Expected return—expected loss approach to optimal portfolio investment

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Abstract

Standard models of portfolio investment rely on various statistical measures of dispersion. Such measures favor returns smoothed over all states of the world and penalize abnormally low as well as abnormally high returns. A model of portfolio investment based on the tradeoff between expected return and expected loss considers only abnormally low returns as undesirable. Such a model has a comparative advantage over other existing models in that a first-order stochastically dominant portfolio always has a higher expected return and a lower expected loss. Expected return—expected loss model of portfolio investment can rationalize the equity premium puzzle. Two random variables are not comoving if there is at least one state of the world in which one random variable yields a positive return and the other—a negative return. Such random variables provide hedging benefits from diversification in portfolio investment according to the expected return—expected loss model. A special case of this model, when an investor linearly trade-offs expected returns and expected losses, is also a special case of the prospect theory when a decision-maker has a piece-wise linear value function without any probability weighting.

1 Introduction

Existing models of portfolio investment (Konno & Yamazaki, 1991; Markowitz, 1952; Yitzhaki, 1982) use statistical measures of dispersion (variance or standard deviation, mean absolute deviation, Gini (1912) mean difference) to capture the

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undesirable attribute (risk) of financial portfolios. Such statistical measures, however, count both unexpectedly low and unexpectedly high returns. Arguably, only the former are undesirable (cf. Roy, 1952) and investors may be even attracted to the latter. A model of portfolio investment based on the tradeoff between expected return and expected loss considers only returns below the reference point as undesirable. Such an approach can be backed by a large literature in behavioral economics finding evidence of loss aversion (Kahneman & Tversky, 1979). For example, willingness to accept often exceeds willingness to pay, a behavioral regularity known as the endowment effect (Kahneman et al., 1990; Thaler, 1980); decision-makers often prefer to retain the status quo (Knetsch, 1989; Samuelson & Zeckhauser, 1988); investors demand higher returns from stocks with downside risk (Ang et al., 2006); traders who experience losses in the morning take extra risks in the afternoon to recover (Coval & Shumway, 2005); investors are prone to the disposition effect (Odean, 1998; Shefrin & Statman, 1985) etc.

Markowitz (1952) mean–variance approach always violates the first-order stochastic dominance (Borch, 1969). Such violations are normatively unappealing and rarely observed in the data (Carbone & Hey, 1995; Loomes & Sugden, 1998, Table 2, p. 591; Hey, 2001, Table 2, p.14; see, however, Tversky & Kahneman, 1986, p. 264; Birnbaum & Navarrete, 1998, p. 61). A model of portfolio investment based on the tradeoff between expected return and expected loss has an important normative advantage over other models—a first-order stochastically dominant portfolio always has a higher expected return and a lower expected loss (Proposition 1 below).

Behavioral finance literature typically models aversion to negative returns by using the elements of Kahneman and Tversky (1979) prospect theory that aggregates positive and negative returns together, with "losses looming larger than gains". For example, Benartzi and Thaler (1995) combine loss aversion with mental accounting to rationalize the equity premium puzzle (Mehra & Prescott, 1985). Barberis et al. (2001) combine loss aversion with a sensitivity to prior outcomes to explain anomalies in asset prices. This paper models aversion to negative returns in the spirit of Markowitz (1952) mean–variance approach: investors tradeoff expected returns versus expected losses.

The remainder is organized as follows. Section 2 presents the expected return–expected loss model of portfolio investment. Section 3 compares this model with other models. Section 4 shows that this model can rationalize the equity premium puzzle. Section 5 concludes.

2 Expected return—expected loss model of portfolio investment

2.1 Set of feasible and efficient portfolios

There is a finite number of the states of the world $n \in N$. Only one state of the world is true *ex post* but an investor does not know *ex ante* which one. States of the world are numbered by subscripts $i \in \{1,\dots,n\}$. Notation $p(s_i)$ denotes the probability of the state of the world s_i, $i \in \{1,\dots,n\}$. There is a finite number of securities $m \in N$.

Securities are numbered by subscripts $j \in \{1,\ldots,m\}$. In particular, the return of security j in the state of the world s_i is denoted by $R_j(s_i)$. *Expected return* of security j is given by Eq. (1).

$$ER_j = \sum_{i=1}^{n} R_j(s_i)p(s_i) \tag{1}$$

Losses (*gains*) are returns below (above) the reference point. For expositional clarity, we consider the simplest case when the reference point is the status quo (zero return) so that losses are simply negative returns. A more general model with a non-zero reference point is qualitatively similar. *Expected loss* of security j is given by Eq. (2).

$$EL_j = - \sum_{\substack{i=1 \\ R_j(s_i)<0}}^{n} R_j(s_i)p(s_i) \tag{2}$$

Portfolio $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ yields return $\sum_{j=1}^{m} \alpha_j R_j(s_i)$ in the state of the world s_i, $\alpha_j \in [0,1]$ for all $j \in \{1,\ldots,m\}$ and $\sum_{j=1}^{m} \alpha_j = 1$. Expected return of any portfolio Eq. (3) is simply the weighted average of expected returns of individual securities in this portfolio.

$$ER(\boldsymbol{\alpha}) = \sum_{j=1}^{m} \alpha_j ER_j \tag{3}$$

Two securities are comoving if they yield losses in the same states of the world (which implies that they yield gains in the same states of the world as well). Note that this concept of comoving securities is different from the concept of (sign) comonotonic securities (cf. Wakker et al., 1994) that is used in rank-dependent utility (Quiggin, 1981, 1982) or cumulative prospect theory (Tversky & Kahneman, 1992). For example, two securities TSLA and NVDA[1] presented in Table 1 are comoving (both securities are in the red in the same states of the world) but not comonotonic (NVDA yields a higher return in state s_1 but TSLA yields a higher return in state s_2). If all securities are comoving then the expected loss of a portfolio is simply the weighted average of expected losses of individual comoving securities Eq. (4).

