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Harmonic choice model

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Abstract

For decades, discrete choice modelling was practically dominated by only two models: multinomial probit and logit. This paper presents a novel alternative—harmonic choice model. It is qualitatively similar to multinomial probit and logit: if one choice alternative greatly exceeds all (falls below at least one of) other alternatives in terms of utility then it is chosen with probability close to one (zero). Compared to probit and logit, the new model has relatively flat tails and it is steeper in the neighborhood of zero (when all alternatives yield the same utility and the decision maker chooses among them at random).

Keywords Probabilistic choice · Independence from irrelevant alternatives · Luce choice model · Fechner model of random errors · Odds ratio

1 Introduction

Any choice model ultimately yields a choice decision and avoids indecisive scenarios/preferences. For example, in the context of standard deterministic choice, a decision maker who weakly prefers *A* over *B*, *B*—over *C*, and *C*—over *A* is indecisive among all three and such preferences are typically ruled out by the transitivity axiom as irrational. In the context of probabilistic choice, a decision maker can choose *A* over *B*, *B*—over *C*, and *C*—over *A* and the literature identifies several possible driving forces behind such revealed cycle: random mistakes (Fechner, 1860; Hey & Orme, 1994), imprecise preferences (Butler & Loomes, 2007, 2011; Falmagne, 1985), lapses of concentration/trembles (Harless & Camerer, 1994), or an asymmetric information when some attributes of choice alternatives are observable to decision makers but not—to outside researchers (McFadden, 1976).

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One possible avenue for building a model of probabilistic choice is to limit the likelihood of a revealed choice cycle such as the one described above. This requires some metric to measure the likelihood and some benchmark to compare it with. A rather natural benchmark is the likelihood of the opposite indecisive cycle when a decision maker chooses alternative *A* over alternative *C*, *C*—over *B*, and *B*—over *A*. Indeed, if the likelihood of the first choice cycle relatively exceeds that of the second, or vice versa, a choice model can be regarded as tilted/biased towards an indecisive scenario.

This leaves us with a question which metric to use for assessing the likelihood of an indecisive cycle. Let *P*(*A*,*B*) denote the probability that a decision maker chooses alternative *A* over alternative *B* in a direct binary choice. The likelihood of the first choice cycle is then some function of *P*(*A*,*B*), *P*(*B*,*C*), and *P*(*C*,*A*). One possibility is to take the *geometric* average of these three binary choice probabilities. This results in the following Eq. (1).

$$\sqrt[3]{P(A,B)P(B,C)P(C,A)} = \sqrt[3]{P(A,C)P(C,B)P(B,A)} \quad (1)$$

Getting rid of the cubic root in (1) yields a condition that is known as the product rule (e.g., Estes, 1960, p. 272; Luce & Suppes, 1965, definition 25, p. 341). Binary choice probabilities satisfy the product rule if and only if they take the form of a binary Luce (1959) choice model (*cf.* Luce & Suppes, 1965, Theorem 48, p. 350): *P*(*A*,*B*)=*u*(*A*)/[*u*(*A*)+*u*(*B*)], where *u*(.) is utility function mapping choice alternatives to strictly positive real numbers. Luce (1959) choice model is also known as strict utility or multinomial logit and it is widely used in economics and psychology. However, utility functions in economics and psychology are typically not restricted to the range of strictly positive real numbers and require some ad hoc monotone transformation (with exponentiation being the most popular one) before they can be embedded into Luce (1959) choice model.

Another possible metric for assessing the likelihood of an indecisive scenario when a decision maker chooses *A* over *B*, *B*—over *C*, and *C*—over *A* is to take the *arithmetic* average of binary choice probabilities *P*(*A*,*B*), *P*(*B*,*C*), and *P*(*C*,*A*). This results in Eq. (2).

$$\frac{P(A,B)+P(B,C)+P(C,A)}{3} = \frac{P(A,C)+P(C,B)+P(B,A)}{3} \quad (2)$$

Getting rid of the common denominator in (2) results in a so-called sum rule, which is the same as the product rule except that probabilities are added rather than multiplied. Under probabilistic completeness (defined in Eq. (4) below), binary choice probabilities satisfy the sum rule if and only if they take the form of an additive choice model: *P*(*A*,*B*)=*0.5*+*v*(*A*)-*v*(*B*)], where *v*(.) is utility function mapping choice alternatives to a bounded interval of real numbers (Blavatskyy, 2023). Additive choice model offers remarkable analytical convenience such as a closed form solution for quantal response equilibria (McKelvey & Palfrey, 1995) but utility functions in economics are typically not restricted in range to a bounded interval.

Another possible metric for assessing the likelihood of an indecisive scenario when a decision maker chooses A over B, B—over C, and C—over A is to take the *harmonic* average of binary choice probabilities $P(A,B)$, $P(B,C)$, and $P(C,A)$. This results in the following Eq. (3). This paper presents a new model of probabilistic choice synonymous with condition (3). Therefore, we call it harmonic choice model. The main comparative advantage of this model is that its utility function is unrestricted in range, *i.e.*, it does not require any ad hoc monotone transformations.

$$\frac{3}{\frac{1}{P(A,B)}+\frac{1}{P(B,C)}+\frac{1}{P(C,A)}}=\frac{3}{\frac{1}{P(A,C)}+\frac{1}{P(C,B)}+\frac{1}{P(B,A)}} \tag{3}$$

2 Modelling framework

Let Ω denote a fixed universal set with $n\geq 2$ choice alternatives (the decision is trivial when the choice set contains only one element). Choice alternatives are labeled by capital Latin letters A, B, C, $D\in\Omega$ and they can be consumption bundles, risky lotteries (probability distributions), uncertain Savage (1954) acts (random variables), streams of intertemporal outcomes, behavioral strategies etc. A menu of two alternatives A, $B\in\Omega$ is denoted by $\{A,B\}$.

