-negatability constitute the necessary and sufficient agenda condition for the oligarchy result. Thus, the triviality result follows from the oligarchy result only under that agenda conditions.

[23] For example, if the agenda is negation connected, which is proven in Wang and Kim (2023) to be the necessary and sufficient agenda condition for the triviality result, the triviality result holds, but the oligarchy result does not hold.

Table 2 Impossibility results in binarizing belief aggregation and judgment aggreagtion

There is no BA with UD of binarizing belief aggregation satisfying
(1) ZP, CP and IND + CDC + Non-oligarchy
(2) ZP, CP and IND + CDC + AN + Non-triviality
(3) CP and IND + CCS and CCP
There is no JA with UD of judgment aggregation satisfying
(1′) ZP, CP and IND + CDC + Non-oligarchy
(2′) ZP, CP and IND + CDC + AN + Non-triviality
(3′) CP and IND + CCS and CCP + Non-dictatorship
(4′) CP and IND + CCS and CCP + AN

results in judgment aggregation or vice versa, by looking at some restrictions or expansions. In what follows, (I) and (II) will argue that those are not the case. In addition, (III) will illustrate the similarities and differences between the proofs in binarizing belief aggregation and judgment aggregation, explaining in detail the originality of our proofs. From now on, BAs and JAs are assumed to satisfy their UD. Our arguments are based on the following observations.

(Observation 1) If a BA F satisfies ZP/CP/IND/CDC/AN/CCP/CCS, then so does the JA $F\upharpoonleft$, respectively.

(Observation 2) If a JA F' satisfies non-oligarchy/non-triviality, then so does any BA $F'\urcorner$, respectively.

(Observation 1) holds because each property mentioned is stated with "for all $\vec{P}$ in the domain" and the universal domain of BA includes the one of JA. (Observation 2) holds because, if a BA F is an oligarchic/trivial function, then the JA $F\upharpoonleft$ is an oligarchic/trivial function, respectively, as demonstrated in Fig. 3.

(I) First, consider whether our results follow "directly" from the results in judgment aggregation. In (Observation 1), which pertains to ZP/CP/IND/CDC/AN/CCP/CCS, each mentioned property of a BA F alone leads to that of $F\upharpoonleft$ in an obvious way without combining other properties or agenda conditions. However, when it comes to non-oligarchy/non-triviality, this is not the case: without combining any other properties—e.g., CP, IND, and CDC—and some agenda conditions—e.g., being a non-trivial algebra—, it does not hold that, if a BA F satisfies non-oligarchy/non-triviality, then so does the JA $F\upharpoonleft$, respectively, in contrast to (Observation 2). If that did hold, then we could, together with (Observation 1), argue as follows: if there were a BA F with UD satisfying (1)/(2)/(3), then $F\upharpoonleft$ would satisfy (1′)/(2′)/[(3′) and (4′)], respectively. This would imply that there were no such BA, allowing us to establish the impossibility of (1)/(2)/(3) directly from the one of (1′)/(2′)/[(3′) or (4′)]. In this sense, we can conclude that our results do not directly follow from the results in judgment aggregation. Indeed, in our proof of Theorem 4, [Step 2] involves showing that for any non-trivial BA

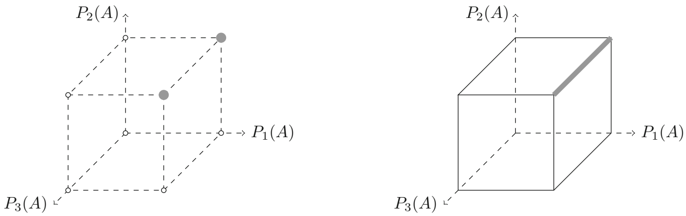

Fig. 3 The left and right figures describe an oligarchy in judgment aggregation and the one in binarizing belief aggregation, respectively, where $N = \{1, 2, 3\}$ and $M = \{1, 2\}$. The gray points represent all and the only vectors to which 1 is assigned

F, $F\restriction$ is a non-trivial JA, using (Fact 1) and (Fact 2) of [Step 1], which can be proved under the assumption of SYS and CDC combined with the agenda condition of being a non-trivial algebra.