$$EL(\boldsymbol{\alpha}) = \sum_{j=1}^{m} \alpha_j EL_j \tag{4}$$

It is conventional to represent securities/portfolios in a two-dimensional diagram with their desirable attribute (expected return) plotted on the vertical axis and their undesirable attribute (expected loss)—on the horizontal axis. A set of portfolios that

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[1] These are monthly changes in close prices (adjusted for dividends and splits) on the first day of the month between June 2019 and May 2020 for Tesla, Inc. (TSLA) and NVIDIA Corporation (NVDA). Data downloaded from https://finance.yahoo.com/

State of the world	TSLA (%)	NVDA (%)
s_1	4.80	9.90
s_2	49.21	10.88
s_3	− 21.56	− 2.34
s_4	2.68	14.23
s_5	55.52	0.48
s_6	26.79	8.64
s_7	4.77	7.82
s_8	30.74	15.48
s_9	6.76	4.02
s_{10}	− 6.62	− 0.72
s_{11}	8.12	2.73
s_{12}	20.68	21.38

Table 1 An example of two comoving securities

can be constructed with two comoving securities is then represented by a straight line connecting these two securities. For example, Fig. 1 shows the set of all feasible portfolios that can be constructed with securities TSLA and NVDA presented in Table 1 (on the assumption that all states of the world are equally likely).

Let us now consider two securities that are not comoving. In other words, there is at least one state of the world s_i in which there is no comovement *i.e.* $R_1(s_i)R_2(s_i) < 0$. For any such state, we calculate a share $\alpha_1(s_i)$ of the first security such that the expected return of the binary portfolio is equal to the reference point (zero return) in state s_i.[2]

$$\alpha_1(s_i) = \frac{R_2(s_i)}{R_2(s_i) - R_1(s_i)} \tag{5}$$

We plot binary portfolios corresponding to all such "break even" shares as well as $\alpha_1 = 0$ and $\alpha_1 = 1$ on the expected return—expected loss plane. Finally, we connect them by straight lines. The obtained piece-wise linear curve represents the set of all feasible portfolios that can be constructed with two securities that are not comoving.[3]

For example, two securities WMT and AAPL[4] presented in Table 2 are not comoving in states s_3 and s_{10} (AAPL is in the red but not WMT) and s_5, s_6, s_8 and s_{11} (WMT is in the red but not AAPL). The last column of Table 2 shows the implied "break even" shares Eq. (5) of WMT for these six states of the world.

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[2] In any state of the world s_i in which there is comovement (i.e. $R_1(s_i)R_2(s_i) \geq 0$) the ratio (5) is not a well-defined share (being either negative or greater than one).

[3] The case with comoving securities that we discussed first is just a special case of this algorithm when there are no "break even" shares.

[4] These are monthly changes in close prices (adjusted for dividends and splits) on the first day of the month between June 2019 and May 2020 for Walmart Inc. (WMT) and Apple Inc. (AAPL). Data downloaded from https://finance.yahoo.com/
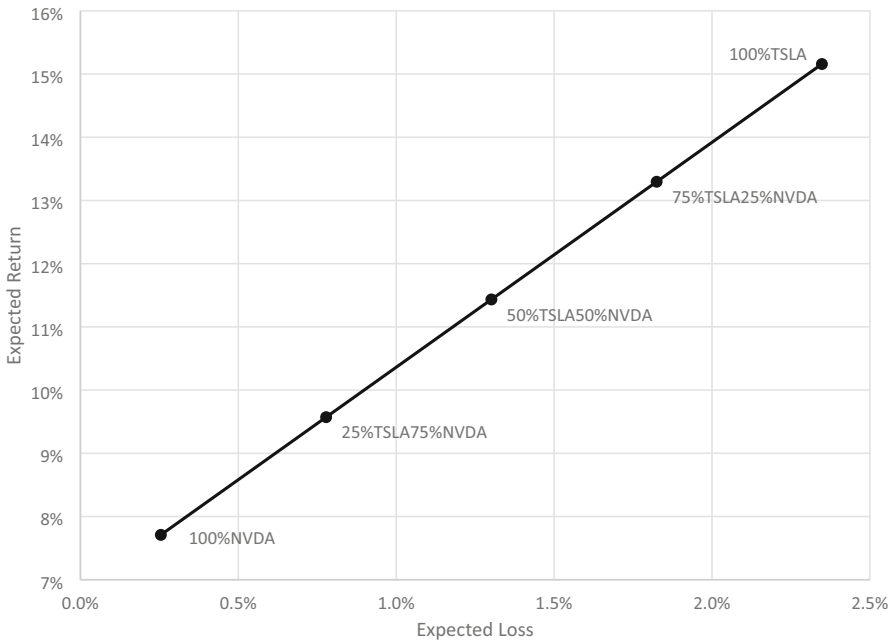
Fig. 1 Feasible portfolios that can be constructed with two securities in Table 1

Table 2 An example of two securities that are not commoving

State of the world	WMT (%)	AAPL (%)	"Break even" share α_1 of WMT
s_1	1.54	5.35	
s_2	7.46	15.54	
s_3	5.51	− 6.76	0.551
s_4	− 5.95	− 11.68	
s_5	− 3.22	5.40	0.626
s_6	− 0.21	10.21	0.980
s_7	1.56	7.43	
s_8	− 1.20	11.07	0.903
s_9	4.38	7.70	
s_{10}	3.52	− 2.02	0.365
s_{11}	− 0.10	7.64	0.987
s_{12}	9.50	13.49	

