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Differential marginality, inessential games and convex combinations of values

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Abstract
The principle of differential marginality (Casajus in Theory and Decis 71(2):163-–174) for cooperative games is a very appealing property that requires equal productivity differentials to translate into equal payoff differentials. In this paper we apply this property to axiomatic characterizations of values. We show that differential marginality implies additivity and symmetry under certain conditions. Based on this result, we propose new characterizations of the equal division and the equal surplus division values. Finally, we characterize two classes of convex combinations of values, i.e., α-egalitarian Shapley values and α-equal surplus division values, by employing differential marginality and establishing the uniqueness of these values on inessential games.

Keywords TU game · Differential marginality · Inessential game · Equal division value · α-Egalitarian Shapley

1 Introduction

The Shapley value (Shapley, 1953) is the most eminent solution for cooperative games with transferable utility (TU games), and it can be characterized by four standard properties: efficiency, symmetry, additivity, and the null player property. To eliminate the controversial additivity, Young (1985) proposes the marginality property, which requires that a player's payoff is only determined by her productivity as measured by her marginal contributions. In the characterization of Young (1985), marginality replaces both additivity and the null player property. In addition, van den Brink (2001) proposes the fairness property saying that the payoffs of two players change by the same amount when adding a game in which these players are symmetric, and van den Brink-fairness can replace additivity and symmetry in the original characterization of the Shapley value. Furthermore, Casajus (2011) proposes

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an equivalent property to van den Brink-fairness and calls it differential marginality. Differential marginality requires that the payoff differences of two players are the same whenever the productivity differences of these players are the same, where the productivity difference is measured by the differences of their marginal contributions.

The equal division (ED) value is a widely recognized value that reflects egalitarianism. van den Brink (2007) characterizes this value in terms of efficiency, symmetry, additivity, and the nullifying player property, which replaces the null player property in the original characterization of the Shapley value. Similarly, Casajus and Huettner (2014) characterizes the equal surplus division (ESD) value by replacing the null player property with the dummifying player property. In this paper, for the same purpose of removing the controversial additivity, we employ the appealing differential marginality property to characterize the ED and ESD values. With more than two players, Casajus (2011) shows that differential marginality, together with efficiency and the null game property, imply additivity and symmetry, where the null game property ensures that any player in the null game gets only a zero payoff. Furthermore, we illustrate the feasibility of replacing the null game property with nullifying player property or dummifying player property, without any restrictions on the number of players. Based on these findings, we propose new characterizations of the ED and ESD values.

For each $\alpha \in [0, 1]$, the associated α-egalitarian Shapley value is the corresponding convex combination of the Shapley value and the ED value, and the α-ESD value is the convex combination of the ESD value and the ED value. They both reflect the reconciliation of marginalism and egalitarianism (van den Brink et al., 2013; Xu et al., 2015). Observe that the α-egalitarian Shapley values on inessential games coincide with the α-ESD values.

In this paper, we provide new characterizations of the α-egalitarian Shapley values and the α-ESD values. To this end, we establish the existence and uniqueness of the values on the inessential games. This shows that the inessential game plays an important role in the characterizations of these two classes of values.

The remainder of this paper is organized as follows. In Sect. 2 we give necessary notation and definitions in TU games. In Sect. 3 we establish the relationship between differential marginality and both symmetry and additivity. We also provide new characterizations of the equal division and the equal surplus division values. Section 4 characterizes the α-egalitarian Shapley values and the α-ESD values by employing differential marginality. Finally, Sect. 5 concludes this paper.

2 Preliminaries

A *TU game* is a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is called the *worth* of the coalition S. Denote by $\mathcal{G}^N$ the set of all TU games on the player set N. Let s represent the cardinality of coalition S. For notation simplicity, we write v instead of (N, v) when refer to a game, and i instead of $\{i\}$.

For any $v \in \mathcal{G}^N$, player $i \in N$ is a *null player* in v if $v(S \cup i) - v(S) = 0$ for all $S \subseteq N \setminus i$; player $i \in N$ is a *dummy player* if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq N \setminus i$.

