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Put–call parity and generalized neo-additive pricing rules

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Abstract

We study price formulas suited for empirical research in financial markets in which put–call parity is satisfied. We find a connection between risk and the bid–ask spread. We further study the compatibility of the model with market frictions, and determine market subsets where the Fundamental Theorem of Asset Pricing applies. Finally, we characterize the price formula.

Keywords Choquet pricing · Fundamental Theorem of Asset Pricing · market frictions · Neo-additive capacity · Put–call parity

1 Introduction

Cerreia-Vioglio et al. (2015) (CMM for brevity) generalized the Fundamental Theorem of Asset Pricing (FTAP) (see Ross 1976; Harrison and Kreps 1979) to financial markets with frictions in which put–call parity is satisfied. They developed an explicit asset-pricing formula, that states, the price must equal the Choquet expectation of the asset payoff with respect to a so-called 'risk-neutral capacity'. We contribute to this discussion by further studying the relationship between the Choquet expectation and asset pricing. We study a special case of CMM financial markets when the set of prices is given by a generalized neo-additive capacity (GNAC). These capacities were developed by Chateauneuf et al. (2007) and Eichberger et al. (2012), resulting in streamlined price formulas with only two parameters and a probability. These limited parameters—the price of an asset is a weighted sum of the expected value (a frictionless price), and the maximal and minimal revenues—make the results simpler to understand and easier to calibrate, and to estimate (see the end of Sect. 3) than a Choquet expectation. The expected

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value parameter measures the effects of friction on pricing. The revenue parameters permit a natural interpretation when they are between 0 and 1, that is, when prices are given by a neo-additive capacity (NAC) but this additional requirement is compelling because it constraints the value of the bid–ask spread (viz. the difference between the price for an immediate purchase and the price for an immediate sell). We provide more insights into interpreting these parameters in both frameworks in Sect. 5.

When prices are given by a GNAC, there is a theoretical connection between asset prices and risk. We show that the bid–ask spread is proportional to the range of an asset's revenues. This is consistent with empirical evidence suggesting that bid–ask spreads vary linearly with risk (see Benston and Hagerman 1974; Stoll 1978, 1985; Amihud and Mendelson 1986), the range being a simple (albeit imperfect) measure of risk. Stoll (1978) and Amihud and Mendelson (1986) showed that this relationship is positive—i.e. the higher the risk, the broader the spread. This relationship can be understood naturally when prices are given by a GNAC: it means that there is no arbitrage opportunity in the spread. In terms of pricing, this is equivalent to placing a higher emphasis on maximal revenues than minimal revenues. In Sect. 6, we analyze the compatibility of a general-capacity price formula with a price formula given by a GNAC with frictions. A general capacity can be represented by the *Weber set*[1] of probabilities. We show that there is no friction among a subset of assets if, and only if, the Weber set probabilities coincide for a specific set of events. We conclude that, even when prices are represented by a general capacity, there might exist subsets of assets without frictions. Put in another way, the FTAP applies only over specific parts of a financial market when put–call parity is satisfied. Naturally, the set of prices given by a GNAC is less flexible. We show that either there is no friction in the market (and the FTAP applies everywhere) or there does not exist a risky *frictionless asset*. This apparent shortcoming is not particularly concerning because, in practice, there is unlikely to be a risky asset which can be added without friction to any other portfolio.

The price formula remains compatible with the absence of friction among a subset of assets. We demonstrate, overall, that prices are given by a GNAC if, and only if, in addition to satisfying put–call parity, there is no friction among assets which yield extreme revenues in the same states of nature.

Our work relates to asset pricing literature which aims to generalize the FTAP to markets with frictions. The principal contributions to this field are from Garman and Ohlson (1981) who proposed a model when prices are linear in the number of shares traded (positively homogeneous); Jouini and Kallal (1995) who generalized the FTAP to sublinear prices; and Prisman (1986) who proposed an extension of the FTAP for markets with convex fees such as taxes and, more recently, to CMM, whose model appears better suited for empirical research. We build on this literature by analyzing a special CMM circumstance when the price is given by a GNAC. The corresponding price formula is simpler and easier to calibrate, and it explicitly

[1] Weber sets are very close in nature to rank-dependent probability assignments (see Nehring 1999) and to Clarke differentials at 0 (see Ghirardato et al. 2004); hence, our results could be translated in these languages.

connects price with risk. We provide a characterization of these price formulas, and we compare them with the price formula given under a general capacity.

This paper is organized as follows. Section 2 presents the framework and Sect. 3 presents the FTAP of Harrison and Kreps (1979) and the theorem of asset pricing of CMM. In Sect. 4, we present the price formula given by a NAC and the price formula given by a GNAC, and study how bid–ask spreads relate to risk. In Sect. 5, we interpret the parameters of a price formula given by a NAC and a price formula given by a GNAC, and we consider situations where price formulas given by a GNAC are better suited for asset pricing. In Sect. 6, we discuss the relationship between pricing formulas, under put–call parity, and the FTAP. Finally, in Sect. 7, we characterize the GNAC pricing formula. The mathematical proofs are presented in Appendix.

2 Framework

We consider a two-period $t \in \{0, 1\}$ financial market with trading occurring only on date $t = 0$. The outcome of the second period is uncertain and is represented by a finite set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ comprising m states of nature. At date 0, agents access the market without costs or constraints. They assemble a portfolio among a finite set of primary assets available for trading, that is, they buy (or sell) the right to receive the payoff $X \in \mathbb{R}^\Omega$ (e.g. the right to receive $X(\omega)$ in state of nature ω at date $t = 1$) and we assume that the market is complete. In particular, put and call options are available for all assets and agents can compose a portfolio which gives a frictionless payoff (or cash) $x_{rf} \in \mathbb{R}^\Omega$ which corresponds to the constant unit vector. In our discussion below, $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ is a pricing rule, which is a non-zero map for which $\tilde{\pi}(X)$ $(-\tilde{\pi}(-X))$ represents the amount of resources an agent should spend (or receive) at date 0 when buying (or selling) the payoff X.

3 The FTAP and the CMM model

The absence-of-friction hypothesis which is at the core of most of the literature on fundamental asset pricing states that a market is frictionless when buying two portfolios separately costs the same as buying them together. In other words, the pricing rule is linear: for all payoffs $X, Y \in \mathbb{R}^\Omega$ and all $\lambda \in \mathbb{R}$,

$$\tilde{\pi}(\lambda X + Y) = \lambda \tilde{\pi}(X) + \tilde{\pi}(Y).$$

It is well known that this relationship does not ensure price equilibrium. For example, if $\tilde{\pi}$ is a negative linear function, then it is optimal for an agent (independently of her preferences) to buy an infinite quantity of assets with positive payoffs, resulting in a sub-optimal portfolio. Harrison and Kreps (1979) demonstrated that *no-arbitrage*, that is, for all $X \in \mathbb{R}^\Omega$,

$$X > 0 \Rightarrow \tilde{\pi}(X) > 0,$$

where $X > 0$ implies that $X(\omega) \geq 0$ for all $\omega \in \Omega$, with at least one strict inequality, is an essential property of frictionless financial markets which require equilibrium. Actually, they show that, if, and only if, the pricing rule is frictionless and has no arbitrage opportunity, then there exists a unique probability such that the price is the expected value of the portfolio's payoffs.