Figure 2 presents six binary portfolios constructed with these "break even" shares on the expected return—expected loss plane (on the assumption that all states of the world are equally likely). A piece-wise linear curve that connects these six portfolios (as well as two degenerate portfolios with 100% of WMT and AAPL)
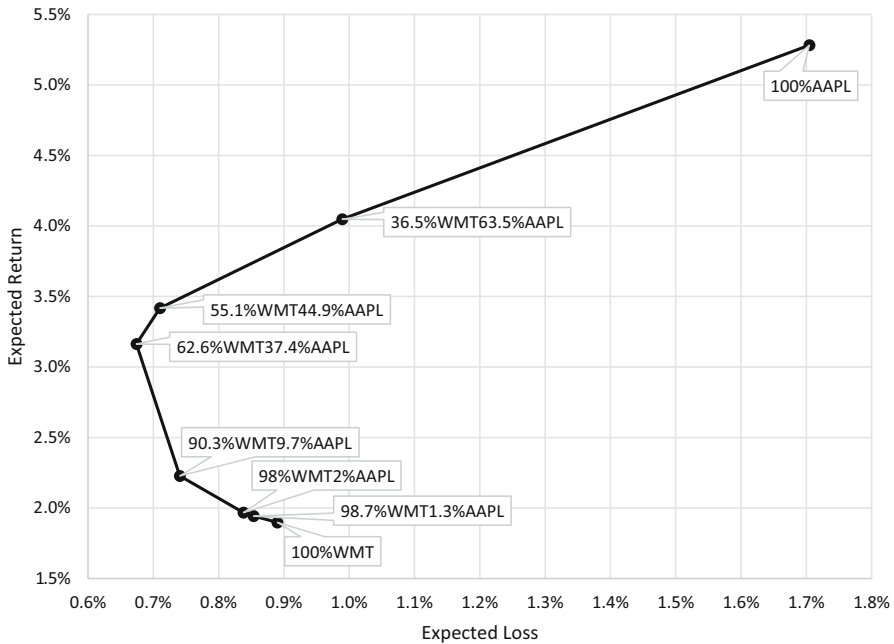
Fig. 2 Feasible portfolios that can be constructed with two securities in Table 2

represents all feasible portfolios that can be constructed with shares of WMT and AAPL.

We assume that an investor prefers portfolios with a higher expected return and a lower expected loss. Thus, a set of efficient portfolios is a subset of feasible portfolios for which there is no other feasible portfolio with a higher expected return and/or lower expected loss. For example, in Fig. 1 the set of efficient portfolios coincides with the set of feasible portfolios. On the other hand, in Fig. 2, the set of efficient portfolios is a subset of feasible portfolios with WMT share being at most 62.6%.

The algorithm of constructing the set of efficient portfolios over two securities is then extended, by iteration, to efficient portfolios over any finite number of securities (comoving or not). For example, Fig. 3 plots the set of feasible portfolios that can be constructed with three securities: TSLA from Table 1, WMT and AAPL from Table 2. This set is bounded by three piece-wise linear curves: (1) a solid curve showing all binary portfolios feasible with only two securities WMT and AAPL (the same as presented in Fig. 2); (2) a dashed curve showing all binary portfolios feasible with WMT and TSLA; and (3) a dashed-dotted curve showing all binary portfolios feasible with AAPL and TSLA.[5] Figure 3 shows that the set of efficient portfolios that can be constructed with three securities TSLA, WMT and AAPL is the same as the set of efficient portfolios that can be constructed with only two

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[5] The dashed-dotted curve on Fig. 3 has only one kink corresponding to the "break even" share of AAPL $\alpha_1 = 0.187$ because AAPL and TSLA are comoving in all but one state (s_4).
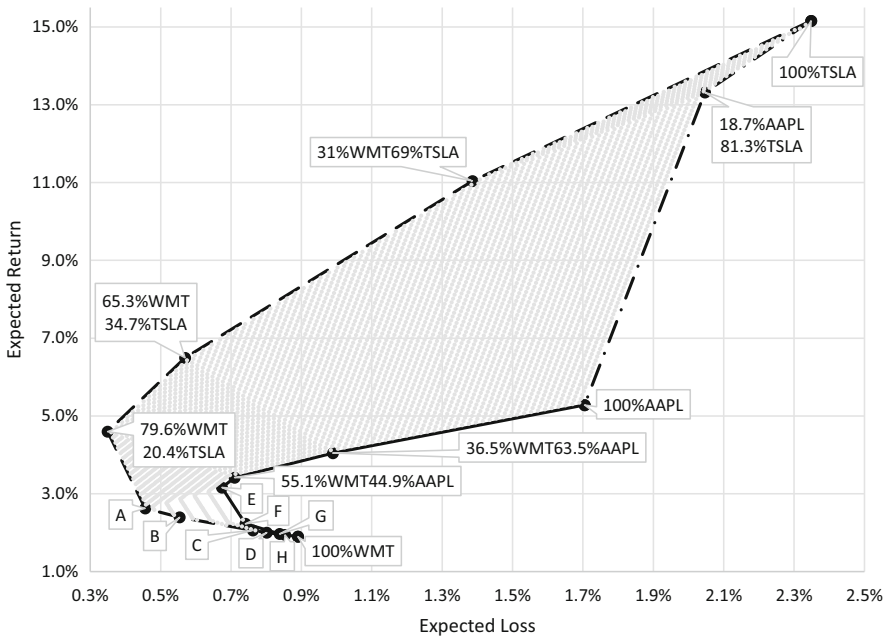
Fig. 3 Feasible portfolios that can be constructed with three securities WMT, AAPL and TSLA from Tables 1, 2 (portfolio **A** is 94.5%WMT and 5.5%TSLA; portfolio **B** is 96.3%WMT and 3.7%TSLA; portfolio **C** is 98.8%WMT and 1.2%TSLA; portfolio **D** is 99.2%WMT and 0.8%TSLA; portfolio **E** is 62.6%WMT and 37.4%AAPL; portfolio **F** is 90.3%WMT and 9.7%AAPL; portfolio **G** is 98%WMT and 2%AAPL; and portfolio **H** is 98.7%WMT and 1.3%AAPL)

securities TSLA and WMT (a section of the dashed curve with WMT share being at most 79.6%).

In the example presented in Fig. 3, three piece-wise linear curves corresponding to feasible binary portfolios do not intersect (they only meet at points representing degenerate portfolios with 100% share of one security). In this case, the set of efficient portfolios contains only binary portfolios (i.e. one security is never used for diversification). Let us now consider a more complex example when the set of efficient portfolios contains fully diversified portfolios (with positive shares of all available securities). We consider securities WMT and AAPL from Table 2 (that are also presented in the second and fifth columns of Table 3) and security REGN[6] presented in the third/sixth column of Table 3. The fourth (seventh) column of Table 3 presents "break even" shares Eq. (5) of WMT (AAPL) in a binary portfolio with REGN.