(II) Next, let's consider the reverse scenario: whether the results in judgment aggregation can be directly derived from our results. For the sake of argument, let's assume that there was a direct and typical way to extend a JA F' satisfying ZP, CP, and IND, together with either CDC, [AN and CDC], or [CCS and CCP] to a BA $F'\upharpoonleft$ satisfying the same properties of binarizing belief aggregation. In such a case, $F'\upharpoonleft$ would either result in an oligarchy, the trivial rule, or the non-existence. Each of these outcomes would, in turn, imply an oligarchy, the trivial rule, or the non-existence of F' according to (Observation 2). However, there is no such direct and typical way.[24]

(III) Nevertheless, by employing the same methods as our proofs, we can derive analogous statements to Lemma 3, Theorem 4, and Theorem 5 for judgment aggregation. It is possible because our reasoning remains valid even when we restrict every $\vec{P}$ in our proofs to profiles of 0/1-valued probability functions. For instance, it is evident that (Fact 1) and (Fact 2) in [Step 1] of our proofs of triviality and oligarchy results imply the corresponding restrictions to the vectors whose components are 0 or 1, denoted by (Fact $1\restriction$) and (Fact $2\restriction$).[25]

In contrast, adopting the corresponding proofs in judgment aggregation does not suffice to obtain our results. First, even though the proofs in judgment aggregation could give (Fact $1\restriction$) and (Fact $2\restriction$), we need to demonstrate that they extend to our domain in order to obtain (Fact 1) and (Fact 2). On top of that, in the proofs of our triviality and oligarchy results, $a_i \neq 1$ (where i is any individual)/$a_j \neq 1$ (where j is not an oligarch) do not necessarily imply that $a_i = 0/a_j = 0$, contrary to the case of judgment aggregation. Thus, we need to prove that

[24] Two natural ways to extend a JA to a BA might include (i) assigning 0 to all profiles of probability functions that are not 0/1-valued or (ii) extending while maintaining monotonicity in a minimal manner. However, Wang and Kim (2023) have demonstrated that neither of these approaches serves as a direct and typical extension method preserving all the properties mentioned above.

[25] (Fact $1\restriction$) corresponds to closure under supersets of winning coalitions (sets of agents whose beliefs and the other agents' non-beliefs are the necessary and sufficient condition for the collective belief) in judgment aggregation.

$G(\vec{a}[a_i \mapsto 0, a_l \mapsto 1 \text{ for all } l \neq i]) = 1/G(\vec{a}[a_j \mapsto 0, a_l \mapsto 1 \text{ for all } l \neq j]) = 1$ in [Step 2], which is the step mentioned in (I). Precisely, this step can be divided into two sub-steps: (i) substitute every $a_l(l \neq i/l \neq j)$ with 1 and prove that G still assigns 1 by (Fact 1); (ii) substitute a_i/a_j with 0 and prove that G still assigns 1 by (Fact 1) and (Fact 2). Therefore, while (i) and (Fact 1) are required both for binarizing belief aggregation and for applying our proofs to judgment aggregation, (ii) is specific to binarizing belief aggregation.[26] Table 3 provides a comparison of the key claims necessary to prove the triviality result in binarizing belief aggregation with the claims needed when applying our proofs to establish the triviality result in judgment aggregation.

Opinion Pooling and Binarizing Belief Aggregation

We cannot apply the proofs from opinion pooling to derive our contagion lemma and the triviality/oligarchy results because we do not assume CCS and CCP to get them, and thus collective beliefs cannot be regarded as 0/1-valued probabilistic beliefs. The pivotal step in the corresponding proofs in opinion pooling hinges on the use of the additivity axiom: $F(\vec{P})(A \cup B) = F(\vec{P})(A) + F(\vec{P})(B)$, assuming $A \cap B = \emptyset$. In binarizing belief aggregation, however, this property does not hold under the assumption of CCS and CDC.