Player $i \in N$ is a *nullifying player* if $v(S \cup i) = 0$ for all $S \subseteq N \backslash i$; and player $i \in N$ is a *dummifying player* if $v(S \cup i) = \sum_{j \in S \cup i} v(j)$ for all $S \subseteq N \setminus i$. Players i and j are *symmetric* if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

For $v, w \in \mathcal{G}^N$ and $a \in \mathbb{R}$, the games $v + w \in \mathcal{G}^N$ and $a \cdot v \in \mathcal{G}^N$ are given by $(v + w)(S) = v(S) + w(S)$ and $(a \cdot v)(S) = a \cdot v(S)$ for all $S \subseteq N$, respectively. The *null game* $\mathbf{0} \in \mathcal{G}^N$ is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. For any nonempty $T \subseteq N$, the *unanimity game* u_T induced by T is defined as $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ otherwise. Any game $v \in \mathcal{G}^N$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N, T \neq \emptyset} \lambda_T^v u_T, \tag{1}$$

where $\lambda_T^v = \sum_{S \subseteq T} (-1)^{t-s} v(S)$ is called the *Harsanyi dividend* of coalition T (Harsanyi, 1959). The *standard game* e_T is given by $e_T(S) = 1$ if $S = T$ and $e_T(S) = 0$ otherwise. Obviously, any game $v \in \mathcal{G}^N$ can also be uniquely represented as

$$v = \sum_{T \subseteq N, T \neq \emptyset} v(T) e_T. \tag{2}$$

A TU game $v \in \mathcal{G}^N$ is *inessential* if $v(S) = \sum_{i \in S} v(i)$ for all $S \subseteq N$. Obviously, each $i \in N$ in an inessential game v is both dummy and dummifying player. The set of all inessential games in $\mathcal{G}^N$ is denoted by $\mathcal{G}_0^N$. Any inessential game $v \in \mathcal{G}_0^N$ can be identified with a vector $x \in \mathbb{R}^N$, and then we represent the inessential game by writing v_x.

We can rewrite equalities (1) and (2) as follows.

$$v = \sum_{i \in N} v(i) u_i + \sum_{T \subseteq N, t \geq 2} \lambda_T^v u_T, \tag{3}$$

and

$$\begin{aligned} v &= \sum_{i \in N} v(i) e_i + \sum_{T \subseteq N, t \geq 2} v(T) e_T \\ &= \sum_{i \in N} v(i) \left(u_i - \sum_{T \subseteq N, T \ni i, t \geq 2} e_T \right) + \sum_{T \subseteq N, t \geq 2} v(T) e_T \\ &= \sum_{i \in N} v(i) u_i - \sum_{T \subseteq N, t \geq 2} \sum_{i \in T} v(i) e_T + \sum_{T \subseteq N, t \geq 2} v(T) e_T \\ &= \sum_{i \in N} v(i) u_i + \sum_{T \subseteq N, t \geq 2} c_T^v e_T, \end{aligned} \tag{4}$$

where $c_T^v := v(T) - \sum_{i \in T} v(i)$. The game $\sum_{i \in N} v(i) u_i$ in (3) and (4) is clearly an inessential game.

A *value* on $\mathcal{G}^N$ is a function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathcal{G}^N$. The *Shapley value* (Shapley, 1953), Sh, is given by

$$\mathrm{Sh}_i(v) := \sum_{T \subseteq N, T \ni i} \frac{\lambda_T^v}{t} \quad \text{for all } i \in N, v \in \mathcal{G}^N. \tag{5}$$

The *equal division value*, ED, is given by

$$\mathrm{ED}_i(v) := \frac{v(N)}{n} \quad \text{for all } i \in N, v \in \mathcal{G}^N. \tag{6}$$

The *equal surplus division value* (Driessen & Funaki, 1991), ESD, which is also called the *center-of-gravity of the imputation set value*, is given by

$$\mathrm{ESD}_i(v) := v(i) + \frac{v(N) - \sum_{i \in N} v(i)}{n} \quad \text{for all } i \in N, v \in \mathcal{G}^N. \tag{7}$$

For each $\alpha \in [0, 1]$, the *α-egalitarian Shapley value* (Joosten, 1996), Sh^α, is given by

$$\mathrm{Sh}_i^\alpha(v) := \alpha \mathrm{Sh}_i(v) + (1 - \alpha)\mathrm{ED}_i(v) \quad \text{for all } i \in N, v \in \mathcal{G}^N. \tag{8}$$

For each $\alpha \in [0, 1]$, the *α-ESD value* (van den Brink & Funaki, 2009), ESD^α, is given by

$$\mathrm{ESD}_i^\alpha(v) := \alpha \mathrm{ESD}_i(v) + (1 - \alpha)\mathrm{ED}_i(v) \quad \text{for all } i \in N, v \in \mathcal{G}^N. \tag{9}$$

We introduce standard axioms for values in TU games.