A solid piece-wise linear curve in Fig. 4 shows all binary portfolios feasible with WMT and AAPL (the same as presented in Fig. 2). A dashed (dashed-dotted) piece-wise linear curve in Fig. 4 shows all binary portfolios feasible with WMT (APPL)

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[6] These are monthly changes in adjusted close prices between June 2019 and May 2020 for Regeneron Pharmaceuticals, Inc. (REGN). Data downloaded from https://finance.yahoo.com/.

Table 3 "Break even" probabilities for binary portfolios WMT-REGN and AAPL-REGN

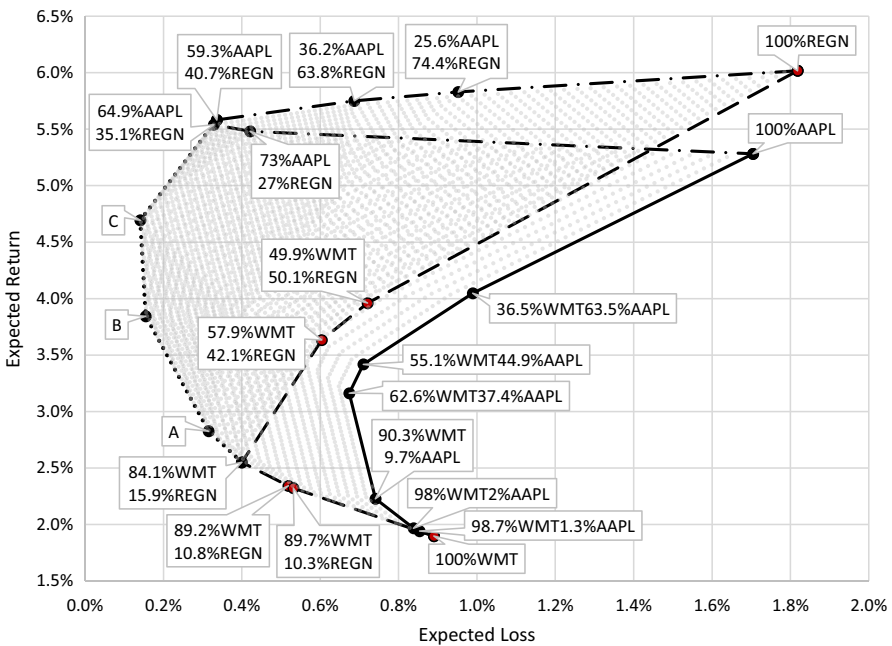| State of the world	WMT (%)	REGN (%)	"Break even" share of WMT	AAPL (%)	REGN (%)	"Break even" share of AAPL
s_1	1.54	8.55		5.35	8.55	
s_2	7.46	7.70		15.54	7.70	
s_3	5.51	9.83		− 6.76	9.83	0.593
s_4	− 5.95	31.55	0.841	− 11.68	31.55	0.730
s_5	− 3.22	− 10.00		5.40	− 10.00	0.649
s_6	− 0.21	1.76	0.892	10.21	1.76	
s_7	1.56	20.48		7.43	20.48	
s_8	− 1.20	10.41	0.897	11.07	10.41	
s_9	4.38	− 4.36	0.499	7.70	− 4.36	0.362
s_{10}	3.52	− 4.83	0.579	− 2.02	− 4.83	
s_{11}	− 0.10	− 2.63		7.64	− 2.63	0.256
s_{12}	9.50	3.74		13.49	3.74	

Fig. 4 Feasible portfolios that can be constructed with three securities WMT, AAPL and REGN (portfolio A is 76.3%WMT, 6.8%AAPL and 16.9%REGN; portfolio B is 47% WMT, 32.2%AAPL and 20.8%REGN; portfolio C is 22.6%WMT, 53.4%AAPL and 24%REGN)

and REGN that is constructed using five "break even" shares from the fourth (seventh) column of Table 3.

The dashed and dashed-dotted curves in Fig. 4 intersect (unlike those in Fig. 3) so that the set of feasible portfolios is bounded not only by solid, dashed and dashed-dotted curves (corresponding to feasible binary portfolios). To construct the set of feasible portfolios we proceed as follows. Table 3 shows that there are eight states of the world where the returns of three available securities are not comoving (s_3–s_6 and s_8–s_{11}). We calculate a share $\alpha_1(s_i, s_j)$ of the first security and a share $\alpha_2(s_i, s_j)$ of the second security such that the expected return of a diversified portfolio over three securities is equal to the reference point (zero return) in two distinct states s_i and s_j where the returns of three available securities are not comoving (*i.e.* $i,j \in \{3$–$6,8$–$11\}$, $i \neq j$). This amounts to solving a system of linear Eq. (6).

$$\begin{cases} \alpha_1(s_i, s_j)R_1(s_i) + \alpha_2(s_i, s_j)R_2(s_i) + \left[1 - \alpha_1(s_i, s_j) - \alpha_2(s_i, s_j)\right]R_3(s_i) = 0 \\ \alpha_1(s_i, s_j)R_1(s_j) + \alpha_2(s_i, s_j)R_2(s_j) + \left[1 - \alpha_1(s_i, s_j) - \alpha_2(s_i, s_j)\right]R_3(s_j) = 0 \end{cases} \quad (6)$$

such that $\alpha_1(s_i, s_j), \alpha_2(s_i, s_j) \in [0, 1]$, $\alpha_1(s_i,) + \alpha_2(s_i, s_j) \leq 1$, $i \neq j$, $\min\{R_1(s_i), R_2(s_i), R_3(s_i)\} < 0 < \max\{R_1(s_i), R_2(s_i), R_3(s_i)\}$,

$\min\{R_1(s_j), R_2(s_j), R_3(s_j)\} < 0 < \max\{R_1(s_j), R_2(s_j), R_3(s_j)\}$.