In contrast, we can employ the proofs of McConway (1981) to obtain the non-existence result in binarizing belief aggregation, given the assumption of CCP and CCS. Under this assumption, our outputs—collective binary beliefs—can be thought of as 0/1-valued finitely additive probabilities, as shown in Lemma 1. Thus, binarizing belief aggregation can be viewed as a form of opinion pooling with the restricted codomain. It is worth recalling that given UD of OP, an OP satisfies CP and IND if and only if it is a linear pooling. This provides the non-existence result, as some linear averages fall outside the codomain of a BA.

Belief Binarization and Binarizing Belief Aggregation

In binarizing belief aggregation, we would usually assume that the number n of individuals is more than one because we would investigate collective epistemic decisions dealing with multiple opinions. However, it is noteworthy that none of the proofs presented thus far have relied on the assumption that $n \geq 2$. Thus, our results can be applied to the case where $n = 1$, which aligns with the problem of belief binarization. It is worth highlighting that, as pointed out by Dietrich and List (2018), anonymous (AN) and proposition-wise independent (IND) judgment aggregation (with individual rationality of consistency and completeness) is the same problem as the problem of belief binarization because the ratio of believers in a proposition can be regarded as the probability of that proposition. Therefore, it should not come as a surprise that our findings for $n = 1$ recover the theorems positing that there exists neither JA with AN satisfying ($2'$) or ($4'$) in Table 2, nor belief binarization satisfying

[26] It does not imply that (Fact $2\upharpoonright$) is unnecessary for the results in judgment aggregation, as (Fact $2\upharpoonright$) is required in [Step 3], which is needed when using our proofs for judgment aggregation.

[27] It is worth noting that the difference between their notions of consistency and deductive closure, which involve the intersection of arbitrary sets, and our weaker notions, which only involve the intersection of finitely many sets, does not impact the proof of ($2''$) and ($4''$). This is because they assume the agenda to be a (non-trivial) algebra, and our weaker notions suffice to establish ($2''$) and ($4''$).

Table 3 Steps for proving the triviality results

	[Step 1]	[Step 2]	[Step 3]
BA	(Fact 1), (Fact 2)	$G(\vec{a}[a_i \mapsto 0, a_l \mapsto 1 \text{ for all } l \neq i]) = 1$ (using (Fact 1) and (Fact 2))	$G(0, ..., 0) = 1$ (using (Fact 2))
JA	(Fact 1↾), (Fact 2↾)	$G(\vec{a}[a_l \mapsto 1 \text{ for all } l \neq i]) = 1$ (using (Fact 1↾))	$G(0, ..., 0) = 1$ (using (Fact 2↾))

Table 4 Impossibility results in belief binarization

There is no BIN with UD of belief binarization satisfying
(2″) CCS[1], CP and IND + CDC + Non-triviality[2]
(4″) CP and IND + CCS and CCP

In Dietrich and List (2021), the assumption of CCS is used rather than ZP. However, as indicated in Lemma 2 (4), CCS is equivalent to ZP here since $\emptyset \in \mathcal{A}$ and we have CDC and CP

Non-triviality here corresponds to non-looseness in Dietrich and List (2021)

(2″) or (4″) in Table 4, which are the results derived from Dietrich and List (2021, 2018).[27]

6 Conclusion

We conclude this paper by highlighting some follow-up studies conducted in Wang and Kim (2023, MS) and Wang (2023). We have weakened the structure of a non-trivial algebra commonly required in opinion pooling and belief binarization and characterized impossibility agendas as in judgment aggregation.[28] In our efforts to find escape routes from the impossibility results, we have primarily focused on relaxing the independence norm and studied specific binarizing belief aggregation procedures. These include collective belief binarization combined with a probabilistic opinion pooling method as well as direct rules generating collective beliefs directly from individual credences. In the realm of collective belief binarization, we have undertaken an analysis of threshold-based methods and proposed distance- and utility-based belief binarization methods. Furthermore, we have introduced and explored direct binarizing belief aggregation procedures based on threshold, distance, and epistemic utility.

[27] It is worth noting that the difference between their notions of consistency and deductive closure, which involve the intersection of arbitrary sets, and our weaker notions, which only involve the intersection of finitely many sets, does not impact the proof of (2″) and (4″). This is because they assume the agenda to be a (non-trivial) algebra, and our weaker notions suffice to establish (2″) and (4″).