A decision maker is an individual or a group of individuals. A decision maker chooses alternative $A\in\Omega$ from the choice set Ω with probability $P(A|\Omega)\in(0,1)$. As mentioned in the introduction, the probability that a decision maker chooses alternative A over alternative B in a direct binary choice is denoted by a conventional simplified notation $P(A,B)\equiv P(A|\{A,B\})$. For simplicity, we assume that there are no dominated alternatives in the choice set, *i.e.* each available alternative is chosen with a strictly positive probability. One possibility could be that a decision maker first detects and discards any dominated alternatives and then chooses among the remaining alternatives in a probabilistic manner, as in Luce (1959) choice model. We assume that choice probabilities add up to one, which is also known as probabilistic completeness. In binary choice probabilistic completeness is $P(A,B)+P(B,A)=1$ for all A, $B\in\Omega$.

$$\sum_{A\in\Omega} P(A|\Omega)=1 \tag{4}$$

Finally, for any choice alternative $A\in\Omega$ the odds against choosing A (5) are defined as the ratio of the probability that A is not chosen to the probability that A is chosen from set Ω. For binary choice, we use a simplified notation $O(A,B)\equiv O(A|\{A, B\})$.

$$O(A|\Omega) \equiv \sum_{\substack{B \in \Omega \\ B \neq A}} P(B|\Omega)/P(A|\Omega) \tag{5}$$

Dogan and Yıldız (2021) recently proved that Luce (1959) choice model is equivalent to odds modularity: $O(A|\Omega) + O(A|\Omega') = O(A|\Omega \cup \Omega')$ for any two choice sets Ω and Ω*'* such that Ω∩Ω '=$\{A\}$. In this paper, we impose a different condition on odds. In particular, we consider the difference in odds against choosing *A* vs. *B* from the same choice set. We show that this condition is equivalent to a new model of probabilistic choice.

3 Independence from irrelevant alternatives

Luce (1959) derived his choice model from a principle of independence from irrelevant alternatives: "… if one is comparing two alternatives according to some algebraic criterion, say preference, this comparison should be unaffected by the addition of new alternatives or the subtraction of old ones (different from the two under consideration). Exactly what should be taken to be the probabilistic analogue of this idea is not perfectly clear, but one reasonable possibility is the requirement that the ratio of the probability of choosing one alternative to the probability of choosing the other should not depend upon the total set of alternatives available". The assumption that choice decision between two alternatives is not influenced by the presence (or absence) of other choice alternatives is intuitively appealing. Yet, the specific form of this principle employed in Luce's choice model—that the ratio of choice probabilities remains unaffected—appears somewhat ad hoc.

In Luce's choice model, the ratio of choice probabilities is equal to the ratio of utilities of the corresponding choice alternatives. Choice probabilities, by definition, cannot be negative. Therefore, the ratio of utilities in Luce's choice model must be non-negative as well. This imposes a restriction on the range of utility function that must map all choice alternatives either to positive real numbers or—to negative real numbers (but not both at the same time).

One possibility to avoid such restriction on the range of utility function is the following. We can assume that the difference (rather than the ratio) of choice probabilities remains independent from irrelevant alternatives. Analogously to Luce's choice model, this assumption implies that the difference in choice probabilities is equal to the difference in utilities of the corresponding alternatives. Since the difference in choice probabilities can be positive or negative, a priori, this does not restrict the range of the utility function only to positive or only to negative real numbers. Yet, choice probabilities, by definition, belong to a bounded interval [0,1]. Therefore, the difference in choice probabilities must be in a bounded interval. This imposes another restriction on the range of utility function that now must map all choice alternatives to a bounded interval.

In sum, the principle of the independence from irrelevant alternatives is traditionally formulated for choice probabilities. This principle links choice probabilities to utilities of choice alternatives. This creates a theoretical inconsistency if choice probabilities are defined as positive bounded numbers, but utility function is unrestricted in range. In this paper we propose a novel form of the independence from irrelevant alternatives to avoid this problem.

Decision theory conventionally characterizes the likelihood of a choice decision by its probability. Yet, choice likelihood can be alternatively described by other means. In a multi-attribute choice in particular, when a decision maker chooses among many alternatives, choice probabilities may not be as practical as the odds in conveying likelihood information. For example, most betting markets use the odds rather than probabilities. This paper formulates the principle of independence from irrelevant alternatives for choice odds rather than choice probabilities. The odds against choosing an alternative (5) convey the same likelihood information as the choice probability but they are not restricted to a bounded interval. Yet, both the odds and choice probabilities are defined as non-negative numbers. Hence, formulating the principle of independence from irrelevant alternatives for the odds ratio leads to similar restrictions as in Luce's choice model. On the other hand, formulating the principle of independence from irrelevant alternatives for differences in odds is rather promising. The difference in odds against choosing two alternatives is an unrestricted real number that can be negative, zero, or arbitrary large. So is the difference in utilities of two choice alternatives. Hence, there is no theoretical inconsistency if these two concepts are linked (as we shall prove below in the subsequent section). Specifically, we assume that the difference in odds against choosing two alternatives is independent from the presence (or absence) of other "irrelevant" choice alternatives (*cf.* definition 1 below).

Definition 1 The odds form of the independence from irrelevant alternatives holds if Eq.(6) is satisfied for any two choice alternatives A, $B \in \Omega$.

$$O(A|\Omega) - O(B|\Omega) = O(A, B) - O(B, A) \tag{6}$$

Ordinal IIA in Fudenberg et al., (2015, Definition 5) with $f(x) = e^{1/x}$ is equivalent to (6).

4 Results

Proposition 1 *If the choice set Ω contains at least three elements and the independence from irrelevant alternatives (6) holds then (3) holds for any three choice alternatives $A, B, C \in \Omega$.*

The proof is presented in the appendix.

It is relatively well-known that the principle of independence from irrelevant alternatives has no bite when the choice set contains only two elements. In this case, choice decision between two alternatives can only depend on these two alternatives and it is automatically independent from irrelevant alternatives (since there are no other available options). Thus, in the special case of a binary choice, we need to assume (3) directly. The harmonic mean is the most conservative of three

Pythagorean means. Thus, we assess the likelihood of an indecisive cycle more conservatively by making assumption (3), compared to (1) and (2).

Proposition 2 *Binary choice probability function* $P : \Omega \times \Omega \to (0, 1)$ *satisfies (3) for any* $A,B,C \in \Omega$ *if and only if there is utility function* $u : \Omega \to \mathbb{R}$ *unique up to addition of a constant such that*

$$P(A,B) = \begin{cases} \frac{1}{2} + \frac{\sqrt{1 + [u(A) - u(B)]^2/4} - 1}{u(A) - u(B)}, & u(A) \neq u(B) \\ \frac{1}{2} & u(A) = u(B) \end{cases} \quad (7)$$

The proof is presented in the appendix.