Theorem 3.1 (FTAP, Harrison and Kreps 1979) *Let $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a non-zero pricing rule. The following statements are equivalent*:

(i) *$\tilde{\pi}$ is frictionless and has no arbitrage opportunity*;
(ii) *there exist a unique risk-neutral probability μ and a riskless rate $r > -1$ such that*

$$\tilde{\pi}(X) = \frac{1}{1+r}\mathbb{E}_{\mu}(X) = \frac{1}{1+r}\sum_{i=1}^{m} X(\omega_i)\mu(\omega_i) \ \forall X \in \mathbb{R}^{\Omega}.$$

This theorem is fundamental because it provides an explicit formula for pricing assets, supported by equilibrium requirements and, in a multiple-period market, it demonstrates the existence of a connection between martingale theory and asset pricing. However, the absence of friction is a big assumption. It ignores the roles played by transaction costs and fees, as well as market impact in financial markets.

Several pricing models have been developed to generalize the FTAP to markets with frictions. The price formula proposed by CMM incorporates frictions such as transaction costs, and establishes a new link between asset pricing and non-linear expectation theory. In a nutshell, a capacity $v : \mathcal{P}(\Omega) \to [0, 1]$, also informally referred to as a non-additive probability, satisfies the following properties $v(\emptyset) = 0$, $v(\Omega) = 1$ (normalization) and $v(A) \leq v(B)$ whenever $A \subseteq B \subseteq \Omega$ (monotonicity). The expected value with respect to a capacity is called the Choquet expected value. It is defined as follows: consider a vector $X \in \mathbb{R}^{\Omega}$ and a permutation of the states of nature $(\omega_1^*, \omega_2^*, \ldots, \omega_m^*)$ such that $X(\omega_1^*) \geq X(\omega_2^*) \geq \ldots \geq X(\omega_m^*)$. Then the Choquet expected value of X with respect to the capacity v is

$$\mathbb{CE}_v(X) := X(\omega_1^*)v(\omega_1^*) + \sum_{i=2}^{m} X(\omega_i^*)[v(\{\omega_1^*, \ldots, \omega_i^*\}) - v(\{\omega_1^*, \ldots, \omega_{i-1}^*\})].$$

The CMM model assumes that put–call parity (see Stoll 1973) is satisfied, that is, for all call options $c_{X,k} := \max(X - kx_{rf}, 0)$ and all put options $p_{X,k} := \max(kx_{rf} - X, 0)$ on the same underlying payoff $X \in \mathbb{R}^{\Omega}$ with strike price $k \in \mathbb{R}$,

$$\tilde{\pi}(c_{X,k}) + \tilde{\pi}(-p_{X,k}) = \tilde{\pi}(X) - k\tilde{\pi}(x_{rf}),$$

where $\max(X, Y) \in \mathbb{R}^{\Omega}$ is the vector such that, $\forall \omega \in \Omega$, $\max(X, Y)(\omega) = \max(X(\omega), Y(\omega))$. In other words, buying a call option and selling a put option on the same underlying payoff with identical strike price k costs the same as buying the underlying payoff and selling k units of the riskless payoff. It is nominally true that when there is no friction, the two strategies earn the same revenues; however, if one strategy were more expensive than the other, then the demand for it would be nil. Thus, at equilibrium, the two strategies must be equally priced. Furthermore, it is typically assumed that an asset with a higher payoff than another must cost at least the same price, that is, for all $X, Y \in \mathbb{R}^{\Omega}$, $X \geq Y$ implies that $\tilde{\pi}(X) \geq \tilde{\pi}(Y)$. Therefore, the pricing rule is monotonic. As is usual in asset pricing literature, this model also assumes that risk-free payoff x_{rf} is *frictionless*, that is, for all $X \in \mathbb{R}^{\Omega}$ and for all $k \in \mathbb{R}$, $\tilde{\pi}(X + kx_{rf}) := \tilde{\pi}(X) + k\tilde{\pi}(x_{rf})$. This last property is labeled, cash-invariance. Their main result is the following characterization of these pricing rules.

Theorem 3.2 (CMM Theorem, CMM) *Let* $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ *be a non-zero pricing rule. The following statements are equivalent*:

(i) $\tilde{\pi}$ *satisfies put–call parity, cash-invariance and monotonicity*;
(ii) *there exists a unique risk neutral capacity* v *and a unique riskless rate* $r > -1$ *such that*

$$\tilde{\pi}(X) = \frac{1}{1+r}\mathbb{CE}_v(X), \forall X \in \mathbb{R}^{\Omega}.$$

CMM showed that the capacity is a probability if, and only if, $\tilde{\pi}$ is frictionless. This elegantly generalizes the FTAP in a two-period model, and the testability of the hypotheses makes it more suitable for empirical research. Furthermore, Choquet expectations and capacities are at the core of innovation in economics theory, especially in decision theory where the Choquet expected utility model was developed to generalize the classical expected-utility model and accommodate the Ellsberg paradox and the Allais paradox. These results have applications in finance (see Chateauneuf et al. 1996; Waegenaere et al. 2003; Chen and Kulperger 2006; Kast et al. 2014), insurance (see Castagnoli et al. 2002, 2004), risk measurement (see Denuit et al. 2006) and investment behavior (see Ludwig and Zimper 2006; Driouchi et al. 2018). Such applications invite study of connections between the Choquet expectation and asset pricing. We propose to do this for a particular family of capacities which is among the most convenient and falls between general capacities and probabilities, the family of so-called neo-additive capacities (NACs) or, more precisely, their generalized form, the so-called generalized neo-additive capacities (GNACs). The NACs were developed by Chateauneuf et al. (2007) to obtain a model of non-linear expected utility more tractable than the Choquet expected utility. Indeed, NACs have fewer parameters needed for calibration than a general capacity, which makes them more suitable for empirical research. They have applications in asset pricing (see Zimper 2012), investment behavior (see Ford et al. 2005), risk (see Chakravarty and Kelsey 2017), game theory (see Eichberger and Kelsey 2011; Jungbauer and Ritzberger 2011; Eichberger and Kelsey 2014), learning behavior (see Zimper and Ludwig 2009), health and retirement (see

Groneck et al. 2016) and for extending the common knowledge theorem of Aumann (see Dominiak and Lefort 2013). In the context of asset pricing, the generalized form of a NAC developed by Eichberger et al. (2012) appears more suitable because it creates no relationship between the *bid–ask spread* and the *power of explanation of a frictionless price*. We introduce more formally the NACs and the GNACs and their associated pricing formulas in the following section. We discuss their interpretation and the previous argument in favour of GNAC pricing rules in Sect. 5.

