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On the existence and stability of equilibria in N-firm Cournot–Bertrand oligopolies

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Abstract

This paper takes a novel approach to studying the existence and stability of Nash equilibria in N-firm Cournot–Bertrand oligopolies. First, we show that such games can be monotonically embedded into a game of strategic heterogeneity, so that each firm best responds to the choices of all other firms in a monotonic way. We then show that this monotonicity can be exploited to derive conditions which guarantee the existence of a unique, dominance solvable Nash equilibrium which is stable under all adaptive dynamics. These conditions constitute a strict improvement over existing results in the literature. Finally, we examine the effect on these conditions resulting from additional firms entering the market.

Keywords Cournot–Bertrand competition · Games of Strategic Heterogeneity · Stability

1 Introduction

Competition between firms and the stability of market equilibria have been some of the most extensively studied topics in economics, starting with Cournot (1838), who considered firms who compete by choosing the amount of product to produce, and Bertrand (1883), who considered firms who compete by choosing which price to set. Modern treatments of these issues (see for example, Amir 1996; Amir and Evstigneev 2018; Vives 2001; Milgrom and Shannon 1994) have focused on the consequences of the monotonicity present in such games, in the sense that Cournot competition can be viewed as a game of strategic substitutes (GSS), where firms best respond to a higher output choice by competitors by choosing a lower quantity, while Bertrand competition can be viewed as a game of strategic complements

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(GSC), where firms best respond to a higher price set by competitors by choosing to set a higher price themselves. This paper exploits the fact that through a simple transformation, markets consisting of both price and quantity setting firms can be viewed as games of strategic heterogeneity (GSH), where each firm is either a strategic complements or a strategic substitutes player, and uses this framework to derive new existence, uniqueness, and stability results in this class of games.

The case of when markets consist both of quantity competitors and price competitors has received much less attention, despite their prevalence. For example, following Tremblay et al. (2011, 2013), such situations persist in the market for small cars, as well as produce markets, where producers compete in quantity while local stores compete in price. One common theme in the theoretical literature has been that the existence and stability of equilibria crucially depend on the degree of product substitutability between firms. Singh and Vives (1984) give conditions under which firms will choose to compete either entirely as quantity competitors or entirely as price competitors, depending on whether the goods are complements or substitutes. Tremblay et al. (2011) study the duopoly setting, and show in the case of perfectly homogeneous goods, a unique equilibrium exists, which is also the competitive equilibrium. Concerning the stability of equilibria, recent results have focused exclusively on the duopoly case, where once again product substitutability plays a prominent role. Tremblay and Tremblay (2011) and Askar (2014) derive a bound on product substitutability which guarantees that an equilibrium will be stable under continuous time dynamics, while Naimzada and Tramontana (2012) give conditions for stability under discrete time best response dynamics.

The approach of this paper is to first show that *N*-player Cournot–Bertrand games can be viewed as GSH. Notice that while the extreme cases of pure Cournot and pure Bertrand markets are GSH (in specific, a GSS and a GSC, respectively), it is not clear that *N*-player Cournot–Bertrand markets satisfy this requirement. To see this, consider a price setting firm k. Notice that if all other price competitors set a higher price, firm k will be induced to also set a higher price. However, if quantity competitors set a higher quantity, firm k will be induced to set a lower price. Thus, firm k does not best respond in a monotonic way to increases in the strategies of all opponents, and hence this game fails to be a GSH. Despite this, we show that such games can be “monotonically embedded” into a GSH, so that firms can be seen as best responding in a monotonic way. To the best of the authors’ knowledge, this paper is the first to apply monotonicity analysis to this class of games.

With this observation, the main contributions of this paper are the following: By applying recent results in the GSH literature which show that many of the existence and stability properties of equilibria in GSH are preserved under monotonic embeddings, we are able to derive conditions on the product substitutability parameter $d \in (-1, 1)$ which guarantee the existence of a unique, dominance solvable equilibrium which is stable under all adaptive dynamics. These conditions offer distinct advantages over many of the results in the existing literature. Firstly, in the case of a duopoly when goods are substitutes and firms have constant marginal costs, both our results and those of Naimzada and Tramontana (2012) guarantee the stability of a unique equilibrium as long as the degree of product substutability is such that $d < \frac{2}{\sqrt{5}}$. However, while Naimzada and Tramontana (2012) are able to guarantee stability under simple best-reply dynamics, we are able to guarantee a unique, dominance solvable equilibrium which is stable under all adaptive dynamics. Recall from Milgrom and Shannon (1994) that a learning process is adaptive as long as agents eventually begin to respond to past play by choosing undominated actions. Hence, adaptive dynamics consist of a very general and broad range of learning rules, which include best-reply dynamics as a simple case.