Choice model (7), like Luce's choice model, is a special case of binary Fechner (1860) model. Figure 1 plots binary choice probability (7) as a function of utility difference $u(A)$-$u(B)$. When this difference is large (small) probability (7) converges to one (zero). As mentioned in the introduction, a comparative advantage of model (7) is its unrestricted range of utility $u(.)$. Probability (7) is always well-defined (between zero and one) even when alternatives yield negative, zero, or arbitrary large utility. For comparison, Fig. 1 plots probability $P(A, B) = F_{0,5}(u(A) - u(B))$ in Fechner (1860) model with normally distributed random errors, where $F_{0,5}(.)$ is the cumulative distribution function of a normal distribution with zero mean and σ=5. Compared to this normal cumulative distribution function, model (7) has relatively flat tails and it is steeper in the neighborhood of (0, ½). Model (7) is relatively less (more) discriminatory in choice between alternatives that differ a lot (little) in terms

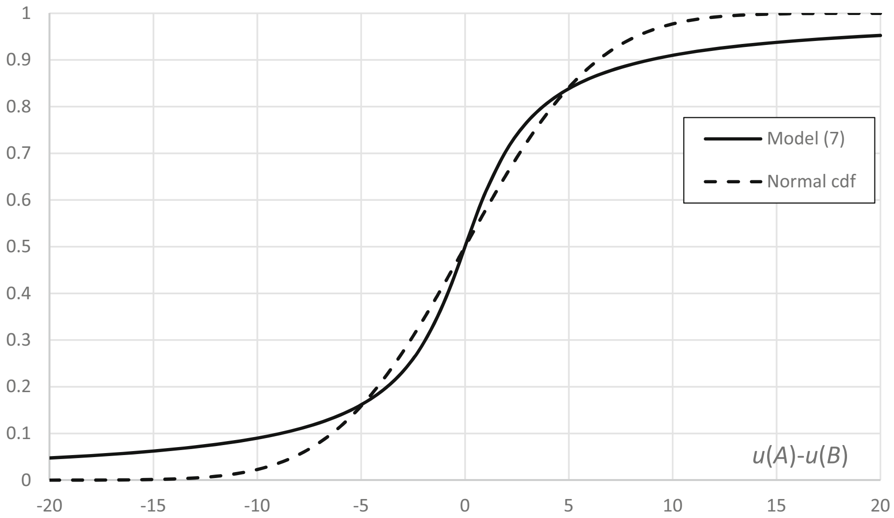

Fig. 1 Binary choice probability (7) as a function of utility difference $u(A)$-$u(B)$ compared to the cumulative distribution function of a normal distribution $F_{0,5}(.)$

of utility. Propositions 1 and 2 are the analogue of Theorem 48 in Luce and Suppes (1965) for Luce's choice model.

We next show that model (7) satisfies the quadruple condition (*cf.* Davidson and Marschak, 1959; Luce & Suppes, 1965, definition 24, p. 341): if $P(A,B) \geq P(C,D)$ then $P(A,C) \geq P(B,D)$ for all $A,B,C,D \in \Omega$.

Proposition 3 *Conditions (3) and (4) imply the quadruple condition.*

The proof is presented in the appendix.

The quadruple condition implies strong stochastic transitivity (Luce & Suppes, 1965, Theorem 39, p. 346). Hence, model (7) satisfies strong stochastic transitivity: if $P(A,B) \geq ½$ and $P(B,C) \geq ½$ then $P(A,C) \geq \max\{P(A,B), P(B,C)\}$ for all $A,B,C \in \Omega$. Strong stochastic transitivity also implies the triangle condition $P(A,B)+P(B,C) \geq P(A,C)$ for all $A,B,C \in \Omega$ (Luce & Suppes, 1965, Theorems 35, 37 and 38). Thus, model (7) also satisfies the triangle condition.

Proposition 4 *Given that the choice set* Ω *contains at least three elements, the independence from irrelevant alternatives (6) holds if and only if there is utility function* $u : \Omega \to \mathbb{R}$ *such that alternative* $A \in \Omega$ *is chosen with probability* $P(A|\Omega) = 1/[x(\Omega) - u(A)]$, *where* $x(\Omega)$ *is the highest root of Eq. (8).*

$$\sum_{A \in \Omega} \frac{1}{x(\Omega) - u(A)} = 1 \tag{8}$$

The proof is presented in the appendix.

Binary choice model (7) can be also viewed as a special case of multinomial model (8) with the highest $x(\{A,B\})$ that solves $[x(\{A,B\}) - u(A)]^{-1} + [x(\{A,B\}) - u(B)]^{-1} = 1$ being (9).

$$x(\{A,B\}) = 1 + \frac{u(A) + u(B)}{2} + \sqrt{1 + [u(A) - u(B)]^2/4} \tag{9}$$

Plugging $x(\{A,B\})$ defined by (9) into $P(A,B) = 1/[x(\{A,B\}) - u(A)]$ yields model (7).

If all choice alternatives yield the same utility v then $x(\Omega)$ that solves (8) is equal to v+ n so that each choice alternative is chosen with the same probability $P(A|\Omega) = 1/n$ for all $A \in \Omega$. Multinomial choice model characterized in Proposition 4 satisfies the regularity condition (*cf.* Luce & Suppes, 1965, definition 26, p. 342): choice probabilities cannot increase when more alternatives are added to the choice set. To see this, note that adding more alternatives to the choice set increases the number of elements in the sum on the left-hand side of (8). Therefore, to keep Eq. (8) satisfied $x(\Omega')$ of an extended set Ω' must exceed $x(\Omega)$ of the original set Ω, which implies that all elements of Ω are chosen with a smaller probability from the extended set Ω'.

Example 1 In ternary choice if alternative A yields the same utility as alternative C then both are chosen with probability (10).

$$P(A|\{A,B,C\}) = \begin{cases} \frac{1}{4} + \frac{\sqrt{8+[u(A)-u(B)+1]^2}-3}{4[u(A)-u(B)]}, & u(A) \neq u(B) \\ \frac{1}{3} & u(A) = u(B) \end{cases} \quad (10)$$

Figure 2 plots choice probability (10) in comparison with bivariate normal distribution (11) with mean vector [0,0] and the covariance matrix $\sigma^2\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$, for σ=9.

$$F(u(A)-u(B),0) = \frac{1}{\sqrt{3}\pi\sigma^2}\int_{-\infty}^{u(A)-u(B)}\int_{-\infty}^{0} e^{-\frac{2(x^2+y^2-xy)}{3\sigma^2}}dxdy \quad (11)$$

Figure 2 illustrates that choice probability $P(A|\{A,B,C\})$ converges to zero when utility $u(A)$ falls substantially below utility $u(B)$. On the other hand, choice probability $P(A|\{A,B,C\})$, as well as $P(C|\{A,B,C\})$, converges to ½ when utility $u(A)$ greatly exceeds utility $u(B)$. Compared to the bivariate normal distribution, ternary choice probability (10) has relatively flat tails and it is steeper in the neighborhood of (0, 1/3), where two distributions intersect. Thus, model (10) is qualitatively similar to bivariate probit but it discriminates less (more) in choice among alternatives that differ a lot (little) in terms of utility.