4 NAC and GNAC pricing rules

For the sake of our exposition, we assume that the set of null events, that is, the set whose events are "impossible to occur" has only one element, the empty set $\emptyset$. However, the validity of the following results does not rely on this assumption. A NAC is a convex combination of a probability and a parameter which takes values between 0 and 1. More formally, the function v is a NAC if there exists a probability $p : \mathcal{P}(\Omega) \rightarrow [0,1]$ and two reals $\alpha \in [0,1]$ and $\delta \in [0,1]$ satisfying $\min_{E \notin \{\emptyset,\Omega\}}[\alpha + \delta p(E)] \geq 0$ and $\max_{E \notin \{\emptyset,\Omega\}}[\alpha + \delta(1 - p(E))] \leq 1$ such that

$$v(E) = \alpha\delta + (1 - \delta)p(E), \forall E \notin \{\emptyset, \Omega\}$$

Chateauneuf et al. (2007) showed that the Choquet expectation with respect to a NAC is a convex combination of the expected value with the maximal and minimal revenues, that is,

$$\tilde{\pi}(X) = (1 - \delta)\mathbb{E}(X \mid p) + \delta(\alpha \max(X) + (1 - \alpha) \min(X)),$$

where $\max(X)$ (or $\min(X)$) is the maximum (or, respectively, minimum) of the coordinates of X. We say that $\tilde{\pi}$ is a NAC pricing rule if it satisfies this equality. Prices given by a NAC are a combination of a frictionless price and of the maximal and the minimal revenues. Eichberger et al. (2012) generalized NACs by letting the parameter α take any real value, and δ take any real value less than 1. The resulting function—which they named a GNAC—is an affine transformation of a probability. The remaining constraint on the parameters makes the GNAC normalized and monotone, and thus, a well-defined capacity. For the sake of discussion, (Eichberger et al. , 2012) substituted two new parameters, a and b for α and δ. We have reproduced their presentation below; note that, except for the constraints on the values taken by the parameters, the two formulas are equivalent when $a = \delta\alpha$ and $b = 1 - \delta$. More formally, the function v is a GNAC if there exists a probability, $p : \mathcal{P}(\Omega) \rightarrow [0,1]$, and two reals, a and $b > 0$, satisfying $\min_{E \notin \{\emptyset,\Omega\}}[a + bp(E)] \geq 0$ and $\max_{E \notin \{\emptyset,\Omega\}}[a + b(1 - p(E))] \leq 1$ such that

$$v(E) = a + bp(E), \forall E \notin \{\emptyset, \Omega\}$$

and $v(\Omega) = 1$ and $v(\emptyset) = 0$. The preceding constraints $\min_{E \notin \{\emptyset,\Omega\}}[a + bp(E)] \geq 0$ and $\max_{E \notin \{\emptyset,\Omega\}}[a + b(1 - p(E))] \leq 1$ simply ensure that the values of a and b are chosen so that the function v is monotone. Eichberger et al. (2012) showed that the

Choquet expectation with respect to a GNAC is a weighted sum of the expected value with the maximal and the minimal revenues, that is,

$$\tilde{\pi}(X) = b\mathbb{E}(X \mid p) + a \max(X) + (1 - a - b) \min(X).$$

We discuss the interpretation of the parameters of NAC pricing rules and GNAC pricing rules in the following section. Overall, GNAC pricing rules require interpreting a smaller number of variables. It is necessary to provide (and justify) the values of $m + 2$ parameters—the values taken by the probability p and the values of a and b—whereas it might be necessary to define as many as $2^m - 2$ values in the general case. General capacities may provide better accuracy in the pricing of assets but that gain in precision may be offset by the additional cost of estimating all the necessary parameters. It is interesting to note that prices given by a GNAC are connected with risk through the bid–ask spread—the difference between the price at which one can immediately buy a payoff and the price at which one can immediately sell it. More formally, the bid–ask spread $B : \mathbb{R}^\Omega \to \mathbb{R}$ is

$$B(X) = \tilde{\pi}(X) + \tilde{\pi}(-X),$$

for all $X \in \mathbb{R}^\Omega$. When the price is given by a GNAC, the bid–ask spread is proportional to the range of asset revenues. Indeed, there exists $\lambda \in \mathbb{R}$ such that, for all $X \in \mathbb{R}^\Omega$,

$$B(X) = \lambda[\max(X) - \min(X)]$$

where $\lambda = 2a + b - 1$, i.e. λ is the difference between the additional weight on the maximal revenue and the additional weight on the minimal revenue. We say that λ is the coefficient of proportionality of B. This interpretation is consistent with empirical evidence which suggests that bid–ask spreads are in a direct relationship with risk (see Benston and Hagerman 1974; Stoll 1978, 1985; Amihud and Mendelson 1986). Indeed, the range is a simple (albeit imperfect) measure of risk. In particular, Stoll (1978) and Amihud and Mendelson (1986) evidenced that the relationship between the spread and the risk is positive: the higher the risk, the broader the spread. Here, that same relationship is natural as the bid–ask spread is necessarily positive, for otherwise a clear arbitrage opportunity exists. The positivity of the spread translates naturally to GNACs. It implies that the coefficient of proportionality λ is positive, i.e. that the additional weight given to the maximal revenue a is greater than the additional weight given to the minimal revenue $1 - a - b$. In the case of a NAC pricing rule, this condition is even simpler: there is no arbitrage opportunity in the spread if, and only if, $\alpha \geq 0.5$. In the case of a general capacity, making sure or verifying that there is no arbitrage opportunity in the bid–ask spread is slightly more demanding: we present the equivalent property in the following proposition.

Proposition 4.1 *Let* $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ *be a Choquet pricing rule with respect to the capacity* $v : \mathcal{P}(\Omega) \to [0, 1]$. *The following statements are equivalent*:

(i) $\tilde{\pi}$ *does not have an arbitrage opportunity in the bid–ask spread*;

(ii) $v(A) + v(A^c) \geq 1$ *for all* $A \in \mathcal{P}(\Omega)$.

In particular, when $\tilde{\pi}$ *is a GNAC pricing rule then there is no arbitrage opportunity in the bid–ask spread if, and only if,* $a \geq 1 - a - b$.

Bid–ask spreads have also been found to be good indicators of the liquidity of the asset (see Garbade 1982; Stoll 1985): the narrower the bid–ask spread, the more liquid the asset. When the pricing rule is Choquet and not necessarily GNAC, the bid–ask spread of riskless payoff is zero and consistent with the perception that riskless assets are the most liquid assets. When the pricing rule is given by a GNAC, it is also the case that payoffs are close to riskless—where their range of revenues is tight—have a small spread which implies that they are more liquid. When we consider a general capacity, the bid–ask spread is not necessarily proportional to the range of revenues: to that end, the following lemma presents the property that a general capacity should satisfy for proportionality to apply.