Secondly, we allow for non-linear cost structures in the case of two firms, and show that if costs are sufficiently convex, then uniqueness and stability are always guaranteed. We further extend the literature on stability in Cournot–Bertrand markets by offering results for the general N-firm case, as well allow for product complementarities. Finally, we study how our condition changes as either an additional quantity or price competitor enters the market. We find that, in general, the addition of a price competitor promotes stability in the case of substitute products, while the addition of a quantity competitor has the opposite effect for both substitutes and complements.

Formulating Cournot–Bertrand markets as monotone games offers several insights and advantages. As evidenced by Example 3 in Barthel and Hoffmann (2019), GSH with at least one strategic substitutes player, which we show to include Cournot–Bertrand markets, may fail to have a stable or a dominance solvable equilibrium, even if it the equilibrium is unique. Hence, methods such as the contraction mapping theorem or the conditions of Bramoullé et al. (2014), all of which guarantee the existence of a unique equilibrium, are not enough to address stability or dominance solvability. However, the monotonicity inherent in such markets can be exploited to construct a best-reply sequence which, when convergent, guarantees the existence of a unique, globally stable equilibrium which is stable under all adaptive dynamics.

This paper is organized as follows: Sect. 2 provides the model description as well as the relevant notions concerning monotone games. Section 3 contains our first result for the case of a Cournot–Bertrand duopoly, which provides a condition which contains as a special case the condition found in Naimzada and Tramontana (2012). Furthermore, we show that this condition not only implies the stability of an equilibrium under best response dynamics, but that it also implies stability under all adaptive dynamics, as well as guarantees the dominance solvability of such equilibria. Section 4 introduces the notion of a monotone embedding which allows us to extend this methodology to the case of N-firms. Finally, we study how this condition changes as more firms of each type enter the market. Section 5 concludes.

2 Model and definitions

We will follow the model in Matsumoto and Szidarovszky (2011) by assuming that there are N firms competing in a market, where the inverse demand function of firm k is given by

$$p_k = \alpha_k - q_k - d\sum_{i \neq k}^{N} q_i.$$

Here, p_k and q_k denote the price and quantity for each firm, respectively, α_k is a demand intercept, and $d \in (-1, 1)$ denotes the degree of substitutability/complementarity between the products of the firms.[1] Let m denote the number of quantity setting firms, and n be the number of price setting firms, so that $n + m = N$. We will assume that each player's action space is some interval $A_k = [\underline{K}_k, \bar{K}_k] \subset \mathbb{R}$. We also assume that each firm k has some cost of production given by

$$C_k(q_k) = c_k^1 q_k^2 + c_k^2 q_k + c_k^3,$$

where $c_k^1, c_k^2, c_k^3 \geq 0$. Also, as allowing for heterogeneous intercept terms α_k does not affect our results, we will assume that $\alpha_k = \alpha$ for all firms k to ease notation.

We will analyze this model in the framework of games of strategic heterogeneity. The next definition follows Barthel and Hoffmann (2019), but has been specifically adapted for our environment. When not distinguishing whether a firm k is a quantity or price competitor, we will denote its action as a_k, and those of its competitors as a_{-k}.