Example 2 Ternary choice among three alternatives none of which yield the same utility.

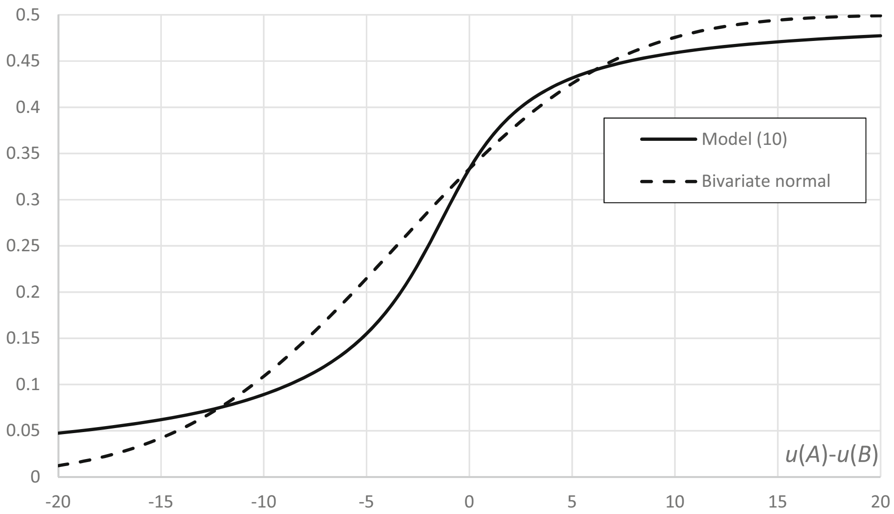

Fig. 2 Ternary choice probability (10) as a function of utility difference $u(A)$-$u(B)$ compared to the bivariate normal distribution (11)

Let $\Delta_1 = 1/[u(A) - u(B)]$ and $\Delta_2 = 1/[u(A) - u(C)]$. Furthermore, let $r = (2\Delta_1 + 2\Delta_2 - 1)^2/9 + \Delta_1/3 + \Delta_2/3 - \Delta_1\Delta_2$ and $q = (2\Delta_1 + 2\Delta_2 - 1)^3/27 + (\Delta_1 + \Delta_2)[\Delta_1/3 + \Delta_2/3 - \Delta_1\Delta_2 - 1/6]$. A decision maker then chooses alternative A among alternatives A, B and C with probability (12).

$$P(A|\{A,B,C\}) = \begin{cases} \frac{1}{3} + 2\sqrt{r}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{r\sqrt{r}}\right)\right) - \frac{2}{3}(\Delta_1 + \Delta_2), & \Delta_1 > 0, \Delta_2 > 0 \\ \frac{1}{3} + 2\sqrt{r}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{r\sqrt{r}}\right) - \frac{2\pi}{3}\right) - \frac{2}{3}(\Delta_1 + \Delta_2), & \Delta_1\Delta_2 < 0 \\ \frac{1}{3} + 2\sqrt{r}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{r\sqrt{r}}\right) - \frac{4\pi}{3}\right) - \frac{2}{3}(\Delta_1 + \Delta_2), & \Delta_1 < 0, \Delta_2 < 0 \end{cases} \tag{12}$$

Figure 3 plots ternary choice probability (12) as a function of utility differences $u(A)$-$u(B)$ and $u(A)$-$u(C)$. When both of these differences are relatively large, the decision maker choses alternative A with probability close to one. When at least one of these differences is relatively small, the decision maker choses alternative A with probability close to zero. When both utility differences are close to zero, the decision maker chooses alternative A with probability close to 1/3. Thus, ternary choice probability (12) qualitatively resembles bivariate normal distribution, but it has relatively flat tails and it is steeper in the neighborhood of (0,0,1/3). Since ternary choice probability (12) is a function of utility differences $u(A)$-$u(B)$ and $u(A)$-$u(C)$, it

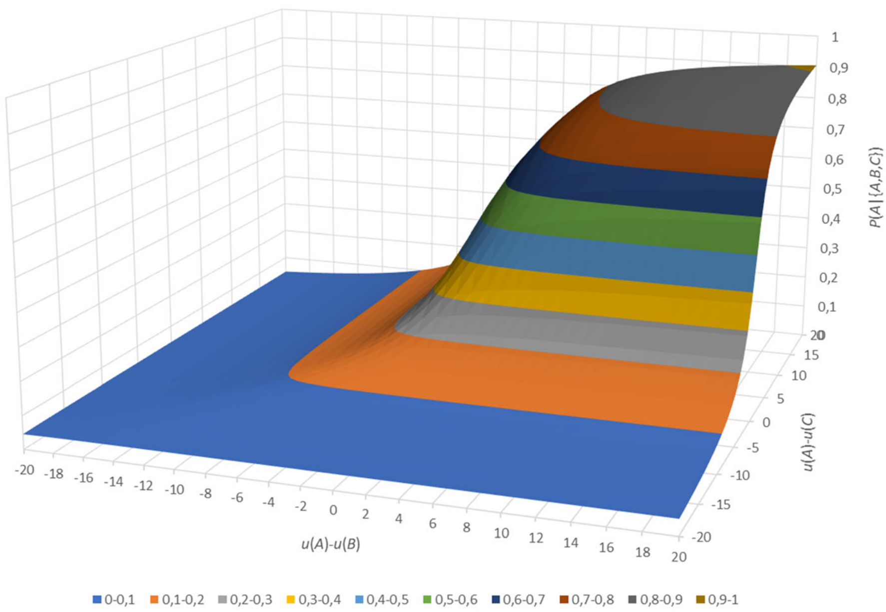

Fig. 3 Ternary choice probability (12) as a function of utility differences $u(A)$-$u(B)$ and $u(A)$-$u(C)$

is a special case of the generalized Fechner model/strong utility (Blavatskyy, 2018, p. 76). Binary choice probability (10) illustrated on Fig. 2 is the asymptotic limit of ternary choice probability (12) when utility difference $u(A)$-$u(C)$ goes to zero, *i.e.* Figure 2 can be obtained from Fig. 3 by cutting along the plane $u(A)$-$u(C)$=0 (the "depth" axis z=0).