Lemma 4.1 *Let* $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ *be a Choquet pricing rule with bid–ask spread* $B : \mathbb{R}^\Omega \to \mathbb{R}$. *Then the following assertions are equivalent*:

(i) *The bid–ask spread is proportional to the range of revenues*;
(ii) $\exists \lambda \in \mathbb{R}, \forall E \notin \{\emptyset, \Omega\}, B(1_E 0) = \lambda$;
(iii) $v(E) + v(E^c) = k, \forall E \notin \{\emptyset, \Omega\}$,

where $x_E y \in \mathbb{R}^\Omega$ *is the vector with coordinates in* $E \in \mathcal{P}(\Omega)$ *equal to* x *and coordinates in* E^c *equal to* y.[2]

5 Interpretation of GNAC pricing rules

In addition to involving fewer parameters needing calibration, the coefficients of a NAC and a GNAC pricing rule are also easier to interpret than a general-capacity pricing rule. In the NAC price formula, the coefficient δ can be interpreted as the power of explanation of frictionless pricing in the market or a measure of how close the market is to frictionless. It contains information about the importance of transaction costs and other frictions on asset pricing. The closer δ is to 1, the less significant the role frictions play. The price of an asset given by a NAC is bounded by the asset revenues. The following inequalities are satisfied for all $X \in \mathbb{R}^\Omega$:

$$\min(X) \leq \tilde{\pi}(X) \leq \max(X).$$

The parameter α indicates whether the price is close to the maximal bound. When $\delta = 1$ and $\alpha = 1$, the asset price is maximal; when $\delta = 1$ and $\alpha = 0$, the asset price is minimal. The first situation captures agents' extreme confidence that the maximal revenue will be delivered in the future. The second situation captures agents' extreme confidence that the minimal revenue will be delivered in the future. As explained in the previous section, arbitrage opportunities are created when $\alpha < 0.5$;

[2] We refer to such vectors as bets.

hence, this last situation should never arise when the price is given by a NAC. Other values of α and δ mediate between these extremes.

Interpreting the parameters of a GNAC pricing rule requires more prudence. The coefficient b can still be regarded as providing information on the importance of the role played by friction in the pricing. However, both b and a can take values greater than 1; therefore, for the interpretation to be meaningful, it is preferable to analyze $b/(|a|+b)$ and $|a|/(|a|+b)$. These values can be interpreted similar to δ and α. The closer $b/(|a|+b)$ to 1, the lesser friction influences the pricing. So, $b/(|a|+b)$ is the power of explanation of a frictionless pricing rule. Similarly, the closer $|a|/(|a|+b)$ is to 1, the greater the effect of frictions on pricing.

It is important to note that GNAC prices are not bounded by the asset revenues. GNAC allows a great deal of pricing flexibility. In particular, GNAC allows calibration of over-confident behavior—when $|a|/(|a|+b)$ is close to 1 and $a > 1$—and under-confident behavior—when $|a|/(|a|+b)$ is close to 1 and $a<0$—where prices are disconnected from the revenues of the asset, i.e. when prices are either greater than the maximal revenue or smaller than the minimal revenue. These two situations cannot be represented in a frictionless environment, nor when prices are given by a NAC. GNAC pricing rules may be used to represent both boom and bust scenarios. Moreover, parameters' calibration for a GNAC is less restrictive than the calibration of a NAC. Indeed, the bounds imposed by a NAC on the parameters' values impose a bound on the value of the coefficient of proportionality of the spread which has to be smaller than 1. In fact, the bounding is even tighter. It also requires λ to be smaller than δ, which creates a strong relationship between the bid–ask spread and the power of explanation of a frictionless pricing. For example, it is not possible to have both asset prices explained at 95% by a frictionless pricing rule and a 10% coefficient of proportionality in the bid–ask spread.

6 Put–call parity and the FTAP

In the previous sections, we argued that GNAC pricing rules are better suited to asset pricing because they incorporate fewer parameters to calibrate and the parameters are easier to interpret. From this perspective, a frictionless market is ideal with only a probability to calibrate. In this section, we ask whether, when put–call parity is satisfied, we can identify a subset of the market in which there is no friction, to apply the FTAP. Intuitively, the FTAP applies to a subset of the market if buying two portfolios of this subset jointly costs the same as buying them separately. To examine this, we must first define a new object, the risky frictionless payoff, as a payoff whose revenues are not known with certainty, and which costs the same to buy whether together with a portfolio or in independent purchases. More formally, the payoff $X \in \mathbb{R}^\Omega$ is frictionless if, for all $Y \in \mathbb{R}^\Omega$ and all $a \in \mathbb{R}$,

$$\tilde{\pi}(aX + Y) = a\tilde{\pi}(X) + \tilde{\pi}(Y).$$

Likewise, if all bets on a particular event $E \in \mathcal{P}(\Omega)$ are frictionless, then we say that this event is frictionless, that is, the payoffs $\mathbb{1}_E 0$ and $\mathbb{1}_{E^c} 0$ are frictionless. Cash invariance is a no-friction property: riskless payoffs are frictionless and Ω is a frictionless event. Moreover, a frictionless asset has no bid–ask spread. We wondered which properties, satisfied by a Choquet pricing rule, are equivalent to the existence of a risky frictionless asset. In particular, we are interested in characterizing the set of payoffs that are frictionless and in determining the form taken by a Choquet pricing rule on this set.

It is well known that Choquet expectations are positively homogeneous, that is, $\forall X \in \mathbb{R}^{\Omega}$, $\forall k \geq 0$, $\tilde{\pi}(kX) = k\tilde{\pi}(X)$, and additive with respect to comonotone vectors (vectors $X, Y \in \mathbb{R}^{\Omega}$ such that for all $\omega, \omega' \in \Omega$, $\omega \neq \omega'$, $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$). We can associate with a vector $X \in \mathbb{R}^{\Omega}$ a ranking of the states of nature ρ (that is a bijection between Ω and $\{1, \ldots, m\}$) which associates 1 with the state ω such that $X(\omega)$ is the highest payoff of X, 2 to the second highest and so on. This is useful because the Choquet expectation of $X \in \mathbb{R}^{\Omega}$ can be regarded as the expectation value of the vector with respect to a probability $\mu_\rho : \mathcal{P}(\Omega) \to [0, 1]$ given by

$$\mu_\rho(E) = \sum_{\omega \in E} [v(P_\rho(\omega) \cup \{\omega\}) - v(P_\rho(\omega))],$$

for all $E \in \mathcal{P}(\Omega)$ where $P_\rho(\omega) := \{\omega' \in \Omega \mid \rho(\omega') < \rho(\omega)\}$ is the set of predecessors of ω. This representation is particularly helpful when attempting to understand why the Choquet expectation is additive with respect to comonotone vectors. The set of probabilities, μ_ρ, is called the Weber set of v (Weber 1988); we denote it $\mathcal{W}(v)$. There exists a connection between the absence of friction and Weber sets. Our first results are valid for general capacities and the corresponding Choquet pricing rules. We show that an event is frictionless if, and only if, the associated capacity is additive with respect to this event and it is equivalent to all probabilities' values for this event in the Weber set being equal to the capacity's value.