The solution $\alpha_1(s_4, s_{11}) = 0.763$ and $\alpha_2(s_4, s_{11}) = 0.068$ to system Eq. (6) is represented as point A in Fig. 4; the solution $\alpha_1(s_4, s_{10}) = 0.47$ and $\alpha_2(s_4, s_{10}) = 0.322$ to system Eq. (6) is represented as point B in Fig. 4; and the solution $\alpha_1(s_3, s_4) = 0.226$ and $\alpha_2(s_3, s_4) = 0.534$ to system Eq. (6) is represented as point C in Fig. 4.[7] Points A, B and C as well as the two nearest binary portfolios (84.1%WMT-15.9%REGN and 59.3%AAPL-40.7%REGN) that form a convex hull with these points are connected by a piece-wise linear dotted curve in Fig. 4. The set of all feasible portfolios is then bounded by this piece-wise linear dotted curve as well as solid, dashed and dashed-dotted curves. The set of efficient portfolios is then the set of feasible binary portfolios AAPL-REGN with AAPL share being at most 59.3% plus the set of (compound/conglomerate) portfolios that can be constructed by mixing a binary portfolio 59.3%AAPL-40.7%REGN with portfolio C (22.6%WMT, 53.4%AAPL and 24%REGN).

2.2 Investor's indifference curves

As the next step, we specify investor's preferences to determine the most preferred efficient portfolio. In general, investor's preferences are represented by the utility function $U(ER(\boldsymbol{\alpha}), EL(\boldsymbol{\alpha}))$ that is increasing in the first argument and decreasing in the second argument. A major theoretical advantage of the expected return— expected loss model of portfolio investment over other models proposed in the literature is that utility function $U(ER(\boldsymbol{\alpha}), EL(\boldsymbol{\alpha}))$ does not violate the first-order stochastic dominance (cf. Proposition 1 below). In contrast, the influential Markowitz (1952) mean–variance approach always violates the first-order stochastic

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[7] In our example, system (6) has nine other solutions that are located inside the convex hull of feasible binary portfolios and hence they cannot be fully diversified boundary portfolios.

dominance (Borch, 1969). The mean-absolute deviation approach (Blavatskyy, 2010; Konno & Yamazaki, 1991) respects the first-order stochastic dominance only when indifference curves are not too steep (with a slope less than 0.5) in the expected return—absolute deviation plane. A similar restriction also applies to the mean-Gini approach (Shalit & Yitzhaki, 1984; Yitzhaki, 1982) that employs Gini (1912) mean absolute difference statistic for measuring statistical dispersion of assets' returns.[8]

Proposition 1 *If portfolio α first-order stochastically dominates portfolio β then $ER(\alpha) \geq ER(\beta)$ and $EL(\alpha) \leq EL(\beta)$.*

The proof is presented in the appendix.

For practical applications a useful parametric form is a quasi-linear utility function $U(ER(\alpha), EL(\alpha)) = ER(\alpha) - a*EL(\alpha)^b$, where $a, b \geq 0$ are constant. When $a = 0$, an investor does not care about expected losses (does not diversify) and picks a portfolio with the highest expected return as manifested by horizontal indifference curves. When a is infinitely large, an investor does not care about expected returns and picks a portfolio with the lowest expected loss (extreme loss aversion) as manifested by vertical indifference curves. Parameter b captures investor's sensitivity to expected losses: if $b > 1$ (< 1) the investor becomes more (less) averse to larger expected losses. Given the set of efficient portfolios and indifference curves representing investor's preferences, the optimal (most preferred) portfolio is an efficient portfolio that is located on the highest indifference curve. Figure 5 illustrates an optimal portfolio (59.3% AAPL and 40.7% REGN) for an example with three securities WMT, AAPL and REGN (presented in Table 3) when investor's preferences are such that expected losses are twice as undesirable as expected gains. Since the set of efficient portfolios is a piece-wise linear curve, "sharp" kink points on this curve are likely to be optimal portfolios for a wide range of preferences.

3 Relationship to other models of portfolio investment

The expected return—expected loss model of optimal portfolio investment is closest to the mean-absolute deviation approach (Blavatskyy, 2010; Konno & Yamazaki, 1991). In the expected return—expected loss model the reference point is constant (the same for all securities). In the mean-absolute deviation approach, the reference point is context-dependent (different for different securities). Specifically, the reference point of security j is the expected return ER_j of this security.[9] The expected loss of a security j is then equal to the mean absolute semideviation of this security (7).

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[8] Mean-Gini approach is equivalent to Yaari (1987) dual model with a quadratic probability weighting function (Blavatskyy, 2016).

[9] This is similar to Gul (1991) theory of disappointment aversion where outcomes are disappointing if they are below a lottery's certainty equivalent.
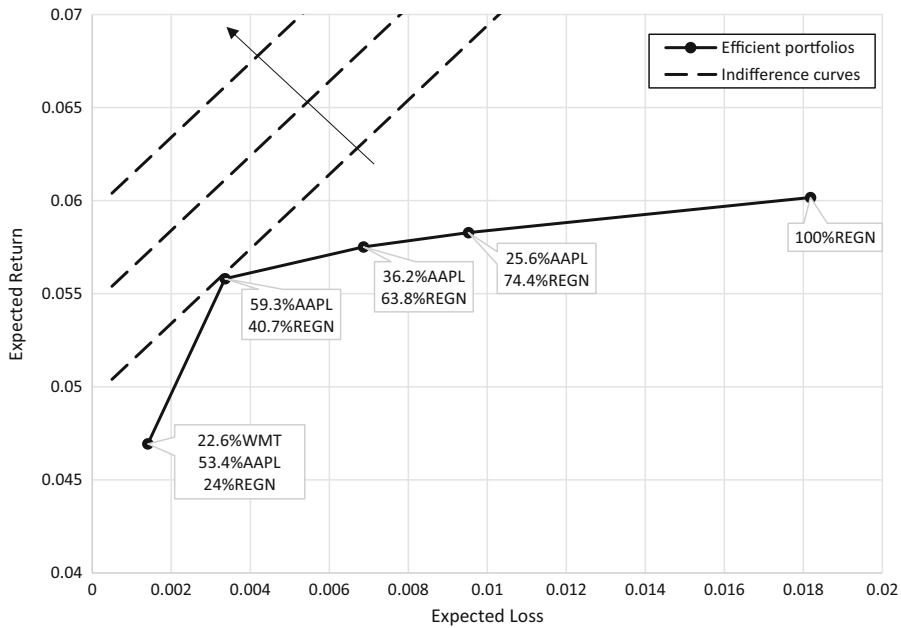
Fig. 5 Optimal portfolio with three securities WMT, AAPL and REGN for utility function $U(ER(\alpha),EL(\alpha)) = ER(\alpha)-2EL(\alpha)$ represented by dashed indifference curves