5 Application: harmonic quantal response equilibrium

This section applies harmonic choice model to normal form games. Specifically, we introduce the concept of harmonic quantal response equilibrium. It is a special parametric form of quantal response equilibrium (McKelvey & Palfrey, 1995, Sect. 2, p. 8). In harmonic quantal response equilibrium, players choose among strategies according to harmonic choice model. Logit quantal response equilibrium is an alternative parametric form of quantal response equilibrium when players choose among strategies according to Luce's choice model (McKelvey & Palfrey, 1995, Sect. 3, p. 11). In quantal response equilibrium players choose in a probabilistic manner, *i.e.*, the best responding strategies are not necessarily chosen with certainty.

We consider a finite normal-form game with $k>1$ players. Every player $i\in\{1,\ldots,k\}$ has a non-empty set Ω_i of pure strategies and a payoff function $u_i:\prod_{i\in\{1,\ldots,k\}}\Omega_i\rightarrow\mathbb{R}$. Players choose among available strategies according to harmonic choice model and they believe that other players do so as well. In harmonic quantal response equilibrium, players' choice probabilities coincide with players' beliefs. Specifically, harmonic quantal response equilibrium is defined by a system of Eq. (13) for every player $i\in\{1,\ldots,k\}$ and every pure strategy $A\in\Omega_i$.

$$P_i(A|\Omega_i)=\frac{1}{x_i(\Omega_i)-u_i(A\times\Omega_{-i})} \tag{13}$$

where $x_i(\Omega_i)$ is the highest root of equation $\sum_{B\in\Omega_i}\frac{1}{x_i(\Omega_i)-u_i(B\times\Omega_{-i})}=1$ and $u_i(A\times\Omega_{-i})$ denotes player i's expected utility of strategy $A\in\Omega_i$ given that other players choose their strategies with harmonic choice probabilities (13).

Example 3 Consider a generalized matching pennies game presented in Table 1 taken from Goeree and Holt (2001, Table 1, p.1406).

Let $v=u(1)-u(0)$ and $\alpha=\frac{u(a)-u(0)}{u(1)-u(0)}$. Harmonic quantal response equilibrium of the generalized matching pennies game is then defined by the following system of Eqs. (14)–(15).

Table 1 The normal form of a generalized matching pennies game

		Player 1	
		Left (p)	Right (1-p)
Player 2	Top (q)	(a, 0)	(0, 1)
	Down (1-q)	(0, 1)	(1, 0)

$$p = \begin{cases} \frac{1}{2} + \frac{\sqrt{1 + v^2[1-2q]^2/4} - 1}{v(1-2q)}, & q \neq \frac{1}{2} \\ \frac{1}{2} & q = \frac{1}{2} \end{cases} \tag{14}$$

$$q = \begin{cases} \frac{1}{2} + \frac{\sqrt{1 + v^2[p(1+\alpha)-1]^2/4} - 1}{v(p(1+\alpha)-1)}, & p \neq \frac{1}{1+\alpha} \\ \frac{1}{2} & p = \frac{1}{1+\alpha} \end{cases} \tag{15}$$

When $a=1$ a unique harmonic quantal response equilibrium is given by $p=q=½$. This corresponds to experimental results reported in Goeree and Holt (2001, p.1407).

When $a\neq1$ harmonic quantal response equilibrium depends on parameter ν. Figure 4 plots harmonic quantal response equilibrium for the case when $\alpha = 4$ and various values of ν as shown on the horizontal axis. If ν is small, then both players randomize between their strategies with probabilities 50–50%. If ν is large, then harmonic quantal response equilibrium converges to the unique mixed strategy Nash equilibrium $p=1/(\alpha+1)=0.2$ and $q=½$. For intermediate values of ν player 1 (2) chooses ‘Right’ (‘Top’) with a relatively high probability. This resembles experimental findings of Goeree and Holt (2001, p.1407).

For comparison, Fig. 4 also plots logit quantal response equilibrium (16)–(17). Figure 4 shows that the set of harmonic quantal response equilibria is quite similar to

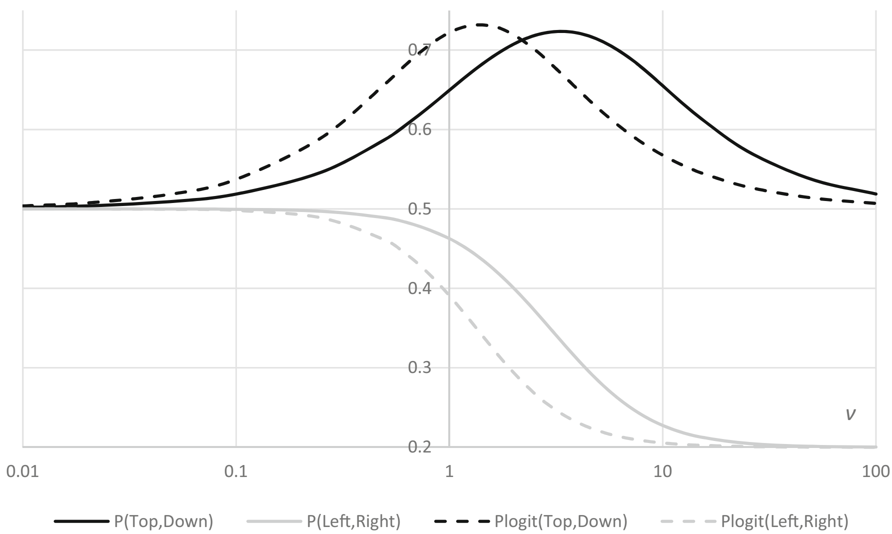

Fig. 4 Harmonic quantal response equilibrium (14)–(15) and logit quantal response equilibrium of the generalized matching pennies game when $\alpha = 4$ and values of ν as shown on the horizontal axis

that of logit quantal response equilibria (the only difference appears to be the level of noise v required for rationalizing these equilibria).

$$P_{Logit}(Left, Right) = \frac{1}{1 + e^{v\left(2P_{Logit}(Top, Down) - 1\right)}} \tag{16}$$

$$P_{Logit}(Top, Down) = \frac{1}{1 + e^{v\left(1 - (1+\alpha)P_{Logit}(Left, Right)\right)}} \tag{17}$$

6 Empirical application: fit to experimental data collected by Hey and Orme (1994)

Hey and Orme (1994) asked 80 experimental subjects to choose twice between 100 pairs of risky lotteries each yielding one of four possible outcomes: £0, £10, £20, and £30. The data set collected by Hey and Orme (1994) is convenient for comparing decision theories and models of stochastic choice and it has been re-examined to that purpose in numerous studies *e.g.*, Hey (1995), Hey and Carbone (1995), Carbone and Hey (2000), Buschena and Zilberman (2000) and Wilcox (2008, 2011). This section compares our proposed harmonic choice model with standard logit model of discrete choice (Luce's choice model) according to their goodness of fit to the experimental data collected by Hey and Orme (1994). We aggregate two repetitions in Hey and Orme (1994) into one data set. Subjects revealing indifference between two lotteries are treated as choosing with probabilities 50–50%.