Efficiency, **E**. For all $v \in \mathcal{G}^N$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Additivity, **A**. For all $v, w \in \mathcal{G}^N$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Symmetry, **S**. For all $v \in \mathcal{G}^N$ and $i, j \in N$, $\varphi_i(v) = \varphi_j(v)$ if i and j are symmetric players in v.

Null player property, **N**. For all $v \in \mathcal{G}^N$ and $i \in N$, $\varphi_i(v) = 0$ if i is a null player in v.

Dummy player property, **D**. For all $v \in \mathcal{G}^N$ and $i \in N$, $\varphi_i(v) = v(i)$ if i is a dummy player in v.

Null game property, **NG**. For the null game $\mathbf{0} \in \mathcal{G}^N$ and all $i \in N$, $\varphi_i(\mathbf{0}) = 0$.

3 Differential marginality, symmetry and additivity

The Shapley value (Shapley, 1953) can be characterized by four properties: efficiency(**E**), the null player property(**N**), symmetry(**S**) and additivity(**A**). These four axioms are widely recognized as the most fundamental ones in cooperative game theory. However, the axiom of additivity is controversial in certain situations. To replace additivity, Casajus (2011) provides the following axiom of differential marginality, and this axiom together with efficiency, the null player property can uniquely characterize the Shapley value.

Differential marginality, **DM**. For all $v, w \in \mathcal{G}^N$ and $i, j \in N$, $\varphi_i(v) - \varphi_j(v) = \varphi_i(w) - \varphi_j(w)$ if $v(S \cup i) - v(S \cup j) = w(S \cup i) - w(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$.

Differential marginality requires the payoff differences of two players to be the same whenever the productivity differences of these players is the same, where the productivity difference is measured by the differences of their marginal contributions. This axiom was actually introduced by van den Brink (2001) earlier and first applied to the characterization of the Shapley value. Obviously, **S** and **A** imply **DM** (van den Brink, 2001), then we explore the conditions under which **DM** can replace **S** and **A**.

The following observation is easily verified.

Proposition 1 *(Casajus, 2011) Differential marginality (**DM**) and the null game property (**NG**) imply symmetry (**S**).*

For **DM** and **A**, neither one of them directly implies the other. However, when $n \neq 2$, Casajus (2011) shows that **DM** together with **E** and **NG** imply **A**.

Proposition 2 *(Casajus, 2011) For $n \neq 2$, efficiency (**E**), differential marginality (**DM**) and the null game property (**NG**) imply additivity (**A**)*

Proposition 2 is limited by the condition $n \neq 2$. To drop the requirement of $n \neq 2$, we replace **NG** by the following somewhat stronger properties.

Inessential game property, **IG**. (Ruiz et al., 1996) For all $v_x \in \mathcal{G}_0^N$ and $i \in N$, $\varphi_i(v_x) = v_x(i)$.

Nullifying player property, **Np**. (van den Brink, 2007) For all $v \in \mathcal{G}^N$ and $i \in N$, $\varphi_i(v) = 0$ if i is a nullifying player in v.

It is easily seen that **IG** or **Np** is stronger than **NG**, since the null game ***0*** is inessential and all $i \in N$ in the null game ***0*** are nullifying players. After replacing **NG** with either **IG** or **Np**, we obtain the following result.

Corollary 1 *Efficiency (**E**), differential marginality (**DM**), and either the inessential game property (**IG**) or the nullifying player property (**Np**) imply additivity (**A**).*

Proof If $n \neq 2$, the assertion follows directly from Proposition 2, because **IG** or **Np** implies **NG**.

We now consider the case when $n = 2$. Set $N := \{i,j\}$.