Definition 1 A game Γ is a **game of strategic heterogeneity**(**GSH**) if the following requirements are satisfied:

1. There are a finite number of players $I = \{1, 2, ...N\}$. Each player k has a strategy space $A_k = [\underline{K}_k, \bar{K}_k] \subset \mathbb{R}$, with common element a_k.
2. Each player k has a continuous profit function $\pi_k : A \to \mathbb{R}$, where A is the Cartesian product of the A_k, with common element a.
3. For each player k, π_k satisfies either increasing differences or decreasing differences in (a_k, a_{-k}).[2]

The notion of increasing (decreasing) differences captures the intuition that when all opponents take a higher strategy, it is beneficial for me to take a higher (lower) strategy. Note that each firm's profit function can be written as

$$\pi_k = p_k q_k - C_k.$$

As pointed out by Amir et al. (2017), further restrictions need to be imposed on d in the case of complementary goods to guarantee that the demand system is well-behaved. Amir et al. (2017) provide such conditions for the case of (pure) Cournot and Bertrand oligopolies. For the remainder of this paper, we will make the following similar regularity assumption for mixed Cournot–Bertrand oligopolies.

[1] If $d = 0$, the goods are unrelated. The goods are substitutes (complements) for $d > 0$ ($d < 0$). If $d = 1$ ($d = -1$), the goods are perfect substitutes (complements).

[2] Recall that π_k satisfies increasing differences in (a_k, a_{-k}) if $\frac{\partial^2 \pi_k}{\partial a_k \partial a_j} \geq 0$ for each $j \neq k$, and decreasing differences in (a_k, a_{-k}) if $\frac{\partial^2 \pi_k}{\partial a_k \partial a_j} \leq 0$ for each $j \neq k$.

Lemma 1 *Suppose that* $d > -\frac{1}{N-m}$ *for the linear inverse demand function given above. Then*

1. π_k *is strictly concave in the strategic variable* a_k *for every firm* k.
2. *The demand functions in terms of each firm's strategic variable satisfy the Law of Demand, and have non-negative intercepts.*

Proof See Appendix. □

In the duopoly case, this assumption implies that we can consider the entire interval $(-1, 1)$, which coincides with the result in Amir et al. (2017).

Then, in the case of a Cournot–Bertrand duopoly ($n = m = 1$), by allowing the quantity competitor to be firm 1 and the price competitor to be firm 2, we have that profits are continuously differentiable, and

$$\frac{\partial^2 \pi_1}{\partial q_1 \partial p_2} = d \geq (<) 0$$

and

$$\frac{\partial^2 \pi_2}{\partial p_2 \partial q_1} = -d(1 + 2c_2^1) \leq (>) 0,$$

for $d \geq (<) 0$ so that the requirements of a GSH are satisfied. One advantage of GSH is the relationship between various solution concepts, which we now define.

We follow along the lines of Barthel and Hoffmann (2019) in defining the set of serially undominated strategies, which are those strategies surviving the process of iteratively deleting strictly dominated strategies. We will say that $a_k \in A_k$ is a **strictly dominated** strategy for player k if for for some $a_k' \neq a_k$, we have that for each a_{-k}

$$\pi_k(a_k', a_{-k}) > \pi_k(a_k, a_{-k}).$$

Then, for some set of actions $M_{-k} \subset A_{-k}$, let us define

$$U_k(M) = \{a_k \in A_k \mid \forall a_k' \in A_k,\ \exists a_{-k} \in M_{-k},\ \pi_k(a_k, a_{-k}) \geq \pi_k(a_k', a_{-k})\}.$$

as the set of undominated responses to M_{-k}. Then, for $M \subset A$, we define

$$U(M) = (U_k(M_{-k}))_{\forall k},$$

where M_{-k} is the projection of M onto A_k. The set of serially undominated strategies is then the result of the following iterative process: Let $M^0 = A$, and for all $z \geq 1$, let $M^z = U(M^{z-1})$. Then the set

$$S = \bigcap_{z \geq 0} M^z$$

is defined to be the set of **serially undominated strategies**. We say that an equilibrium $a^* \in A$ is **dominance solvable** if it is the only serially undominated strategy.