Proposition 6.1 *Let $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a Choquet pricing rule with respect to the capacity $v : \mathcal{P}(\Omega) \to [0, 1]$. Let E be an event in $\mathcal{P}(\Omega)$. The following statements are equivalent*:

(i) *E is frictionless*;
(ii) *$v(A) = v(A \cap E) + v(A \cap E^c)$ for all A in $\mathcal{P}(\Omega)$*;
(iii) *$\mu(E) = v(E)$ for all $\mu \in \mathcal{W}(v)$.*

Furthermore, a payoff is frictionless if, and only if, it can be decomposed as a sum of bets on frictionless events.

Proposition 6.2 *Let $X = \sum_{i=1}^{n} x_i \mathbb{1}_{E_i} \in \mathbb{R}^{\Omega}$, where for all i $x_i > x_{i+1}$, $x_i \in \mathbb{R}$ and $E_i \in \mathcal{P}(\Omega)$. Let $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a Choquet pricing rule. The following statements are equivalent*:

(i) *X is frictionless*;
(ii) for all $i \in \{1, \ldots, n\}$, *E_i is a frictionless event.*

We deduce from Proposition 6.2 that the set of frictionless events forms a linear space. We denote it Φ. On Φ, the FTAP applies and any probability of the Weber set of v can be used to price payoffs: for all $X \in \Phi$,

$$\tilde{\pi}(X) = \mathbb{E}_{\mu}(X) \quad \text{where } \mu \in \mathcal{W}(v).$$

If we can determine that a risky payoff is frictionless, then it is possible to price a large set of payoffs using the FTAP and any probability within the Weber set of the capacity, and at the same time to price payoffs which are not frictionless with the capacity. From another perspective, we can easily incorporate the existence of frictionless payoffs when calibrating a capacity by letting the corresponding values of the probabilities of the Weber set coincide.

It is not possible to have both risky frictionless payoffs and payoffs with friction when prices are given by a GNAC: GNAC pricing rules require that either the market is frictionless or that there exists no risky frictionless payoff on the market. However, this loss of generality will turn out not to be an argument against GNAC pricing rules. Indeed, it is very unlikely in practice that a risky frictionless payoff will be encountered. Our first result is slightly more compelling: it demonstrates that the absence of a bid–ask spread for a risky bet is necessary and sufficient for the absence of bid–ask spreads on the whole market.

Lemma 6.1 *Let $\tilde{\pi} : \mathbb{R}^{\Omega} \rightarrow \mathbb{R}$ be a GNAC pricing rule with respect to the GNAC $v : \mathcal{P}(\Omega) \rightarrow [0,1]$ with bid–ask spread $B : \mathbb{R}^{\Omega} \rightarrow \mathbb{R}$. The following assertions are equivalent*:

(i) $\exists A \in \mathcal{P}(\Omega)$ *such that* $A \notin \{\emptyset, \Omega\}$ *and* $v(A) + v(A^c) = 1$;
(ii) $B(X) = 0$ *for all* $X \in \mathbb{R}^{\Omega}$.

To account for the presence of a bid–ask spread on a risky payoff, it is necessary to assume that all other risky payoffs present in the market have a bid–ask spread. In practice, this condition does not seem unrealistically demanding because bid–ask spreads are the most common type of frictions present in financial markets. Our second result shows that a GNAC pricing rule is frictionless if, and only if, there exists a frictionless event.

Proposition 6.3 *Let $\tilde{\pi} : \mathbb{R}^{\Omega} \rightarrow \mathbb{R}$ be a GNAC pricing rule and let $E \notin \{\emptyset, \Omega\}$ be an event. The following assertions are equivalent*:

(i) *E is a frictionless event*;
(ii) *$\tilde{\pi}$ is frictionless.*

We deduce from Propositions 6.2 and 6.3 that if there exists a frictionless risky payoff, then the market is frictionless. Theoretically, this may seem a demanding restriction but, in practice, it is unrealistic to assume that a risky payoff is frictionless when the market is complete since it implies that this payoff may be added to any other portfolio without friction. In conclusion, for practical valuation matters, the loss of flexibility incurred by a GNAC pricing rule is not problematic when all risky payoffs have a bid–ask spread. Moreover, as we show in the next section, GNAC pricing rules are compatible with the existence of a frictionless subset of payoffs, namely those with matching extreme revenues. GNAC pricing requires the reasonable assumption that any payoff of this subset can be added to a portfolio composed of other payoffs of the subset without additional costs.

7 Characterization of GNAC pricing rules

From the definition of a Choquet expectation, when prices are given by a general capacity, there is no friction among comonotone payoffs. This is a necessary characteristic of Choquet pricing rules along with monotonicity and an additional trait called constant modularity, viz. $\forall k \in \mathbb{R}$, $\forall X \in \mathbb{R}^{\Omega}$, $\tilde{\pi}(\max(X, kx_{rf})) + \tilde{\pi}(\min(X, kx_{rf})) = \tilde{\pi}(X) + k\tilde{\pi}(x_{rf})$. This result was demonstrated by Greco (1982). In this section, we show that GNAC pricing rules can be characterized by put–call parity, monotonicity, cash invariance and the absence of friction among payoffs which yield extreme revenues in the same states of nature. Since GNACs are a subset of capacities, we only have to demonstrate that a Choquet pricing rule is GNAC if, and only if, there is no friction among payoffs which yield extreme revenues in the same states of nature. The rest of the proof results from the main theorem of CMM that we recalled in Sect. 3. We denote $\arg\max X \in \mathcal{P}(\omega)$ (or, $\arg\min X \in \mathcal{P}(\Omega)$) the arguments of the maxima of X, that is, the set of states of nature E where the coordinates of X are maximal (or, respectively, minimal). We say that two payoffs $X, Y \in \mathbb{R}^{\Omega}$ have matching extreme revenues if their maximal and minimal revenues occur in the same states of nature, that is, if

$$\arg\max X \cap \arg\max Y \neq \emptyset; \text{ and}$$
$$\arg\min X \cap \arg\min Y \neq \emptyset.$$

We also expand the definition of frictionless payoffs, to absence of friction among payoffs with matching extreme revenues in the following way. Let $X, Y \in \mathbb{R}^{\Omega}$ be two payoffs with matching extreme revenues, in this instance, there is no friction among payoffs with matching extreme revenues if

$$\tilde{\pi}(X + Y) = \tilde{\pi}(X) + \tilde{\pi}(Y).$$

We show that a Choquet pricing rule is a GNAC if, and only if, there is no friction among matching extreme payoffs.