$$EL_j = \sum_{\substack{i=1 \\ R_j(s_i)<ER_j}}^{n} \left[ER_j - R_j(s_i)\right]p(s_i) = \frac{1}{2}\sum_{i=1}^{n}\left|R_j(s_i) - ER_j\right|p(s_i) \tag{7}$$

To understand the relationship between the mean-Gini approach (Shalit & Yitzhaki, 1984; Yitzhaki, 1982) and the expected return—expected loss model we need to assume that the reference point is both context-dependent and state-dependent (i.e. the reference point is stochastic). Specifically, the reference point RP_{ij} of the j-th security in the i-th state of the world is the expected return of this security in all states that bring a higher return than j-th return in state i: $G_{ij} \equiv \left\{k \in \{1,\ldots,n\}|R_j(s_k) > R_j(s_i)\right\}$.

$$RP_{ij} = \sum_{k \in G_{ij}} R_j(s_k)\frac{p(s_k)}{\sum_{h \in G_{ij}}p(s_h)} \tag{8}$$

The expected loss of the j-th security is then equal to one-half of Gini (1912) mean absolute difference statistic Eq. (9) with mean differences weighted by the inverse of the j-th decumulative distribution function in the state with a lower return.

$$EL_j = \sum_{i=1}^{n} \left[RP_{ij} - R_j(s_i) \right] p(s_i)$$

$$= \sum_{i=1}^{n} \sum_{k \in S_{ij}} \frac{\left[R_j(s_k) - R_j(s_i) \right]}{\sum_{h \in G_{ij}} p(s_h)} p(s_i) p(s_k) \qquad (9)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\left| R_j(s_k) - R_j(s_i) \right|}{\sum_{h \in G_{\min\{i,k\}j}} p(s_h)} p(s_i) p(s_k)$$

The expected return—expected loss model is overlapping with Kahneman and Tversky (1979) original prospect theory as well as Tversky and Kahneman (1992) cumulative prospect theory. The special case when an investor tradeoffs the expected return vs. the expected loss in a linear manner, i.e. $U(ER(\alpha), EL(\alpha)) = ER(\alpha) - a*EL(\alpha)$, corresponds to the special case of both versions of the prospect theory with a piece-wise linear value function with a loss aversion coefficient $\lambda = 1 + a$ and without non-linear probability weighting.

There is no obvious mathematical relationship between the expected return—expected loss model and Markowitz (1952) mean–variance approach. The relationship between these two models can be illustrated with the example of three securities WMT, AAPL and REGN. The solid line in Fig. 6 shows the set of efficient portfolios that can be constructed with these three securities according to the expected return—expected loss model (the same as in Figs. 4 and 5). The set of efficient portfolios according to Markowitz (1952) mean–variance approach is the
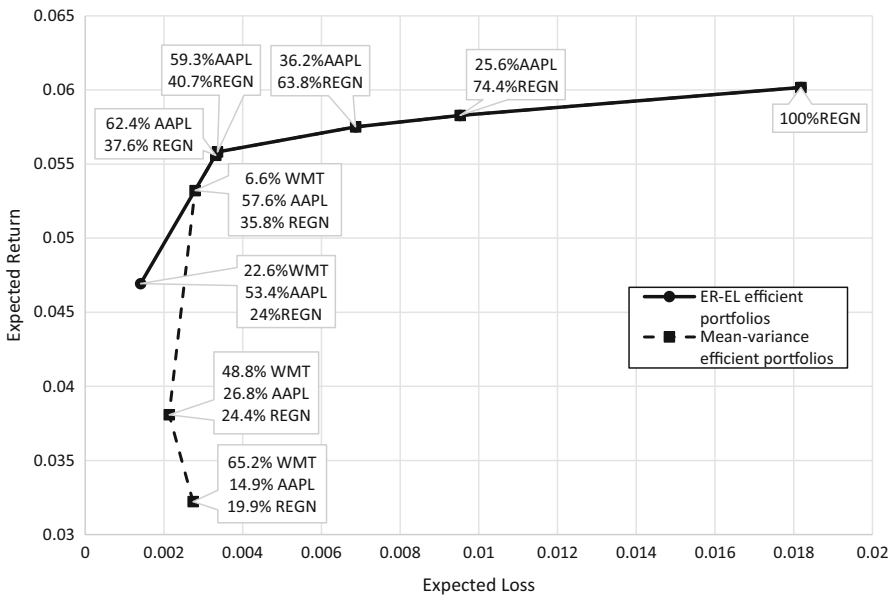

Fig. 6 Efficient portfolios with three securities WMT, AAPL and REGN according to the expected return—expected loss model and mean variance approach

set of binary portfolios AAPL-REGN with AAPL share being at most 62.4% plus the set of (compound/conglomerate) portfolios that can be constructed by mixing a binary portfolio 62.4%AAPL-37.6%REGN with a fully diversified portfolio 65.2%WMT, 14.9%AAPL and 19.9%REGN. This set is represented by a dashed line in Fig. 6.