[28] Wang and Kim (2023) have demonstrated that the agenda conditions for the oligarchy/triviality/non-existence results are path-connectedness and even-negatability/negation-connectedness/blockedness, respectively.

Appendix — Proofs

Lemma 1 *Let W be a non-empty set, $\mathcal{A} \subseteq \mathcal{P}(W)$ be an agenda, and $Bel : \mathcal{A} \to \{0,1\}$ be a binary belief function. Then, (i) Bel is complete and consistent iff Bel is extendable to a finitely additive 0/1-valued probability function on the algebra $\mathfrak{a}(\mathcal{A})$ generated by $\mathcal{A}$. (ii) Bel is complete and σ-consistent iff Bel is extendable to a σ-additive 0/1-valued probability function on the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$.*

Proof Proof of part (i)

($\to$) It is well-known that any $A \in \mathfrak{a}(\mathcal{A})$ has the form of $\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i}^{\circ}$ where $E_{ij_i}^{\circ}$ is $E_{ij_i} (\in \mathcal{A})$ or $E_{ij_i}^{c}$. Since the agenda $\mathcal{A}$ is closed under complement, we let $A = \bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i}$. Define $Bel^* : \mathfrak{a}(\mathcal{A}) \to \{0,1\}$ as

$$Bel^*\left(\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i}\right) := \max_{i=1,\dots,n} \min_{j_i=1,\dots,m_i} Bel(E_{ij_i})$$

(a) Bel^* is well defined: Assume $\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i} = \bigcup_{k=1}^{l} \bigcap_{q_k=1}^{r_k} F_{kq_k}$. Then $(\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i})^c \cap (\bigcup_{k=1}^{l} \bigcap_{q_k=1}^{r_k} F_{kq_k}) = \emptyset$. Since $(\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i})^c = \bigcup_{(j_i)_i \in J} \bigcap_{i=1}^{n} E_{ij_i}^c$ where $J = \{1,\dots,m_1\} \times \dots \times \{1,\dots,m_n\}$, we have

$$\bigcup_{(j_i)_i \in J} \bigcup_{k=1}^{l} \bigcap_{i=1}^{n} \bigcap_{q_k=1}^{r_k} (E_{ij_i}^c \cap F_{kq_k}) = \emptyset \tag{1}$$

Now suppose $Bel^*(\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i}) = 0$ and $Bel^*(\bigcup_{k=1}^{l} \bigcap_{q_k=1}^{r_k} F_{kq_k}) = 1$. Then for all $i \in \{1,\dots,n\}$ we can pick a $j_i \in \{1,\dots,m_i\}$ such that $Bel(E_{ij_i}^c) = 1$. Note that this tuple $(j_1,\dots,j_n)$ is contained in J defined above. Moreover, there exists $k \in \{1,\dots,l\}$ such that for all $q_k \in \{1,\dots,r_k\}$, $Bel(F_{kq_k}) = 1$. Since Bel is consistent, $\bigcap_{i=1}^{n} \bigcap_{q_k=1}^{r_k} (E_{ij_i}^c \cap F_{kq_k}) \neq \emptyset$, which contradicts Equation (1). Similarly, we can prove that it is not the case that $Bel^*(\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i}) = 1$ and $Bel^*(\bigcup_{k=1}^{l} \bigcap_{q_k=1}^{r_k} F_{kq_k}) = 0$. Therefore, $Bel^*(\bigcup_{i=1}^{n} \bigcap_{j_i=1}^{m_i} E_{ij_i}) = Bel^*(\bigcup_{k=1}^{l} \bigcap_{q_k=1}^{r_k} F_{kq_k})$.

(b) By construction, $Bel^*(A) = Bel(A)$ for all $A \in \mathcal{A}$.