Proposition 7.1 *Let* $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ *be a Choquet pricing rule. The following assertions are equivalent*:

(i) $\tilde{\pi}$ *satisfies no friction among payoffs with matching extreme revenues;*
(ii) $\tilde{\pi}$ *satisfies no friction among bets with matching extreme revenues*;
(iii) $\tilde{\pi}$ *is a GNAC pricing rule.*

GNAC pricing rules are compatible with markets in which put–call parity is satisfied and there is no friction between matching extreme payoffs. NAC pricing rules are slightly more restrictive. When the prices are given by a NAC, the events in which the payoff yields its maximal revenue and its minimal revenue are both overweighted. This adds constraints on the prices of non-matching extreme payoffs. Indeed, Chateauneuf et al. (2007) showed that NACs imply that there exist $E, F, G, H \neq \emptyset$ with $E \cup F \neq \Omega$, $G \cup H \neq \Omega$ and $E \cap F = \emptyset = G \cap H$ such that

$$\tilde{\pi}(1_{E\cup F}0) \leq \tilde{\pi}(1_E 0) + \tilde{\pi}(1_F 0);$$
$$\tilde{\pi}(1_{G\cup H}0) \geq \tilde{\pi}(1_G 0) + \tilde{\pi}(1_H 0).$$

Thus, the set of prices given by a NAC is not compatible with financial markets in which it is always more expensive to buy assets separately due to frictions. It is also not compatible with a financial market in which buying assets separately is always less expensive.

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Appendix

Proof of Lemma 4.1 We first assume that the bid–ask spread is proportional to a constant. It follows immediately that the bid–ask spread of bets which yield 1 if some event occurs, and 0 if the complementary event occurs, is constant.

Now, we assume that the bid–ask spread of bets of the form $1_E 0$, where E is an event of Ω, is equal to a constant $\lambda \in \mathbb{R}$. We are going to show that the capacity values of complementary events sum to a constant. For all $E \notin \{\emptyset, \Omega\}$, we have

$$B(1_E 0) = v(E) + v(E^c) - 1.$$

Thus,

$$v(E) + v(E^c) = \lambda + 1 \qquad \text{for all } E \notin \{\emptyset, \Omega\}.$$

Finally, we assume that the capacity values of complementary events sum to a constant $k \in \mathbb{R}$. We are going to show that the bid–ask spread is proportional to the range of revenues. We let $X \in \mathbb{R}^\Omega$. We denote $x_1, \ldots, x_n$ the n coordinates of X such that $x_1 \geq x_2 \geq \ldots \geq x_n$. Up to reindexing the states of nature, we assume that the payoff X yields x_i in ω_i for all $i \in \{1, \ldots, n\}$. By definition, the bid–ask spread of X equals

$$\sum_{i=1}^{m} x_i [v(\{\omega_j \in \Omega \mid j \leq i\}) - v\big(\{\omega_j \in \Omega \mid j < i\}\big) - v\big(\{\omega_j \in \Omega \mid j \geq i\}\big) + v\big(\{\omega_j \in \Omega \mid j > i\}\big)]$$

which simplifies to

$$x_1 [v(\{\omega_1\}) + v(\{\omega_2, \ldots, \omega_m\}) - 1] - x_m [v(\{\omega_1, \ldots, \omega_{m-1}\}) + v(\{\omega_m\}) - 1].$$

By applying the above assumption and by substituting $\lambda = k - 1$, we obtain the desired result

$$B(X) = \lambda(x_1 - x_m).$$

Proof of Proposition 4.1 We first assume that there is no arbitrage in the bid–ask spread. We are going to show that the capacity values of complementary events sum to a real greater than 1. By assumption, we have

$$\tilde{\pi}(X) \geq -\tilde{\pi}(-X), \text{ for all } X \in \mathbb{R}^{\Omega}.$$

In particular, we have

$$\tilde{\pi}(1_A 0) \geq -\tilde{\pi}(-1_A 0), \text{ for all } A \in \mathcal{P}(\Omega)$$

which implies

$$v(A) + v(A^c) \geq 1, \text{ for all } A \in \mathcal{P}(\Omega).$$

Now, we assume that the capacity values of complementary events sum to a real greater than 1. We are going to show that there is no arbitrage in the bid–ask spread. We let $X \in \mathbb{R}^{\Omega}$ be a payoff. We denote $x_1, \ldots, x_n$ the n coordinates of X such that $x_1 \geq x_2 \geq \ldots \geq x_n$. Up to reindexing the states of nature, we assume that the payoff X yields x_i in ω_i for all $i \in \{1, \ldots, n\}$. Then by definition of a Choquet expectation, we have

$$\tilde{\pi}(X) = \sum_{i=1}^{m} x_i [v\big(\{\omega_j \in \Omega \mid j \leq i\}\big) - v\big(\{\omega_j \in \Omega \mid j < i\}\big)]$$

which, by assumption, is greater than

$$\sum_{i=1}^{m} x_i [1 - v\big(\{\omega_j \in \Omega \mid j > i\}\big) - (1 - v\big(\{\omega_j \in \Omega \mid j \geq i\}\big)].$$

This sum simplifies to

$$\sum_{i=1}^{m} x_i[v(\{\omega_j \in \Omega \mid j \geq i\}) - v(\{\omega_j \in \Omega \mid j > i\})]$$

which is equal to $-\tilde{\pi}(-X)$. We hence obtain the desired result, for all $X \in \mathbb{R}^{\Omega}$,

$$\tilde{\pi}(X) \geq -\tilde{\pi}(-X).$$

Proof of Proposition 6.1 We are going to show that an event E is frictionless if, and only if, the capacity is additive with respect to this event. We first assume that E is a frictionless event. We are going to show that the capacity is additive with respect to E. By assumption, we have

$$\tilde{\pi}(1_E0) + \tilde{\pi}(-1_E0) = \tilde{\pi}(1_E0 + (-1_E0))$$

which implies

$$v(E) + v(E^c) = 1.$$

Now, we let $A \in \mathcal{P}(\Omega)$ such that $A \cap E \neq \emptyset$ and $A \cap E^c \neq \emptyset$. Then by assumptions, we have

$$\tilde{\pi}(1_E0 + 1_A0) = \tilde{\pi}(1_E0) + \tilde{\pi}(1_A0)$$

and

$$\tilde{\pi}(1_{E^c}0 + 1_A0) = \tilde{\pi}(1_{E^c}0) + \tilde{\pi}(1_A0).$$

It implies

$$v(A \cap E) + v(E \cup A \cap E^c) = v(E) + v(A) \tag{1}$$

and

$$v(A \cap E^c) + v(E^c \cup A \cap E) = v(E^c) + v(A). \tag{2}$$

We replace $v(E^c)$ by $1 - v(E)$, and we combine Eqs. 1 and 2 to get

$$v(A \cap E^c) + v(A \cap E) + v(E^c \cup A \cap E) + v(E \cup A \cap E^c) = 1 + 2v(A).$$

We now substitute $v(E^c \cup A \cap E)$ with $\tilde{\pi}(1_{E^c \cup A \cap E}0)$ and $v(E \cup A \cap E^c)$ with $\tilde{\pi}(1_{E \cup A \cap E^c}0)$. By assumption, we get

$$v(A \cap E^c) + v(A \cap E) + \tilde{\pi}(1_{E^c}0) + \tilde{\pi}(1_{A \cap E}0) + \tilde{\pi}(1_E0) + \tilde{\pi}(1_{A \cap E^c}0) = 1 + 2v(A).$$