The set of efficient portfolios according to the expected return—expected loss model (a solid line in Fig. 6) overlaps with that according to Markowitz (1952) mean–variance approach (a dashed line in Fig. 6) when efficient portfolios have a relatively high expected return and a relatively high expected loss. In this case, according to both models, efficient portfolios are binary portfolios AAPL-REGN with AAPL share being at most 59.3%. Thus, if investors prefer stocks with high returns and are relatively less averse to expected losses (or variance of returns), the optimal portfolio is likely to be the same according to both models.

If investors are sensitive to expected losses, the optimal portfolio differs in two models. Optimal portfolio according to the mean–variance approach generally has a lower expected return and higher expected losses compared to the optimal portfolio according to our model. This happens because a portfolio with low expected losses may have abnormally high returns, which makes it too "volatile" according to the mean–variance approach. The latter favors portfolios with returns smoothed over all states of the world (at the cost of lower returns).

4 The equity premium puzzle

The equity premium puzzle (Mehra & Prescott, 1985) refers to an empirical finding that an unreasonably high degree of risk aversion (under expected utility theory) is required to rationalize a diversified portfolio that includes a stock market index and relatively low-interest governmental bonds. To illustrate the equity premium puzzle, the second column of Table 4 shows yields on a 13 week US treasury bill between June 1st, 2019 and May 1st, 2020 and the third column of Table 4 shows monthly changes in adjusted close prices of NASDAQ Composite Index for the same time period.[10] Let us assume that each of the twelve months listed in Table 4 is an equally likely state of the world. We ignore the possibility of different taxation rates for treasury bills and traded stocks. Under these assumptions, an investor maximizing expected utility with a constant relative risk aversion utility function

$$u(x) = \frac{x^{1-r}}{1-r} \tag{10}$$

holds both a 13 week US treasury bill and NASDAQ Composite Index in her portfolio if her coefficient of relative risk aversion is $r > 2.41$. For example, an investor with a coefficient of relative risk aversion $r = 2.5$ holds a diversified portfolio 4% T-bills and 96% NASDAQ index. A decision-maker with $r = 2.5$ sells a 50–50% chance to receive either $100 or $1 for only $1.58, which might be an unreasonably low certainty equivalent. Indeed, many empirical studies find lower

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[10] Data (^IRX and ^IXIC) downloaded from https://finance.yahoo.com/.

Table 4 Yields on a 13 week US treasury bill and monthly changes in NASDAQ index

Date	Yield on a 13 week Treasury bill (%)	NASDAQ (%)	"Break even" share α_1 of T-bill
May 01 2020	0.128	6.753	
Apr 01 2020	0.090	15.447	
Mar 01 2020	0.030	− 10.123	0.997
Feb 01 2020	1.230	− 6.377	0.838
Jan 01 2020	1.510	1.988	
Dec 01 2019	1.505	3.544	
Nov 01 2019	1.540	4.499	
Oct 01 2019	1.498	3.663	
Sep 01 2019	1.770	0.458	
Aug 01 2019	1.933	− 2.600	0.574
Jul 01 2019	2.030	2.113	
Jun 01 2019	2.035	7.421	

coefficients of relative risk aversion. For instance, Blavatskyy and Pogrebna (2010, Table III, p.973) estimate coefficients of relative risk aversion to be between − 0.12 and 0.45 (depending on the econometric model of probabilistic choice) in a natural experiment with Italian contestants standing to win up to €500000. de Roos & Sarafidis (2010, Table XIV) estimate coefficients of relative risk aversion to be between 0.46 and 0.65 (depending on econometric model) for Australian contestants standing to win up to AUD 200000.

To rationalize even a small share of treasury bonds in a diversified portfolio that includes a stock market index, an expected utility maximizer must exhibit the levels of risk aversion that are descriptively unjustifiable. In contrast, the expected return—expected loss model of portfolio investment can rationalize the equity premium puzzle without unrealistic assumptions about investor's preferences. The last column of Table 4 lists "break even" shares of treasury bills needed for constructing the set of efficient binary portfolios over a 13 week US Treasury bill and NASDAQ Composite Index. This set of efficient binary portfolios is shown as a solid piece-wise linear curve in Fig. 7. Note that binary portfolios with 100% or 99.7% share of treasury bills have no negative returns, i.e. their expected loss is zero.

Figure 7 shows that the optimal binary portfolio includes 57.4% of a 13 week US treasury bill when the slope of the investor's indifference curve touching the solid line in the expected return—expected loss plane is between 0.52 and 0.64. As discussed in the previous section, this corresponds to a coefficient of loss aversion between 1.52 and 1.64 in a version of prospect theory without non-linear probability weighting. Coefficients of loss aversion greater than two are not uncommon in the empirical literature (e.g., Tversky & Kahneman, 1992). Hence, we can conclude that a relatively modest aversion to expected losses is sufficient for rationalizing a
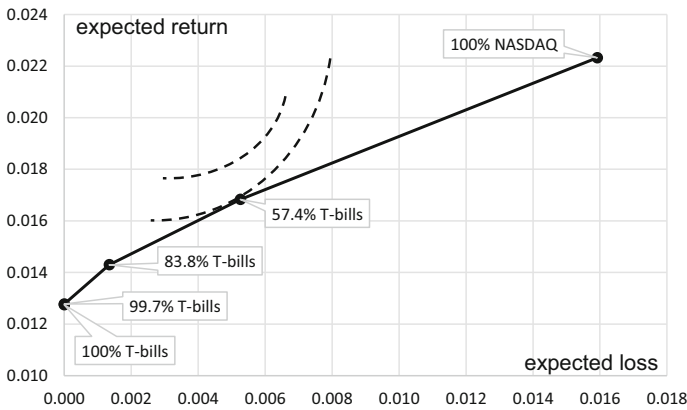
Fig. 7 Set of efficient binary portfolios of a 13 week US treasury bill and NASDAQ Composite Index (solid line) and indifference curves (dashed line) rationalizing the optimal portfolio with 57.4% share of treasury bills

relatively large share of treasury bonds in a diversified portfolio together with a stock market index.