To study stability, we define what it means for a sequence to be an adaptive dynamic. To that end, let

$$\mathcal{P}(T,t) = \{a^s \mid T \leq s < t\}$$

denote any sequence of play between time periods T and t. Also, for any set S, let inf S and sup S define the infimum and supremum of S in $\mathbb{R}^N$, respectively. Then, for a given $\mathcal{P}(T,t)$, we have that $\hat{P}(T,t) = [\inf \mathcal{P}(T,t), \sup \mathcal{P}(T,t)]$ is the smallest interval containing the sequence of play $\mathcal{P}(T,t)$. Then, a sequence of play $(a^s)_{s=0}^{\infty}$ is an **adaptive dynamic** if, for each $T \geq 0$, there exists some $T' \geq 0$, such that for all $t \geq T'$,

$$a^t \in [\inf U([\hat{P}(T,t)), \sup U(\hat{P}(T,t))].$$

Intuitively, a sequence of play is adaptive as long as it eventually falls within the bounds defined by the highest and lowest undominated responses to previous play. Note that this is a very inclusive definition, and incorporates learning processes such as best-response dynamics and fictitious play, among many others. We will call an equilibrium strategy profile $a^\star \in A$ **globally stable** if every non-trivial[3] adaptive dynamic $(a^s)_{s=0}^{\infty}$ converges to it. That is, starting anywhere in the strategy space, as long as players behave adaptively, they will come to learn to play the globally stable equilibrium. Proposition 1 below draws a connection between dominance solvability and global stability in GSH:

Proposition 1 (Barthel and Hoffmann 2019) *Suppose that Γ is a GSH. Then there exist lowest and highest serially undominated strategies (y and z, respectively) such that all other serially undominated strategies are included in the interval $[y, z]$. Furthermore, Γ is dominance solvable if and only if there exists a globally stable Nash equilibrium $a^\star \in A$, in which case $a^\star = y = z$.*

Proof See Theorem 1 in Barthel and Hoffmann (2019). □

Proposition 1 highlights the advantage of exploiting the monotonicity present in Cournot–Bertrand oligopolies when evaluating the stability and uniqueness of equilibria over other traditional methods. For example, while approaches such as the contraction mapping theorem or the conditions of Bramoullé et al. (2014) guarantee when a unique equilibrium exists, Example 3 in Barthel and Hoffmann (2019) shows that in a GSH, a unique equilibrium may exist which is not dominance solvable, and hence not globally stable by Proposition 1. Hence, the approach of this paper will be to derive conditions under which Γ exhibits a unique, dominance solvable equilibrium. To that end, let $I_C \subset I$ be those firms whose profit function satisfies increasing differences, and $I_S \subset I$ be those players whose profit function satisfies decreasing differences, and let $BR_k(a_{-k})$ be the best response function for firm k. Finally, let $\bar{a}_k$ and $\underline{a}_k$ be the largest and smallest elements in firm k's action space, respectively. Then, consider the following **iterative procedure**:

- $z^0 = (\bar{a}_k)_{k \in I}$, $y^0 = (\underline{a}_k)_{k \in I}$,

[3] $(a^s)_{s=0}^{\infty}$ is non-trivial as long as it is not a constant sequence.

- $z^1 = \left(\left(BR_k(\underline{a}_{-k}) \right)_{k \in I_S}, \left(BR_k(\bar{a}_{-k}) \right)_{k \in I_C} \right)$,
  $y^1 = \left(\left(BR_k(\bar{a}_{-k}) \right)_{k \in I_S}, \left(BR_k(\underline{a}_{-k}) \right)_{k \in I_C} \right)$.
- In general, for $s \geq 1$,
  $z^s = \left(\left(BR_k(y_{-k}) \right)_{k \in I_S}, \left(BR_k(z_{-k}) \right)_{k \in I_C} \right)$,
  $y^s = \left(\left(BR_k(z_{-k}) \right)_{k \in I_S}, \left(BR_k(y_{-k}) \right)_{k \in I_C} \right)$.

Lemmas 2 and 4 in Barthel and Hoffmann (2019) then show:

1. $(z^s)_{s=0}^{\infty}$ and $(y^s)_{s=0}^{\infty}$ are decreasing and increasing sequences, respectively. Furthermore, for each $s \geq 0$, $z^s \geq y^s$.
2. $z^s \to z$ and $y^s \to y$.

Therefore, as long as we can derive conditions under which z^s and y^s converge to the same point, Proposition 1 can be applied. We apply this intuition in Sect. 3 in the case of a duopoly. Section 4 then extends this analysis to the case of N firms.