Let φ satisfy **E**, **DM**, and **IG**. For any $v \in \mathcal{G}^{\{i,j\}}$, we set $w := \lambda_i^v u_i + \lambda_j^v u_j$. Clearly, w is inessential. We obtain $\varphi(w) = \mathrm{Sh}(w)$ because of **IG**. Moreover, since i and j are symmetric in $v - w$, it follows from **DM** that $\varphi_i(v) - \varphi_j(v) = \varphi_i(w) - \varphi_j(w) = \lambda_i^v - \lambda_j^v$. Next, since **E** means that $\varphi_i(v) + \varphi_j(v) = \lambda_i^v + \lambda_j^v + \lambda_{\{i,j\}}^v$, we have $\varphi_k(v) = \lambda_k^v + \frac{\lambda_{\{i,j\}}^v}{2} = \mathrm{Sh}_k(v), \forall k \in \{i,j\}$. Therefore, $\varphi(v) = \mathrm{Sh}(v)$.

Similarly, we let ψ satisfy **E**, **DM**, and **Np**. For any $v \in \mathcal{G}^{\{i,j\}}$, we set $w' := \big(v(i) - v(j)\big)e_i$. Since j is nullifying in w' and ψ satisfies **E** and **Np**, we have $\psi_k(w') = 0, \forall k \in \{i,j\}$. Next, i and j are symmetric in $v - w'$, so **DM** means that $\psi_i(v) - \psi_j(v) = \psi_i(w') - \psi_j(w') = 0$. Finally, **E** means that $\psi_i(v) = \psi_j(v) = \frac{v(N)}{2}$. Therefore, $\psi(v) = \mathrm{ED}(v)$.

Because the Shapley and ED values meet **A**, we get that φ and ψ also meet **A**. □

van den Brink (2007) shows that the ED value is uniquely determined by the axioms of efficiency (**E**), symmetry (**S**), additivity (**A**) and the nullifying player property (**Np**). Casajus and Huettner (2014) further show that the ESD value can be uniquely determined by replacing the nullifying player property (**Np**) in the axiomatization of van den Brink (2007) with the dummifying player property (**Dp**) below.

Dummifying player property, **Dp**. (Casajus & Huettner, 2014) For all $v \in \mathcal{G}^N$ and $i \in N$, $\varphi_i(v) = v(i)$ if i is a dummifying player in v.

Proposition 1 and Corollary 1 tell us that **DM** together with **E** and either **IG** or **Np** implies **S** and **A**. Note that **Dp** implies **IG**. By the results of van den Brink (2007) and Casajus and Huettner (2014) above, we immediately obtain the following characterizations of the ED and ESD values.

Theorem 1 *The ED value is the unique value that satisfies efficiency (**E**), differential marginality (**DM**), and the nullifying player property (**Np**).*

Theorem 2 *The ESD value is the unique value that satisfies efficiency (**E**), differential marginality (**DM**), and the dummifying player property (**Dp**).*

Remark 1 Note that both the Shapley and ESD values satisfy **E**, **DM** and **IG**, so we can not weaken **Dp** into **IG** in Theorem 2.

4 α-Egalitarian Shapley values and α-ESD values

In this section, we characterize the α-egalitarian Shapley values and the α-ESD values by employing differential marginality.

In fact, when $n > 3$, Casajus and Yokote (2019) have established a characterization of the α-egalitarian Shapley values by means of efficiency, weak differential monotonicity and the average dummy player property below.

The sign function, $\text{sign} : \mathbb{R} \to \{-1, 0, 1\}$, is given by $\text{sign}(c) = -1$ if $c < 0$, $\text{sign}(0) = 0$, and $\text{sign}(c) = 1$ if $c > 0$.

Weak differential monotonicity, **DMo** $^-$. (Casajus & Yokote, 2019) For all $v, w \in \mathcal{G}^N$ and $i,j \in N$ such that $v(S \cup i) - w(S \cup i) \geq v(S \cup j) - w(S \cup j)$ for all $S \subseteq N \setminus \{i,j\}$, $\text{sign}\big(\varphi_i(v) - \varphi_i(w)\big) \geq \text{sign}\big(\varphi_j(v) - \varphi_j(w)\big)$.

Weak differential monotonicity requires that if the change of one player in the productivity is weakly greater than that of another, then the direction of the change of these players' payoffs coincides with the changes in their productivities.

Average dummy player, **AD**. (Casajus & Yokote, 2019) For all $v \in \mathcal{G}^N$ and $i \in N$ such that i is a dummy player in v, we have

(i) $v(i) \geq v(N)/n$ implies $\varphi_i(v) \leq v(i)$,
(ii) $v(i) \leq v(N)/n$ implies $\varphi_i(v) \geq v(i)$.