(c) Bel^* is a finitely additive probability function: $Bel^*(W) = Bel(E \cup E^c) = max\{Bel(E)\ , Bel(E^c)\} = 1$ for some $E \in \mathcal{A} (\neq \emptyset)$ since Bel is complete. Let $A = \bigcup_{i=1}^{n} A_i$ where $A_i = \bigcap_{j_i=1}^{m_i} E_{ij_i}$ and $B = \bigcup_{k=1}^{l} B_k$ where $B_k = \bigcap_{q_k=1}^{r_k} E_{kq_k}$ and assume that A and B is disjoint. Then we have

$$\begin{aligned} Bel^*(A \cup B) &= Bel^*\left(\bigcup_{i=1}^{n}\bigcap_{j_i=1}^{m_i} E_{ij_i} \cup \bigcup_{k=1}^{l}\bigcap_{q_k=1}^{r_k} F_{kq_k}\right) \\ &= \max\{Bel^*(A_1), ..., Bel^*(A_n), Bel^*(B_1), ..., Bel^*(B_l)\} \\ &= \max\{Bel^*(A), Bel^*(B)\} = Bel^*(A) + Bel^*(B) \end{aligned}$$

The last equality follows from $Bel^*(A) = 0$ or $Bel^*(B) = 0$. This holds, for otherwise there would exist $i \in \{1, ..., n\}$ such that for all $j_i \in \{1, ..., m_i\}$ $Bel(E_{ij_i}) = 1$ and also $k \in \{1, ..., l\}$ such that for all $q_k \in \{1, ..., r_k\}$ $Bel(F_{kq_k}) = 1$, from which it follows, by consistency of Bel, that $\bigcap_{j_i=1}^{m_i} E_{ij_i} \cap \bigcap_{q_k=1}^{r_k} F_{kq_k} \neq \emptyset$, a contradiction to $A \cap B = \emptyset$.

($\leftarrow$) Let P be a finitely additive 0/1-valued probability function extending Bel. (a) P is complete and so is Bel since $P(A) = 1$ or $P(A) = 0$, i.e., $P(A^c) = 1$. (b) Let $P(A_1), ..., P(A_k) = 1$. Then $P(\bigcap_{i=1}^{k} A_i) = 1$ and hence $\bigcap_{i=1}^{k} A_i \neq \emptyset$. So P is consistent and so is Bel.

Proof of part (ii)

($\rightarrow$) (a) If Bel is complete and σ-consistent, then the extension Bel^* on $\mathfrak{a}(\mathcal{A})$ defined in the proof of part (i)(a) is also complete and σ-consistent: completeness of Bel^* follows from the finite-additivity of Bel^* in the proof of part (i)(c). For σ-consistency, assume $Bel^*(A_n) = 1$ for all $n \in \mathbb{N}$. Let $A_n = \bigcup_{i=1}^{l(n)} \bigcap_{j=1}^{m_i} E_{ij}$. Then there exists $i(n) \in \{1, ..., l(n)\}$ such that $Bel(E_{i(n)j}) = 1$ for all $j \in \{1, ..., m_{i(n)}\}$. Note that $A_n \supseteq \bigcap_{j=1}^{m_{i(n)}} E_{i(n)j}$ and thus $\bigcap_{n\in\mathbb{N}} A_n \supseteq \bigcap_{n\in\mathbb{N}} \bigcap_{j=1}^{m_{i(n)}} E_{i(n)j}$. Since Bel is σ-consistent, $\bigcap_{n\in\mathbb{N}} \bigcap_{j=1}^{m_{i(n)}} E_{i(n)j}$ is not empty and neither is $\bigcap_{n\in\mathbb{N}} A_n$. Thus Bel^* is σ-consistent. (b) Bel^* is σ-additive: since it follows from completeness and σ-consistency of Bel^* that $Bel^*(\bigcup_{n\in\mathbb{N}} A_n) = 0$ iff $Bel^*(A_n^c) = 1$ for all $n \in \mathbb{N}$, where A_ns are pairwise disjoint, we obtain $Bel^*(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} Bel^*(A_n)$. (c) Combining this with Carathéodory extension theorem yields the claim. ($\leftarrow$) It can be shown similarly to the proof of ($\leftarrow$) in part (i). $\square$

Lemma 2 *In binarizing belief aggregation and judgment aggregation, the following holds*:

(1) Given CCS, CP implies ZP.
(2) Given CCP, ZP implies CP.
(3) Let $W, \emptyset \in \mathcal{A}$. Then, $F(\vec{P})(W) = 1$ by CDC or CP, and $F(\vec{P})(\emptyset) = 0$ by CCS or ZP.
(4) Let $\emptyset \in \mathcal{A}$ and F satisfy CDC and CP. Then, F satisfies ZP iff F satisfies CCS.