5 Conclusion

The problem of optimal portfolio investment amounts to finding the most desirable convex combination of several random variables. Often regarded as a normative benchmark, expected utility theory (von Neumann & Morgenstern, 1947) employs a non-linear Bernoulli utility function. Thus, in general, under expected utility theory there is no closed-form solution for an optimal convex combination of several random variables. This limitation carries over to many well-known generalizations of expected utility theory such as rank-dependent utility (Quiggin, 1981, 1982) or cumulative prospect theory (Tversky & Kahneman, 1992). The possibility to characterize a closed-form solution for an optimal convex combination is valued in practical applications in finance. Hence, several models of optimal portfolio investment were proposed with a linear utility function, e.g. Markowitz (1952) mean–variance approach. Such models produce a closed-form solution for an optimal convex combination but lack a solid preference foundation. For example, Markowitz (1952) mean–variance approach may lead to violations of the first-order stochastic dominance. This state of the literature demands for a model of optimal portfolio investment that, on the one hand, is based on intuitive micro-economic preferences (*e.g.* respects the first-order stochastic dominance) and, on the other hand, produces a closed-form solution for an optimal convex combination (which essentially boils down to avoiding any non-linear transformations of state-contingent payoffs). The main contribution of this paper is to advance one such model.

The idea that investors weight expected returns vis-à-vis expected losses is intuitively appealing. Such a model also has normatively attractive properties. For

example, it does not lead to violations of the first-order stochastic dominance. On the descriptive side, the model can be supported by a large behavioral literature finding evidence of loss aversion. To construct a set of efficient portfolios in the expected return—expected loss plane we only need to solve (a system of) linear equations. Thus, a closed-form solution for an optimal convex combination under this model is even simpler compared to the classic mean–variance approach (which requires solving a quadratic optimization problem).

An important concept in the expected return—expected loss model is that of comoving random variables. Two random variables are comoving if there is no state of the world in which one random variable yields a positive return and the other—a negative return. Comoving random variables do not provide any additional benefit from diversification. The expected return (loss) of a convex combination of comoving random variables is equal to the weighted average of the expected returns (losses) of individual comoving random variables. In contrast, random variables that are not comoving provide additional benefits from diversification. The expected loss of a convex combination of such variables is less than the corresponding weighted average of the expected losses of individual random variables. In other words, securities that are not comoving provide hedging benefits in portfolio investment.

The efficiency frontier under expected return—expected loss model is qualitatively similar to that under the mean–variance approach. The mean-absolute deviation approach (Konno & Yamazaki, 1991) can be viewed as the expected return—expected loss model with an endogenous reference point that is equal to the expected return of a security. The mean-Gini approach (Shalit & Yitzhaki, 1984; Yitzhaki, 1982) can be viewed as the expected return—expected loss model with an endogenous state-dependent reference point. Finally, the expected return—expected loss model is overlapping with prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992). A special case of the former, when an investor linearly trade-offs expected returns and expected losses, is also a special case of the latter, when a decision-maker has a piece-wise linear value function without any probability weighting.

One possible criticism of the expected return—expected loss model is that the model predicts no diversification in a bear market when all securities yield negative returns (since the security with the smallest loss is then also the security with the highest expected return). Yet, in such a relatively rare stock market where all securities yield negative returns, the assumption that investors maintain zero return as their reference point is not realistic. On such a bear market the investors would lose money anyway and they might be concerned with avoiding larger losses so that their reference point is a negative return. With a negative reference point, the expected return—expected loss model predicts diversification on a bear market when all securities yield negative returns.

Appendix

Proof of **proposition 1**

For any portfolio $\boldsymbol{\alpha}$ we denote the return of this portfolio in state of the world s_i as

$$R_i(\boldsymbol{\alpha}) \overset{\text{def}}{=} \sum_{j=1}^{m} \alpha_j R_j(s_i)$$

For any two portfolios $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ let X denote the set of their possible returns:

$$X \overset{\text{def}}{=} \bigcup_{i=1}^{n} \{R_i(\boldsymbol{\alpha}), R_i(\boldsymbol{\beta})\}$$

Let N be the number of elements in set X and let us number those elements by subscript k in the ascending order so that x_1 is the minimum element and x_N is the maximum element in X. Finally. We denote the cumulative distribution function of portfolio $\boldsymbol{\alpha}$ as

$$F_k(\alpha) \overset{\text{def}}{=} \sum_{\substack{i=1 \\ R_i(\alpha) \le x_k}}^{n} p(s_i)$$

If portfolio $\boldsymbol{\alpha}$ first-order stochastically dominates portfolio $\boldsymbol{\beta}$ then $F_k(\boldsymbol{\alpha}) \le F_k(\boldsymbol{\beta})$ for all $x_k \in X$.

Expected return (3) of portfolio $\boldsymbol{\alpha}$ can be then written as

$$ER(\boldsymbol{\alpha}) = \sum_{j=1}^{m} \alpha_j ER_j = \sum_{i=1}^{n} R_i(\boldsymbol{\alpha}) p(s_i)$$

$$= x_N + \sum_{k=1}^{N-1} F_k(\boldsymbol{\alpha})[x_k - x_{k+1}] \ge x_N$$

$$+ \sum_{k=1}^{N-1} F_k(\boldsymbol{\beta})[x_k - x_{k+1}] = ER(\boldsymbol{\beta})$$

Let X_- denote the set of all losses (negative returns) in X and zero:

$$X_- \overset{\text{def}}{=} \{x \in X | x < 0\} \cup \{0\}$$

Let L be the number of elements in set X_-. As before. We keep the numbering in the ascending order so that x_1 is the minimum element (the biggest loss) in X_- and $x_L = 0$. Expected loss of portfolio $\boldsymbol{\alpha}$ can be then written as

$$EL(\alpha) = - \sum_{\substack{i=1 \\ R_i(\alpha) < 0}}^{n} R_i(\alpha) p(s_i) = \sum_{k=1}^{L-1} F_k(\alpha)[x_{k+1} - x_k] \le \sum_{k=1}^{L-1} F_k(\beta)[x_{k+1} - x_k] = ER(\beta)$$

Q.E.D.

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