This property states that whenever a dummy player is weakly more (less) productive than the average of all players, then her payoff is not greater (lower) than her productivity.

Proposition 3 *(Casajus & Yokote, 2019) For $n > 3$, a value φ satisfies efficiency (**E**), weak differential monotonicity (**DMo** $^-$) and the average dummy player property (**AD**) if and only if there exists an $\alpha \in [0, 1]$ such that $\varphi = \text{Sh}^\alpha$.*

With the help of **E** and **DM**, when using the **AD** of Casajus and Yokote (2019) to characterize the α-egalitarian Shapley values, the reasonable range of the coefficient α would not be $[0,1]$. For example, let $\beta = -1$ and $\varphi(v) = \beta \mathrm{Sh}(v) + (1-\beta)\mathrm{ED}(v)$ for all $v \in \mathcal{G}^N$. It can be verified that the value φ obeys **E**, **DM**, and **AD**. Therefore, in order to characterize the convex combinations of values, Sh^α, we need to employ a stronger property.

The **AD** property requires the dummy player, whose productivity is weakly higher than the average of all players, to subsidize other players or not. But the subsidy should not exceed the difference between her productivity and the average. This is a more sensible approach, as exceeding this limit would result in the dummy player losing her status as a more productive individual and no longer being eligible to subsidize others. Next, it is also important to ensure that subsidy to the dummy player with weakly lower productivity are appropriate. As a result, we propose and utilize the strong average dummy player property.

Strong average dummy player, **AD** $^+$. For all $v \in \mathcal{G}^N$ and $i \in N$ such that i is a dummy player in v, we have

(i) $v(i) \geq v(N)/n$ implies $v(i) \geq \varphi_i(v) \geq v(N)/n$,
(ii) $v(i) \leq v(N)/n$ implies $v(i) \leq \varphi_i(v) \leq v(N)/n$.

This property states that, whenever a dummy player is weakly more (less) productive than the average of all players, then her payoff is weakly lower (greater) than her productivity, and weakly greater (lower) than the average.

To characterize the α-egalitarian Shapley values and the α-ESD values, we first show the following result for inessential games.

Corollary 2 *For $n \neq 2$, if the value φ satisfies efficiency (**E**), the strong average dummy player property (**AD** $^+$) and differential marginality (**DM**), there exists an $\alpha \in [0,1]$ such that $\varphi(v_x) = \mathrm{Sh}^\alpha(v_x)$ for all $x \in \mathcal{G}_0^N$.*

Proof One can easily check that **AD** $^+$ implies **NG**, so φ satisfies **S** and **A** by Propositions 1 and 2.

For $n = 1$, let $N := \{i\}$. By **E**, it is clear that $\varphi_i(v) = v(i) = \mathrm{Sh}^\alpha(v)$.

Let now $n > 2$. For any $c \in \mathbb{R}$ and $i \in N$, the game cu_i is determined. Since φ satisfies **E** and **S**, we can write

$$\varphi_k(cu_i) := \begin{cases} f(c,i) & k = i \\ \dfrac{c - f(c,i)}{n-1} & k \neq i \end{cases}. \tag{10}$$

In a way similar to (10), for any $c \in \mathbb{R}$ and $j \in N$, we have

$$\varphi_k(cu_j) := \begin{cases} f(c,j) & k = j \\ \dfrac{c - f(c,j)}{n-1} & k \neq j \end{cases}. \tag{11}$$

Because φ meets **S** and **A**, we get $\varphi_i(cu_i + cu_j) = f(c,i) + \frac{c-f(c,j)}{n-1} = f(c,j) + \frac{c-f(c,i)}{n-1} = \varphi_j(cu_i + cu_j)$, which means that $f(c,i)=f(c,j)$. Thus $f(c,\ i)$ does not depend on the choice of $i\in N$, and so $f(c,i)=f(c)$.