Proof

(1) Assume $\vec{P}(A) = \vec{0}$, which is equivalent to $\vec{P}(A^c) = \vec{1}$ since $\vec{P}$ is a profile of probabilistic beliefs. By CP, we have $F(\vec{P})(A^c) = 1$, from which it follows that $F(\vec{P})(A) = 0$ by CCS.

(2) Assume $\vec{P}(A) = \vec{1}$, which is equivalent to $\vec{P}(A^c) = \vec{0}$ since $\vec{P}$ is a profile of probabilistic beliefs. By ZP, we have $F(\vec{P})(A^c) = 0$, from which it follows that $F(\vec{P})(A) = 1$ by CCP.
(3) As noted already, CDC and CCS imply $F(\vec{P})(W) = 1$ and $F(\vec{P})(\emptyset) = 0$, respectively. The rest parts hold since $\vec{P}(W) = \vec{1}$ and $\vec{P}(\emptyset) = \vec{0}$.
(4) ($\leftarrow$) ZP follows from CCS and CP by part (1). ($\rightarrow$) When $\emptyset \in \mathcal{A}$, from ZP it follows that $F(\vec{P})(\emptyset) = 0$ by part (3), which is equivalent to CCS when we have CDC.

□

Lemma 3 (*Contagion Lemma*) *Let $\mathcal{A}$ be a non-trivial algebra and F be a BA with UD. If F satisfies CDC, ZP, CP, and IND, then it satisfies SYS.*

Proof By IND, we can let $F(\vec{P})(A) = G_A(\vec{P}(A))$ for all $\vec{P}$. We need to show that $G_A = G_B$ for all $A, B \in \mathcal{A}$.

(Case 1) $\emptyset \neq A \subseteq B \neq W$

By UD, $\vec{P}$ given as in Fig. 4 can be an argument of F: since there exist possible worlds in A and B^c, which are represented by dots in Fig. 4, due to UD, we can assign the probabilities $\vec{a}$ and $\vec{1} - \vec{a}$ to A and B^c, respectively. Then B has the probabilities $\vec{a}$. This gives the following:

(i) If $G_A(\vec{a}) = F(\vec{P})(A) = 1$, then $F(\vec{P})(B) = G_B(\vec{a}) = 1$ by CDC (closure under superset).
(ii) Since $F(\vec{P})(A \cup B^c) = 1$ by CP, if $G_B(\vec{a}) = F(\vec{P})(B) = 1$, then $F(\vec{P})(A) = G_A(\vec{a}) = 1$ by CDC (closure under intersection) since $(A \cup B^c) \cap B = A$.

(Case 2) $A \backslash B \neq \emptyset$ and $B \backslash A \neq \emptyset$

Let $v \in A \backslash B$ and $w \in B \backslash A$ as in Fig. 5. We can use $C \in \mathcal{A}$ such that $\{v, w\} \subseteq C \neq W$ since $\mathcal{A}$ is a non-trivial algebra. (Take the union of two elements in $\mathcal{A}$ one of which includes v and one of which includes w. We can find such two elements whose union is not W because $\mathcal{A}$ is non-trivial: if there were no such two elements, it means that $(A - B) \cup (B - A) = W$, hence $\mathcal{A} = \{\emptyset, A, B, W\}$ where $B = A^c$, which contradicts the non-triviality of $\mathcal{A}$.) By the result of (Case 1), we have $G_A = G_{A \cap C} = G_C = G_{C \cap B} = G_B$.