Then, again by assumption, we get the desired result

$$v(A) = v(A \cap E) + v(A \cap E^c).$$

Now we assume that the capacity is additive with respect to an event E. We are going to show that E is frictionless: we are going to show that for all $a \in \mathbb{R}$ and all $X \in \mathbb{R}^{\Omega}$,

$$\tilde{\pi}(X + a_E 0) = \tilde{\pi}(X) + a\tilde{\pi}(1_E 0). \qquad (3)$$

We fix $X \in \mathbb{R}^{\Omega}$ and we denote $x_1, \ldots, x_n$ its n coordinates such that $x_1 \geq x_2 \geq \ldots \geq x_n$. We denote $A_{2i-1} \cup A_{2i}$ the event in which the payoff yields x_i with $(A_{2i-1} \cup A_{2i}) \cap E = A_{2i-1}$, as in the following table

	x_1	x_2	...	x_n
E	A_1	A_3	...	A_{2n-1}
E^c	A_2	A_4	...	A_{2n}

so that all events in E have an odd subscript and all events in E^c have an even subscript. Events A_i can be empty. We denote $\mathcal{E}$ the set of even integers in $\{1, \ldots, n\}$ and $\mathcal{O}$ the set of odd integers in $\{1, \ldots, n\}$ and we fix $i \in \mathcal{O}$. We first show that Eq. 3 is satisfied for $a > 0$. We denote ρ the ranking associated with X, and μ the corresponding probability in the Weber set. We consider another payoff, $Y = X + a_E 0$, denoting $\rho^{\star}$ the ranking associated with this payoff, and $\mu^{\star}$ the corresponding probability in the Weber set. We can now show that $\mu(A_i \cup A_{i+1}) = \mu^{\star}(A_i \cup A_{i+1})$. By assumption, we can decompose $v(\{A_j \mid Y(A_j) \leq Y(A_i)\})$ with respect to E, that is with respect to its odd and even events. In other words, we have $v(\{A_j \mid Y(A_j) \leq Y(A_i)\})$ equal to

$$v(\{A_j \mid Y(A_j) \geq Y(A_i),\ j \in \mathcal{O}\}) + v(\{A_j \mid Y(A_j) \geq Y(A_i),\ j \in \mathcal{E}\}). \qquad (4)$$

Similarly, we can decompose $v(\{A_j \mid Y(A_j) > Y(A_i)\})$ with respect to E. It is equal to

$$v(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{O}\}) + v(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{E}\}). \qquad (5)$$

Since i is odd, we have

$$v(\{A_j \mid Y(A_j) \geq Y(A_i),\ j \in \mathcal{E}\}) = v(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{E}\}). \qquad (6)$$

By definition, the probability $\mu^{\star}(A_i \cup A_{i+1})$ is equal to

$$v(\{A_j \mid Y(A_j) \geq Y(A_i)\}) - v(\{A_j \mid Y(A_j) > Y(A_i)\})$$

which, by Eq. 4, 5 and 6, is equal to

$$v(\{A_j \mid Y(A_j) \geq Y(A_i),\ j \in \mathcal{O}\}) \\ -v(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{O}\}).$$

By construction, the equalities

$$v(\{A_j \mid Y(A_j) \geq Y(A_i),\ j \in \mathcal{O}\}) = v(\{A_j \mid X(A_j) \geq X(A_i),\ j \in \mathcal{O}\})$$

and

$$v(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{O}\}) = v(\{A_j \mid X(A_j) > X(A_i),\ j \in \mathcal{O}\})$$

are satisfied. Thus, the probability $\mu^{\star}(A_i \cup A_{i+1})$ is equal to

$$v(\{A_j \mid X(A_j) \geq X(A_i),\ j \in \mathcal{O}\}) - v(\{A_j \mid X(A_j) > X(A_i),\ j \in \mathcal{O}\})$$

which, in turn, by assumption, is equal to $\mu(A_i \cup A_{i+1})$, yielding

$$\tilde{\pi}(Y) = \tilde{\pi}(X) + a\tilde{\pi}(1_E 0).$$

We also have

$$\tilde{\pi}(X + a_{E^c}0) = \tilde{\pi}(X) + a\tilde{\pi}(1_{E^c}0).$$

We replace $a_{E^c}0$ by $a(\mathbb{1}_\Omega - 1_E 0)$ and we use the assumption to replace $\tilde{\pi}(1_{E^c}0)$ by $1 - \tilde{\pi}(1_E 0)$ to get

$$\tilde{\pi}(X + a(\mathbb{1}_\Omega - 1_E 0)) = \tilde{\pi}(X) + a(1 - \tilde{\pi}(1_E 0)).$$

Hence,

$$\tilde{\pi}(X - a_E 0) = \tilde{\pi}(X) - a\tilde{\pi}(1_E 0).$$

It follows that, for all $a \in \mathbb{R}$ and all $X \in \mathbb{R}^\Omega$,

$$\tilde{\pi}(Y) = \tilde{\pi}(X) + a\tilde{\pi}(1_E 0),$$

that is, E is a frictionless event.

Now, we can show that the capacity is additive with respect to an event E if, and only if, all probability values in the Weber set coincide with the value of the capacity for this event. We first assume that the capacity is additive with respect to an event E. We are going to show that all the probabilities in the Weber set coincide with the value taken by the capacity on E. We fix a probability μ in the Weber set. We consider a vector X associated with this probability, that is, there exists a ranking ρ such that ρ is associated with X and μ is associated with X. We denote $x_1, x_2, \ldots, x_n$ the coordinates of X such that $x_1 \geq x_2 \geq \ldots \geq x_n$. As shown in the following table, we denote $A_{2i-1} \cup A_{2i}$ the event in which the payoff yields x_i such that $(A_{2i-1} \cup A_{2i}) \cap E = A_{2i-1}$, so that all events in E have an odd subscript.

	x_1	x_2	...	x_n
E	A_1	A_3	...	A_{2n-1}
E^c	A_2	A_4	...	A_{2n}

The relationship

$$v(\{A_j \mid X(A_j) \geq X(A_i)\}) - v(\{A_j \mid X(A_j) > X(A_i)\})$$

simplifies to

$$v(\{A_j \mid X(A_j) \geq X(A_i), j \in \mathcal{O}\}) - v(\{A_j \mid X(A_j) > X(A_i), j \in \mathcal{O}\})$$

when i is odd and $\mu(E)$ is equal to

$$\sum_{\substack{i=1 \\ i \in \mathcal{O}}}^{2n} [v(\{A_j \mid X(A_j) \geq X(A_i), j \in \mathcal{O}\}) - v(\{A_j \mid X(A_j) > X(A_i), j \in \mathcal{O}\})]$$

and simplifies to $v(E)$.