By **A**, (10) and (11), we have that $f(c+d)=\varphi_i\big((c+d)u_i\big)=\varphi_i(cu_i)+\varphi_i(du_i)=f(c)+f(d)$ for all $c,d\in\mathbb{R}$. Thus f is additive. Then we note that if $d>0$, it is clear that $du_i(i)>\frac{du_i(N)}{n}$ and $f(d)=\varphi_i(du_i)\overset{\mathbf{AD}^+}{\geq}\frac{du_i(N)}{n}>0$. Hence $f(c+d)=f(c)+f(d)>f(c)$ and f is monotonic. We deduce that f is linear and $f(c)=cf(1)$ by Theorem 1 in Aczél and Oser (1966).[1]

We set $\alpha:=\frac{n}{n-1}f(1)-\frac{1}{n-1}$, which also means that $f(1)=\frac{n-1}{n}\alpha+\frac{1}{n}=\alpha+(1-\alpha)\frac{1}{n}$. For any $c\in\mathbb{R}$ and $i\in N$, we can rewrite (10) as

$$\varphi_k(cu_i)=\begin{cases} cf(1) & k=i\\ \dfrac{c-cf(1)}{n-1} & k\neq i\end{cases}=\begin{cases}\alpha c+(1-\alpha)\dfrac{c}{n} & k=i\\ (1-\alpha)\dfrac{c}{n} & k\neq i\end{cases}, \tag{12}$$

which means $\varphi(cu_i)=\mathrm{Sh}^{\alpha}(cu_i)$. Because of $\mathbf{AD}^+$, we have $\frac{c}{n}\leq\varphi_i(cu_i)\leq c$, then one can easily get that $\alpha\in[0,1]$.

Finally by **A** and (12), for all $v_x\in\mathcal{G}_0^N$, we have $\varphi(v_x)=\varphi(\sum_{i\in N}v_x(i)u_i)\overset{\mathbf{A}}{=}\sum_{i\in N}\varphi(v_x(i)u_i)\overset{(12)}{=}\sum_{i\in N}\mathrm{Sh}^{\alpha}(v_x(i)u_i)\overset{\mathbf{A}}{=}\mathrm{Sh}^{\alpha}(v_x)$. We note that there exists a unique $\alpha\in[0,1]$ such that, for any $v_x,v_y\in\mathcal{G}_0^N$, $\varphi(v_x)=\mathrm{Sh}^{\alpha}(v_x)$ and $\varphi(v_y)=\mathrm{Sh}^{\alpha}(v_y)$. □

By Corollary 2, we are ready to give an axiomatization of the α-egalitarian Shapley values.

Theorem 3 *For $n\neq 2$, a value φ on $\mathcal{G}^N$ satisfies efficiency (**E**), strong average dummy player property ($\mathbf{AD}^+$), and differential marginality (**DM**) if and only if there exists an $\alpha\in[0,1]$ such that $\varphi=\mathrm{Sh}^{\alpha}$.*

Proof Each Sh^{α} meets **E** and **DM**, which follows directly from that the Shapley and ED values satisfy **E** and **DM**. By (8) and the fact that Sh satisfies **D**, each Sh^{α} satisfies $\mathbf{AD}^+$.

Let φ be a value on $\mathcal{G}^N$ that satisfies **E**, $\mathbf{AD}^+$ and **DM**. Note that $\mathbf{AD}^+$ implies **NG**. By Propositions 1 and 2, φ satisfies **S** and **A**.

For $n=1$, it is clear that $\varphi=\mathrm{Sh}^{\alpha}$ by **E**. We now assume that $n>2$.

Set $w_T:=u_T-\frac{\sum_{j\in T}u_j}{t}$. For each $v\in\mathcal{G}^N$, by (3) and (5), we have

[1] In Theorem 1 of Aczél and Oser (1966), they showed that f is linear if f is additive and monotonic.

$$
\begin{aligned}
v =& \sum_{i\in N} \lambda_i^v u_i + \sum_{T\subseteq N, t\geq 2} \lambda_T^v u_T \\
=& \sum_{i\in N} \lambda_i^v u_i + \sum_{T\subseteq N, t\geq 2} \lambda_T^v \left(u_T - \frac{\sum_{i\in T} u_i}{t} \right) + \sum_{T\subseteq N, t\geq 2} \lambda_T^v \frac{\sum_{i\in T} u_i}{t} \\
=& \sum_{i\in N} \sum_{T\subseteq N, t=1, T\ni i} \frac{\lambda_T^v}{t} u_i + \sum_{T\subseteq N, t\geq 2} \lambda_T^v w_T + \sum_{i\in N} \sum_{T\subseteq N, t\geq 2, T\ni i} \frac{\lambda_T^v}{t} u_i \\
=& \sum_{i\in N} \mathrm{Sh}_i(v) u_i + \sum_{T\subseteq N, t\geq 2} \lambda_T^v w_T.
\end{aligned} \tag{13}
$$