(Case 3) We can let $G_\emptyset = G_A$ and $G_W = G_A$, for any $A (\neq \emptyset, W) \in \mathcal{A}$ since $G_A(\vec{0}) = 0$ by ZP and $G_A(\vec{1}) = 1$ by CP. □

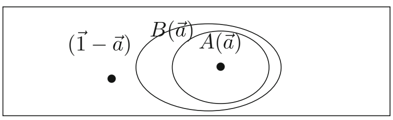

Fig. 4 A dot • represents a possible world

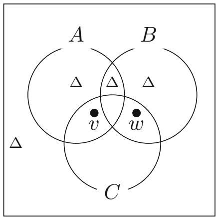

Fig. 5 The triangles $\vartriangle$ indicate all possible locations at one of which a possible world is ensured to exist so that $C \neq W$

Theorem 4 (*Triviality Result*) *Let $\mathcal{A}$ be a non-trivial algebra. The only BA satisfying UD, ZP, CP, IND, CDC, and AN is the unanimity rule.*

Proof It is easily seen that the unanimity rule satisfies all mentioned properties. For the other direction, by Lemma 3 we have SYS and thus, we can let $F(\vec{P})(A) = G(\vec{P}(A))$ where $G(\vec{1}) = 1$ by CDC (in particular $F(\vec{P})(W) = 1$) or CP. Now suppose that $G(\vec{a}) = 1$ for some $\vec{a} \neq \vec{1}$ and pick up any $a_i \neq 1$ in $\vec{a}$. To derive a contradiction, we take the following three steps.

[Step 1] We show the following:

(Fact 1) if $\vec{a} \leq \vec{b}$ and if $G(\vec{a}) = 1$, then $G(\vec{b}) = 1$

(Fact 2) if $\vec{a} + \vec{b} - \vec{1} \geq \vec{0}$ and if $G(\vec{a}) = 1$ and $G(\vec{b}) = 1$, then $G(\vec{a} + \vec{b} - \vec{1}) = 1$

Since $\mathcal{A}$ is non-trivial, we have at least 3 non-empty elements of $\mathcal{A}$ that have no intersections with each other. We represent each possible world of such elements by a dot in Fig. 6. Since $\mathcal{A}$ is an algebra, there are A and B in $\mathcal{A}$ as in the left figure and A, B and $A \cap B$ as in the right figure. By UD, we can assign to A and B, respectively, the probabilities $\vec{a}$ and $\vec{b}$ such that $\vec{a} \leq \vec{b}$, as in the left figure. In the right figure, by UD we can assign to A, B and $A \cap B$, respectively, the probabilities $\vec{a}$, $\vec{b}$, and $\vec{a} + \vec{b} - \vec{1}$ where $\vec{a} + \vec{b} - \vec{1} \geq \vec{0}$. The left figure gives us (Fact 1) by closure under superset from CDC and the right figure gives us (Fact 2) by closure under intersection from CDC.

[Step 2] We show that $G(\vec{a}[a_i \mapsto 0, a_l \mapsto 1 \text{ for all } l \neq i]) = 1$.

By (Fact 1), we can substitute a_i and a_l with any higher value keeping the value of G as 1 and by mixed applications of (Fact 1) and (Fact 2), we can substitute a_i with any lower value using the fact that for any $\vec{a} \geq (0.5, ..., 0.5)$,

$$\text{if } G(\vec{a}) = 1 \text{ then } G(\vec{a} + \vec{a} - 1) = 1 \tag{2}$$

as follows. By (Fact 1), we can substitute all other components a_l (i.e., $l \neq i$) that are not 1 with 1 and so we have $G((1, ..., 1, a_i, 1, ..., 1)) = 1$. This process enables us to focus on the i-th component of vectors when we apply (2) because $1 + 1 - 1 = 1$. Now employ (Fact 1) and (2). (i) If $a_i \leq 0.5$, we have $G(1, ..., 1, 0.5, 1, ..., 1) = 1$ by

Fig. 6 The left one is for (Fact 1) and the right one is for (Fact 2)

(Fact 1) and $G(1, ..., 1, 0, 1, ..., 1) = 1$ by (2). (ii) Now let $a_i > 0.5$. After applying (2) to $G((1, ..., 1, a_i, 1, ..., 1)) = 1$ k times, we have $G((1, ..., 1, a_i^{(k)}, 1, ..., 1)) = 1$ where $a_i^{(k)} = 1 - 2^k(1 - a_i)$ and there must be k such that $a_i^{(k)} \leq 0.5$. Then, we can use (i) and obtain $G(1, ..., 1, 0, 1, ..., 1) = 1$.