Under harmonic choice model, the likelihood that a subject chooses lottery $L : \{£0, £10, £20, £30\} \rightarrow [0, 1]$ over lottery $R : \{£0, £10, £20, £30\} \rightarrow [0, 1]$ is given by

$$P(L, R) = \frac{1}{2} + \frac{\sqrt{1 + \Delta EU(L, R)^2/4} - 1}{\Delta EU(L, R)} \tag{18}$$

where the difference in lotteries' expected utility is $\Delta EU(L, R) \equiv \sigma(L - R)u'$ and $u \equiv (0, u_1, u_2, 1)$ is a vector of normalized Bernoulli utilities. Three subjective parameters $\sigma = u(£30)$, $u_1 = u(£10)/u(£30)$ and $u_2 = u(£20)/u(£30)$ are estimated to maximize total log-likelihood of all revealed binary choices. Estimation is done separately for each subject. Optimization is performed in MatlabR2017b based on the Nelder–Mead simplex algorithm.

For comparison, the same estimation is done for logit model where the likelihood that lottery L is chosen over lottery R is given by (19).

$$P(L, R) = \frac{1}{1 + e^{-\Delta EU(L, R)}} \tag{19}$$

For 43 out of 80 subjects (53.8%) harmonic choice model provides a better goodness of fit than standard logit model. The two models are non-nested, and they can be compared using Vuong's likelihood ratio test (Vuong, 1989; Loomes et al., 2002, p.128). Vuong's statistic z has a limiting standard normal distribution when

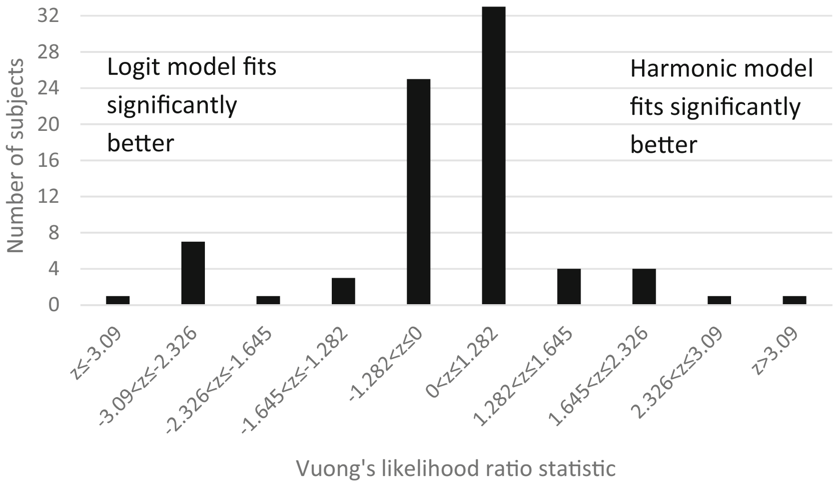

Fig. 5 Distribution of Vuong's likelihood ratio statistic

two models make equally good predictions. A significant positive (negative) value of Vuong's z indicates that harmonic (logit) choice model fits the data better. Since both models have the same number of free parameters, there is no need for adjustment via Akaike or Schwarz information criterion. Figure 5 shows the distribution of Vuong's likelihood ratio statistic across 80 subjects. For a great majority of subjects there is no statistically significant difference between models (18) and (19) under conventional significance levels. Estimated subjective parameters are also very similar across two models. For example, Fig. 6 shows the scatterplot of parameter u_2 (normalized utility of £20) estimated using harmonic and logit models (across all 80 subjects).

7 Discussion

The main theoretical advantage of harmonic choice model is that its assumptions do not conflict with the mathematical properties of a standard microeconomic utility function. Thus, there is no need for any ad hoc transformation of the latter (*e.g.*, exponentiation to map utilities only to positive reals). This brings extra clarity in applications. For example, a so-called noise parameter in quantal response equilibrium or the volatility of random errors in econometric estimation are clearly the same as the scale parameter of utility function. If the scale parameter is relatively small (so that all choice alternatives are mapped to similar utilities), then a decision maker chooses almost at random. If the scale parameter is relatively large, then a decision maker is nearly certain to choose the most desirable alternatives. In standard microeconomic theory derived from a (deterministic) binary preference relation, the scale parameter is irrelevant (multiplying all utilities by the same factor has no effect on preferences). In a model of probabilistic choice, the scale parameter becomes quite important as it captures noise, *i.e.*, departure from deterministic choice to random choice.

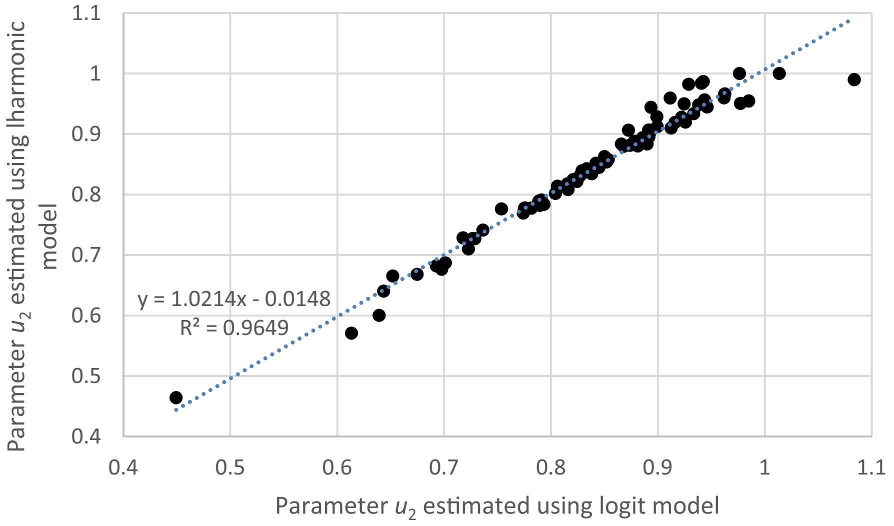

Fig. 6 Scatterplot of parameter u_2 estimated using harmonic and logit models

Harmonic choice model, like standard logit model, is derived from the well-known principle of the independence from irrelevant alternatives. The main difference between the two is that the former imposes independence from irrelevant alternatives on choice odds whereas the latter—on choice probabilities. One of the consequences is that logit model restricts the *geometric* average of binary choice probabilities whereas harmonic choice model restricts the *harmonic* average of binary choice probabilities, which explains its name. This also offers the possibility of a clean empirical test for model selection.