For all $T \subseteq N$ with $t \geq 2$, each $i \in N \setminus T$ in the game $\lambda_T^v w_T$ is a dummy player and $\lambda_T^v w_T(i) = 0 = \frac{\lambda_T^v w_T(N)}{n}$. By **AD** $^+$, we get

$$
\varphi_i\left(\lambda_T^v w_T\right) = 0 \quad \text{for all } i \in N \setminus T. \tag{14}
$$

Furthermore, note that any two players $i, j \in T$ are symmetric in game $\lambda_T^v w_T$. Equality (14) together with **E** and **S** implies that $\varphi_i(\lambda_T^v w_T) = \frac{\lambda_T^v w_T(N)}{t} = 0$, $\forall i \in T$. Therefore, $\varphi_i(\lambda_T^v w_T) = 0$ for all $i \in N$ and $T \subseteq N$ with $t \geq 2$. Thus, by (13), Corollary 2, **A** of φ and **E** of Sh,

$$
\begin{aligned}
\varphi_i(v) =& \alpha \mathrm{Sh}_i(v) + (1-\alpha) \frac{\sum_{j\in N} \mathrm{Sh}_j(v)}{n} \\
=& \alpha \mathrm{Sh}_i(v) + (1-\alpha) \frac{v(N)}{n} \\
=& \mathrm{Sh}_i^\alpha(v)
\end{aligned} \tag{15}
$$

for all $i \in N$. □

We now turn our attention to axiomatization of the α-ESD values. To realize this, we first propose a new property.

Strong average dummifying player, **ADp** $^+$. For all $v \in \mathcal{G}^N$ and $i \in N$ such that i is a dummifying player in v, we have that

(i) $v(i) \geq v(N)/n$ implies $v(i) \geq \varphi_i(v) \geq v(N)/n$,
(ii) $v(i) \leq v(N)/n$ implies $v(i) \leq \varphi_i(v) \leq v(N)/n$.

In the first case, the dummifying player with weakly more productivity will consider whether to contribute to subsidizing other players or not, and in the second, she will be subsidized by the other players or not.

Theorem 4 *For $n \neq 2$, a value φ on $\mathcal{G}^N$ satisfies efficiency (**E**), strong average dummifying player property (**ADp** $^+$), and differential marginality (**DM**) if and only if there exists an $\alpha \in [0,1]$ such that $\varphi = \mathrm{ESD}^\alpha$.*

Proof Since both the ED and ESD values meet **E** and **DM**, it is clear that any ESD^α value, $\alpha \in [0,1]$, satisfies **E** and **DM**. That the ESD value meets **Dp** (Casajus &

Huettner, 2014) implies that any ESD^α meets **ADp** $^+$.

Note that **ADp** $^+$ implies **NG**, so the value φ satisfies **S** and **A** by Propositions 1 and 2.

If $n=1$, **E** directly implies that $\varphi = \mathrm{ESD}^\alpha$. We next assume that $n > 2$.

For all $v \in \mathcal{G}^N$, by (4), we get

$$v = \sum_{i \in N} v(i) u_i + \sum_{T \subseteq N, t \geq 2} c_T^v e_T,$$

where $c_T^v = v(T) - \sum_{i \in T} v(i)$.

In any game $c_T^v e_T$ with $T \subsetneq N$ and $t \geq 2$, note that each player $i \in N \setminus T$ is a dummifying player and $c_T^v e_T(i) = 0 = \frac{c_T^v e_T(N)}{n}$. By **ADp** $^+$, we have

$$\varphi_i(c_T^v e_T) = 0, \ \forall i \in N \setminus T.$$

On the other hand, note that any pairs of players of T are symmetric in $c_T^v e_T$. By **E** and **S**, we get

$$\varphi_i(c_T^v e_T) = \frac{c_T^v e_T(N)}{t} = 0 \quad \text{for} \quad \text{all } i \in T.$$