[Step 3] We show, by induction, $G((0, ..., 0)) = 1$, which contradicts ZP.

We have $G((0, 1, ..., 1)) = G((1, 0, 1, ..., 1)) = ... = G((1, ..., 1, 0)$ by UD, AN and (Step 2). Let $\vec{a}_k$ be a vector (0, ..., 0, 1, ..., 1) where the first k components are 0 and the others are 1. For $k = 1$, we have $G(\vec{a}_1) = 1$. Assume $G(\vec{a}_k) = 1$. Since $G((1, ..., 1, 0, 1, ..., 1)) = 1$ where all components are 1 except for the (k+1)-th one, which is 0, we have $G(\vec{a}_{k+1}) = 1$ by (Fact 2). □

Theorem 5 (*Oligarchy Result*) *Let $\mathcal{A}$ be a non-trivial algebra. The only BAs satisfying UD, ZP, CP, IND, and CDC are the oligarchies.*

Proof It is obvious that an oligarchy satisfies the properties. For the other direction, to construct the set M of oligarchs in Definition 11, we employ [Step 1] and [Step 2] in the proof of Theorem 4. (By UD, ZP, CP, IND and CDC, we have SYS by Lemma 3, and [Step 1] and [Step 2] follow from UD and CDC. Note that in the proof of Theorem 4, we did not use AN except in [Step 3].) Consider the set $G^{-1}(1) := \{\vec{a} | \; G(\vec{a}) = 1\}$ where G is a function satisfying $F(\vec{P})(A) = G(\vec{P}(A))$. We collect individuals i such that $a_i = 1$ for all $\vec{a} \in G^{-1}(1)$ and define the set M of such individuals: $M := \{i \in N | \; a_i = 1 \text{ for all } \vec{a} \text{ such that } G(\vec{a}) = 1\}$. We will show (i) and (ii) in the following.

(i) M is non-empty.
Suppose M is empty. Then $G(0, 1, ..., 1) = G(1, 0, 1, ..., 1) = ... = G(1, ..., 1, 0) = 1$ by [Step 1] and [Step 2], and we have $G(0, ..., 0) = 1$ using the same way of [Step 3], which contradicts ZP.

(ii) $a_i = 1$ for all $i \in M$ iff $G(\vec{a}) = 1$.
($\leftarrow$) It is obvious by the construction of M. ($\rightarrow$) Since we have (Fact 1) in [Step 1], it is enough to show that $G((\mathbb{1}_M(i))_{i \in N}) = 1$ where $\mathbb{1}_M(i) = 1$ if $i \in M$, otherwise $\mathbb{1}_M(i) = 0$. For any $j \notin M$, there is $\vec{a}$ such that $G(\vec{a}) = 1$ and $a_j \neq 1$, by definition of M. By [Step 2],

$$G(\vec{a}[a_j \mapsto 0, a_l \mapsto 1 \text{ for all } l \neq j]) = 1 \tag{3}$$

Now, we proceed by induction analogously to [Step 3]. Enumerate individuals who are not in M, like $j_1, j_2, ..., j_{|N|-|M|}$ and let $\vec{a}_k$ be a profile where $a_{j_1} = 0, ..., a_{j_k} = 0$ and other components are all 1. For $k = 1$, we have $G(\vec{a}_1) = 1$. Assume $G(\vec{a}_k) = 1$. Since by Equation (3) we have $G(1, ..., 1, 0, 1, ..., 1) = 1$ where 0 is the j_{k+1}-th component, by (Fact 2) in [Step 1], we have $G(\vec{a}_{k+1}) = 1$. Therefore, we have $G(\vec{a}_{|N|-|M|}) = G((\mathbb{1}_M(i))_{i \in N}) = 1$

□

Corollary 6 (*Non-existence Result*) *Let $\mathcal{A}$ be a non-trivial algebra. There is no BA satisfying UD, CP, IND, CCP, and CCS.*

Proof Since CDC follows from CCP and CCS, and ZP follows from CCS and CP, the only possible BAs satisfying the above conditions would be the oligarchies by Theorem 5, which do not satisfy collective completeness. □

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Declarations

Conflict of interest Author have no conflicts of interest to disclose.

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