Since harmonic choice model, like logit, relies on the principle of independence, it is also vulnerable to the well-known criticism of this principle. Specifically, the new model is ill-suited in situations when there are strong substitution effects between choice alternatives. Debreu (1960) criticized logit model using the following three choice alternatives: the Debussy quartet (A), the 8th symphony of Beethoven (B) and the same symphony with a different conductor (C). Debreu (1960) argued that a decision maker who reveals a slight preference for French music in a direct binary choice and no preference between different conductors, so that $P(A,B)=P(A,C)=0.6$ and $P(B,C)=0.5$, must have $P(A|\{A,B,C\}=3/7$ in Luce (1959) choice model. In other words, a slight preference for French music in binary choice is reversed in ternary choice. Beethoven pieces with different conductors are nearly perfect substitutes for each other, which undermines the principle of independence in this example. Harmonic choice model is also prone to Debreu's critique. If $P(A,B)=P(A,C)=0.6$ and $P(B,C)=0.5$ then Eq. (7) implies $u(B)=u(C)$ and $u(A)=u(B)+5/6$. Equation (10) then implies that $P(A|\{A,B,C\}=0.3$ and $P(B|\{A,B,C\}=P(C|\{A,B,C\}=0.35$.

Debreu's critique applies not only to models of probabilistic choice that are explicitly built on the independence principle but also—to very different approaches to probabilistic choice such as the random preference (also known as random utility or random parameter) approach (*e.g.*, Falmagne, 1985; Loomes & Sugden, 1995). To

Table 2 Random preference example

Preference ordering	Probability of preference ordering
A>B>C	7/30
A>C>B	7/30
B>C>A	2/15
B>A>C	2/15
C>B>A	2/15
C>A>B	2/15

illustrate this, let us consider six possible preference orderings of Debreu's musical pieces, listed in Table 2. Consider a decision maker who has a slight preference for the French music so that the first two orderings in Table 2 (where the Debussy quartet is the most preferred option) are picked with probability 7/30 each and the remaining four orderings are picked with equal probability (2/15). In binary choice, this decision maker behaves as in the original Debreu's example: A is chosen over B (or C) with probability 0.6, and B is chosen over C with probability 0.5. In ternary choice, this decision maker also reverses his preference for the French music, similar to the original Debreu's example: A is chosen among all three alternatives with probability $7/15<0.5$ and B (or C) is chosen among all three alternatives with probability 4/15. In sum, Debreu's critique challenges harmonic choice model but it also challenges the whole literature on probabilistic choice including standard logit, probit, and random utility models.

8 Conclusion

Standard microeconomic theory takes a binary preference relation as the primitive of choice so that revealed choices are nearly always deterministic. More often than not such simple theory cannot be taken directly to data. For example, the same decision maker may not repeat his or her choice decision when presented with the same decision problem for the second time. Camerer (1989), Starmer and Sugden (1989), Hey and Orme (1994), Ballinger and Wilcox (1997) report respectively inconsistency rates of 31.6%, 26.5%, 25%, and 20.8%. Models of probabilistic choice are developed to capture such stochastic decision making.

This paper presents a new model of probabilistic choice, which we call harmonic choice model. The proposed model has standard properties—it is regular, satisfies strong stochastic transitivity, the quadruple condition, and the triangle inequality. Harmonic quantal response equilibrium is qualitatively similar to the well-known logit quantal response equilibrium. In empirical application, the new model yields results comparable with standard logit model.

Harmonic choice model is not only very similar to logit, but it is also derived from the same principle of the independence of irrelevant alternatives. The crucial difference is that we impose this principle on choice odds whereas Luce (1959) imposed independence on choice probabilities. The latter approach yields a somewhat simpler algebra but requires a restricted range of utility function. In practice, this restricted range is obtained by some ad hoc transformation of utilities such as exponentiation or using a power function (*e.g.*, Holt & Laury, 2002, Eq. (1), p.1652). This is not needed in the model presented in this paper.

Assuming that the independence of irrelevant alternatives holds for choice odds rather than choice probabilities yields harmonic choice model that is very similar to logit but does not require a restricted range of utilities. Compared to multinomial probit or logit, harmonic choice model has relatively flat tails and a steeper slope in the neighborhood of zero (when all alternatives yield the same utility). In other words, compared to standard models, harmonic choice model discriminates less (more) when alternatives differ a lot (little) in terms of utility. Harmonic choice model is vulnerable to Debreu's critique just like logit, probit, and random utility models.

Appendix

Proof of Proposition 1 If the choice set contains at least three elements, then we can select three alternatives A, B, $C \in \Omega$. If the independence from irrelevant alternatives (6) holds, then we must have

$$O(A|\Omega) - O(B|\Omega) = O(A,B) - O(B,A)$$

$$O(B|\Omega) - O(C|\Omega) = O(B,C) - O(C,B)$$

$$O(C|\Omega) - O(A|\Omega) = O(C,A) - O(A,C)$$

Adding these three equations together yields

$$O(A,B) + O(B,C) + O(C,A) = O(A,C) + O(C,B) + O(B,A)$$

Using the definition of choice odds (5) we can rewrite this equation as

$$\frac{P(B,A)}{P(A,B)} + \frac{P(C,B)}{P(B,C)} + \frac{P(A,C)}{P(C,A)} = \frac{P(C,A)}{P(A,C)} + \frac{P(B,C)}{P(C,B)} + \frac{P(A,B)}{P(B,A)}$$

Finally, using probabilistic completeness, we can rewrite this equation as

$$\frac{1-P(A,B)}{P(A,B)} + \frac{1-P(B,C)}{P(B,C)} + \frac{1-P(C,A)}{P(C,A)} = \frac{1-P(A,C)}{P(A,C)} + \frac{1-P(C,B)}{P(C,B)} + \frac{1-P(B,A)}{P(B,A)}$$