Now, we assume that all probabilities in the Weber set coincide with the capacity value for an event E, and we will show that the capacity is additive with respect to E. We let E_1, E_2 be two distinct subsets of Ω such that $E = E_1 \cup E_2$ and we consider two events A and B such that $A = B \cup E_1$ and $B \cap E = \emptyset$. We let ρ be a ranking such that $\rho(E_1) > \rho(B) > \rho(E_2) > \rho(\Omega \backslash (E_1 \cup B \cup E_2))$ with the convention that $\rho(A) > \rho(B)$ if $\rho(\omega_i) > \rho(\omega_j)$ for all $\omega_i \in A$ and all $\omega_j \in B$. We let μ be the probability associated with ρ. We have $\mu(E)$ equal to

$$v(E_1) + v(E_1 \cup B \cup E_2) - v(E_1 \cup B)$$

which is, in turn, equal to

$$v(A \cap E) + v(A \cup E) - v(A).$$

We let $\rho^\star$ be a ranking such that

$$\rho^\star(B) > \rho^\star(E_1) > \rho^\star(E_2) > \rho^\star(\Omega \backslash (E_1 \cup B \cup E_2)$$

and we let $\mu^\star$ be the associated probability. We have $\mu^\star(E)$ equal to

$$v(B \cup E_1 \cup E_2) - v(B) \text{ which is equal to } v(A \cup E) - v(A \cap E^c)$$

and we get the desired result:

$$v(A) = v(A \cap E) + v(A \cap E^c), \text{ for all } A \in \mathcal{P}(\Omega).$$

Proof of Proposition 6.2 We assume that X is a frictionless payoff. We are, therefore, going to show that we can decompose it as a sum of frictionless events, in part, by contradiction. We write X with the following form:

$$X = \sum_{i=1}^{n} x_{iE_i}0.$$

We are going to prove that the E_i are frictionless. We assume that there exist some A_i $i \in \{1,\ldots,n\}$ that are not frictionless. Up to reindexing, we decompose X into two sums. The left sum groups all x_i's on frictionless events and the right one groups x_i's on events with frictions:

$$X = \sum_{i=1}^{k} x_{iE_i}0 + \sum_{i=k+1}^{n} x_{iE_i}0.$$

We have $\tilde{\pi}(X+Y)$ equal to

$$\tilde{\pi}\left(X - \sum_{i=1}^{k} x_{iE_i}0 + \sum_{i=1}^{k} x_{iE_i}0 + Y\right).$$

By assumption, this is not equal to

$$\tilde{\pi}\left(\sum_{i=k+1}^{n} x_{iE_i}0\right) + \tilde{\pi}\left(\sum_{i=1}^{k} x_{iE_i}0 + Y\right).$$

By additivity, the preceding equation is equal to

$$\tilde{\pi}\left(\sum_{i=k+1}^{n} x_{iE_i}0 + \sum_{i=1}^{k} x_{iE_i}0\right) + \tilde{\pi}(Y).$$

We can now recognize $\tilde{\pi}(X) + \tilde{\pi}(Y)$, a contradiction.

Now, we assume that X can be decomposed as a sum of frictionless events. We are going to show that X is frictionless. We have

$$X = \sum_{i=1}^{n} x_{iE_i}0,$$

where for all $i \in \{1,\ldots,n\}$ $x_i \in \mathbb{R}$, the events E_i are frictionless and

$$\sum_{i=1}^{n} 1_{E_i}0 = \mathbb{1}_{\Omega}.$$

If we let $Y \in \mathbb{R}^{\Omega}$, we have $\tilde{\pi}(X+Y)$ equal to

$$\tilde{\pi}\left(\sum_{i=1}^{n} x_{iE_i}0 + Y\right).$$

This is, by assumption, equal to

$$\sum_{i=1}^{n} \tilde{\pi}(x_{iE_i}0) + \tilde{\pi}(Y)$$

We get the desired result: $\tilde{\pi}(X) + \tilde{\pi}(Y)$ for all $Y \in \mathbb{R}^{\Omega}$.

Proof of Lemma 6.1 We assume that there exists an event $A \notin \{\emptyset, \Omega\}$ such that $v(A) + v(A^c) = 1$. We are going to show that the bid–ask spread is nil. From Lemma 4.1, we have $\lambda = v(A) + v(A^c) - 1$. Thus, $\lambda = 0$ which entails $B(X) = 0$ for all $X \in \mathbb{R}^{\Omega}$.

Now, we assume that the bid–ask spread is null. By definition, the bid–ask spread, $B(1_A 0) = 0$ implies $\lambda = 0$ with $\lambda = v(A) + v(A^c) - 1$.

Proof of Proposition 6.3 We assume that E is a frictionless event; we can show that $\tilde{\pi}$ is frictionless. We consider an event $A \in \mathcal{P}(\Omega)$, we have

$$v(A) = v(A \cap E) + v(A \cap E^c).$$

This implies

$$a + bp(A) = 2a + bp(A).$$

Hence, $a = 0$. Moreover,

$$v(E) + v(E^c) = b = 1.$$

Thus $a = 0$ and $b = 1$. Therefore, for all $A \in \mathcal{P}(\Omega)$,

$$v(A) = p(A).$$

Now, if we assume that $\tilde{\pi}$ is frictionless, then v is additive.

Proof of Proposition 7.1 First, we assume that the capacity is pairwise additive for payoffs with matching extreme revenues. Then it is, in particular, additive for bets with matching extreme revenues. We will now show that the capacity is a GNAC. To do so, we consider the following property, which we call Property A.

Definition 8.1 (Property A, Eichberger et al. 2012) $v(E \cup F) - v(F) = v(E \cup G) - v(G)$ is satisfied for all events $E, F, G \in \mathcal{P}(\Omega)$ such that $E \cup F \neq \Omega$, $E \cup G \neq \Omega$, $E \cap F = \emptyset = E \cap G$, $F \neq \emptyset$, $G \neq \emptyset$.

Eichberger et al. (2012) showed in Lemma 3 that Property A is satisfied if, and only if, the capacity is a GNAC. We will show that Property A is satisfied. We let $A, B \in \mathcal{P}(\Omega)$, such that $A \cap B \neq \emptyset$ and $A \cup B \neq \Omega$. The bets $1_A 0, 1_B 0 \in \mathbb{R}^{\Omega}$ have matching extreme revenues. Hence, by assumption

$$\tilde{\pi}(1_A 0 + 1_B 0) = v(A \cap B) + v(A \cup B)$$

which is equal to $v(A) + v(B)$. Hence, the result is

$$v(A \cup B) - v(B) = v(A) - v(A \cap B).$$

We denote $E = A \backslash A \cap B$, $F = A \cap B$ and $G = B$. We get Property A with $F \subset G$:

$$v(E \cup G) - v(G) = v(E \cup F) - v(F).$$

Moreover, if we let $F_1, F_2 \subset G$ then

$$v(E \cup F_1) - v(F_1) = v(E \cup F_2) - v(F_2).$$

Now, we assume that the capacity is a GNAC then by the definition of a GNAC pricing rule, it is immediate that it is additive among payoffs with matching extreme revenues. □

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