Thus, if $T \subsetneq N$ with $t \geq 2$

$$\varphi_i(c_T^v e_T) = 0, \forall i \in N.$$

Then in the game $c_N^v e_N$, by **E** and **S**, it can be easily seen that

$$\varphi_i\left(c_N^v e_N\right) = \frac{c_N^v}{n} = \frac{v(N) - \sum_{j \in N} v(j)}{n} \quad \text{for} \quad \text{all } i \in N.$$

Therefore, (4) and **A** imply that

$$\varphi_i(v) = \varphi_i\left(\sum_{j \in N} v(j) u_j\right) + \frac{v(N) - \sum_{j \in N} v(j)}{n} \quad \text{for} \quad \text{all } i \in N.$$

Finally, **AD** $^+$ coincides with **ADp** $^+$ for inessential games. By Corollary 2, we get $\varphi_i(v) = \alpha v(i) + \frac{v(N) - \alpha \sum_{j \in N} v(j)}{n} = \mathrm{ESD}_i^\alpha(v)$. So $\varphi(v) = \mathrm{ESD}^\alpha(v)$. □

The following example illustrates that Theorems 3 and 4 are not true for $n = 2$. Let $N := \{i, j\}$ and consider the value ϕ given by

$$\left(\phi_i(v), \phi_j(v)\right) = \begin{cases} \left(\frac{v(N) + v(i) - v(j)}{2}, \frac{v(N) - v(i) + v(j)}{2}\right) & v(i) \geq v(j) \\ \left(\frac{v(N)}{2}, \frac{v(N)}{2}\right) & \text{otherwise.} \end{cases}$$

which satisfies **E**, **AD** $^+$/**ADp** $^+$, and **DM**, but not $\mathrm{Sh}^\alpha/\mathrm{ESD}^\alpha$.

Furthermore, the axioms in Theorems 3 and 4 are non redundant. The Sh, ED and ESD values all satisfy **E** and **DM**, but Sh does not meet **ADp** $^+$, and ESD does not meet **AD** $^{+}$. The value φ^E defined by $\varphi_i^E(v) = v(i)$ satisfies **AD** $^{+}$/**ADp** $^{+}$ and **DM** but not **E**. For all $v \in \mathcal{G}^N$, let $C^v = \{i \in N \mid i \text{ is dummy player in } v\}$, and $D^v = \{i \in N \mid i \text{ is dummifying player in } v\}$. The values φ^{DM} and $\hat{\varphi}^{DM}$ are respectively given by

$$\varphi_i^{DM}(v) = \begin{cases} v(i) & i \in C^v \\ \dfrac{v(N) - \sum_{j \in C^v} v(i)}{n - |C^v|} & \text{otherwise} \end{cases}$$

and

$$\hat{\varphi}_i^{DM}(v) = \begin{cases} v(i) & i \in D^v \\ \dfrac{v(N) - \sum_{j \in D^v} v(i)}{n - |D^v|} & \text{otherwise.} \end{cases}$$

Clearly, the value φ^{DM} satisfies **E** and **AD** $^{+}$ but not **DM**, while the value $\hat{\varphi}^{DM}$ satisfies **E** and **ADp** $^{+}$ but not **DM**.

5 Conclusions

In this paper, we investigate several applications of the differential marginality axiom to axiomatic characterizations of values. Casajus (2011) shows that differential marginality together with efficiency and the null game property implies additivity under this condition $n \neq 2$. By replacing the null game property, we show that differential marginality together with efficiency and either the inessential game property or the nullifying player property implies additivity. Based on this result, we propose characterizations of the equal division and the equal surplus division values.

For $n \neq 2, 3$, Casajus and Yokote (2019) characterize the α-egalitarian Shapley values with weak differential monotonicity. We remove the restriction $n \neq 3$ in our new characterization of the α-egalitarian Shapley values by replacing weak differential monotonicity with differential marginality. In fact, we can also characterize the α-egalitarian Shapley values by using the axioms of efficiency, differential monotonicity of Casajus and Huettner (2013) and the average dummy player property of Casajus and Yokote (2019) in a similar way to this paper, and the only difference to note is that differential monotonicity is required to prove $f(d) > 0$ and $\alpha \in [0, 1]$ in Corollary 2. Moreover, we also establish an axiomatization of the α-ESD values by using differential marginality and determining the uniqueness of the value on inessential games.

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Data availability No data is used in our paper.

Declarations

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