Simplifying and rearranging then yields (3). □

Proof of Proposition 2 Consider first the case when binary choice probability function $P: \Omega \times \Omega \to (0,1)$ satisfies (3) for any $A,B,C \in \Omega$. Using probabilistic completeness (4), we can rewrite (3) as follows:

$$\frac{1}{P(A,B)}+\frac{1}{P(B,C)}+\frac{1}{1-P(A,C)}=\frac{1}{1-P(A,B)}+\frac{1}{1-P(B,C)}+\frac{1}{P(A,C)}$$

Rearranging this equation yields

$$\frac{1}{P(A,B)}-\frac{1}{1-P(A,B)}=\frac{1}{1-P(B,C)}-\frac{1}{P(B,C)}-\left[\frac{1}{P(A,C)}+\frac{1}{1-P(A,C)}\right]$$

The left-hand side of this equation does not depend on C. Hence, the right-hand side must also not depend on C. Let us then fix C and define a real-valued function

$$u(.)\equiv\frac{1}{1-P(.,C)}-\frac{1}{P(.,C)}$$

We obtain then

$$\frac{1}{P(A,B)}-\frac{1}{1-P(A,B)}=u(B)-u(A)$$

Rearranging yields quadratic equation

$$P^2(A,B)[u(B)-u(A)]+P(A,B)[u(A)-u(B)-2]+1=0$$

If $u(A)=u(B)$ then we have an immediate solution $P(A,B)=1/2$. Otherwise, the solution to this quadratic equation is given by

$$P(A,B)=\frac{1}{2}+\frac{\sqrt{1+[u(A)-u(B)]^2/4}-1}{u(A)-u(B)}$$

Note that utility function $u(.)$ is unique up to addition of a constant. If we fix the third alternative to be C', then this corresponds to a different real-valued utility function:

$$u'(.)\equiv\frac{1}{1-P(.,C')}-\frac{1}{P(.,C')}=u(A)-u(C')$$

Reversely, if there is utility function $u:\Omega\rightarrow\mathbb{R}$ such that binary choice probability is given by (7) for any $A,B\in\Omega$, then a relatively straightforward algebra yields

$$\frac{1}{P(A,B)}=\frac{1}{P(B,A)}+u(B)-u(A)$$

$$\frac{1}{P(B,C)} = \frac{1}{P(C,B)} + u(C) - u(B)$$

$$\frac{1}{P(C,A)} = \frac{1}{P(A,C)} + u(A) - u(C)$$

Adding these three equations together and rearranging then yields (3). □

Proof of Proposition 3 Let us consider four choice alternatives $A,B,C,D \in \Omega$ such that $P(A,B) \geq P(C,D)$.

If condition (3) holds for $A,B,C \in \Omega$, then we have

$$\frac{1}{P(A,B)} + \frac{1}{P(B,C)} + \frac{1}{P(C,A)} = \frac{1}{P(A,C)} + \frac{1}{P(C,B)} + \frac{1}{P(B,A)}$$

If condition (3) holds for $B,C,D \in \Omega$, then we have

$$\frac{1}{P(D,B)} + \frac{1}{P(B,C)} + \frac{1}{P(C,D)} = \frac{1}{P(D,C)} + \frac{1}{P(C,B)} + \frac{1}{P(B,D)}$$

Subtracting one of these equalities from another then yields

$$\frac{1}{P(A,B)} - \frac{1}{P(C,D)} + \frac{1}{P(D,C)} - \frac{1}{P(B,A)} = \frac{1}{P(A,C)} - \frac{1}{P(B,D)} + \frac{1}{P(D,B)} - \frac{1}{P(C,A)}$$

If $P(A,B) \geq P(C,D)$, then $\frac{1}{P(A,B)} - \frac{1}{P(C,D)} \leq 0$. Moreover, if probabilistic completeness holds, then we also have $\frac{1}{P(D,C)} - \frac{1}{P(B,A)} \leq 0$. Therefore, we must have

$$\frac{1}{P(A,C)} - \frac{1}{P(B,D)} + \frac{1}{P(D,B)} - \frac{1}{P(C,A)} \leq 0$$

If probabilistic completeness holds, then this inequality can be rearranged as

$$\frac{1}{P(A,C)} - \frac{1}{1 - P(A,C)} \leq \frac{1}{P(B,D)} - \frac{1}{1 - P(B,D)}$$

Since function $1/x$-$1/(1$-$x)$ is strictly decreasing in x, we must then have $P(A,C) \geq P(B,D)$. □

Proof of Proposition 4 We first prove the sufficiency part. If the independence from irrelevant alternatives (6) holds then

$$O(A|\Omega) - O(B|\Omega) = O(A,B) - O(B,A)$$

Using probabilistic completeness, this equation can be rewritten as

$$\frac{1}{P(A|\Omega)} - \frac{1}{P(B|\Omega)} = \frac{1}{P(A,B)} - \frac{1}{1 - P(A,B)}$$

By proposition 1 and 2 there is utility function $u : \Omega \to \mathbb{R}$ such that the right-hand side of this equation is equal to $u(B) - u(A)$. Rearranging then yields

$$\frac{1}{P(A|\Omega)} + u(A) = \frac{1}{P(B|\Omega)} + u(B)$$

A similar argument implies that $\frac{1}{P(A|\Omega)} + u(A)$ is constant for any other choice alternative. Let us denote this constant by $x(\Omega)$. Then choice probability is given by $P(A|\Omega) = 1/[x(\Omega) - u(A)]$. Summing over all choice alternatives $A \in \Omega$ and using (4) then yields Eq. (8) that implicitly defines constant $x(\Omega)$. In general, Eq. (8) has n real roots but only the highest root is such that all choice probabilities are strictly positive.

For the "necessity" part, if alternative $A \in \Omega$ is chosen with probability $P(A|\Omega) = 1/[x(\Omega) - u(A)]$ then

$$\frac{1}{P(A|\Omega)} - \frac{1}{P(B|\Omega)} = u(B) - u(A) = \frac{1}{P(A,B)} - \frac{1}{1 - P(A,B)}$$

Using probabilistic completeness, the equality between the left-most and the right-most part can be rewritten as the independence from irrelevant alternatives (6). □

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Data availability Data and code are available in the online supplementary appendix.

Declarations

Conflict of interest The author declares that he has no relevant or material financial interests that relate to the research described in this paper.

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