3 Cournot–Bertrand duopoly

We now apply our observations above to the case of a Cournot–Bertrand duopoly. From now on, we will refer to firm 1 as the quantity competing firm, and firm 2 as the price competitor. After solving demand for each firm's strategic variable, best response functions in the case of unconstrained strategy spaces are given by

$$q_1^{\star}(p_2) = \frac{(1-d)\alpha - c_1^2}{2((1-d^2)+c_1^1)} + \frac{d}{2((1-d^2)+c_1^1)} p_2, \quad \forall p_2 \in \mathbb{R}$$
$$p_2^{\star}(q_1) = \frac{\alpha(1+2c_2^1)+c_2^2}{2(1+c_2^1)} - \frac{d(1+2c_2^1)}{2(1+c_2^1)} q_1, \quad \forall q_1 \in \mathbb{R}.$$

To define the strategy spaces, we assume that for each firm $k = 1, 2$, $A_k = [\underline{K}_k, \bar{K}_k]$, where $\bar{K}_k$ and $\underline{K}_k$ can be taken to be arbitrarily large and small, respectively. Notice that because the main results of this paper give conditions under which a unique, stable equilibrium is guaranteed to exist, this is without loss of generality. By concavity of profit functions, best responses can then be expressed as

$$BR_1(p_2) = \begin{cases} \bar{K}_1, & q_1^{\star}(p_2) > \bar{K}_1 \\ q_1^{\star}(p_2), & \text{otherwise}, \\ \underline{K}_1, & q_1^{\star}(p_2) < \underline{K}_1 \end{cases} \quad \forall p_2 \in A_2,$$
$$BR_2(q_1) = \begin{cases} \bar{K}_2, & p_2^{\star}(q_1) > \bar{K}_2 \\ p_2^{\star}(q_1), & \text{otherwise}, \\ \underline{K}_2, & p_2^{\star}(q_1) < \underline{K}_2 \end{cases} \quad \forall q_2 \in A_1.$$

With A_1 and A_2 defined, and recalling from previous discussion that for $d \geq (<) 0$, firm 1 is a strategic complements (substitutes) player, while firm 2 is a strategic substitutes (complements) player, we have now formulated the Cournot–Bertrand duopoly as a GSH.

Our first result shows that following the iterative procedure described above allows us to establish bounds on product substitutability which allow us to apply Proposition 1. We show as an example that our condition subsumes that of Naimzada and Tramontana (2012) as a special case, and more importantly allows us to draw much stronger conclusions about the resulting equilibrium. Finally, our result implies that if the costs of the quantity competitor are sufficiently convex, then the hypothesis holds regardless of degree of substitutability.

Theorem 1 *Let Γ be the Cournot–Bertrand duopoly described above and let $d \in (-1, 1)$.*[4] *Then, Γ has a unique, dominance solvable equilibrium which is globally stable under all adaptive dynamics as long as*

$$\left(\frac{d^2(1 + 2c_2^1)}{4((1 - d^2) + c_1^1)(1 + c_2^1)} \right) < 1.$$

Proof We proceed by studying the limits of the iterative procedure described above for the case when $d \geq 0$. The case when $d < 0$ follows identically, the only difference being which player is the strategic complements or substitutes player. Let $z^0 = (\bar{K}_1, \bar{K}_2)$, and $y^0 = (\underline{K}_1, \underline{K}_2)$. When $d \geq 0$, firm 1 is the strategic complements player, and hence it follows by the definition of the iterative procedure and best responses that

$$z_1^1 - y_1^1 = BR_1(\bar{K}_2) - BR_1(\underline{K}_2) \leq q_1^\star(\bar{K}_2) - q_1^\star(\underline{K}_2) \leq \frac{d}{2((1 - d^2) + c_1^1)} K,$$

where $K = \max\{\bar{K}_1 - \underline{K}_1, \bar{K}_2 - \underline{K}_2\}$. Likewise, we have for firm 2 that

$$z_2^1 - y_2^1 = BR_2(\underline{K}_1) - BR_2(\bar{K}_1) \leq p_2^\star(\underline{K}_1) - p_2^\star(\bar{K}_1) \leq \frac{d(1 + 2c_2^1)}{2(1 + c_2^1)} K.$$

By way of induction, suppose that for $s \geq 1$ odd, we have for firm 1 that

$$z_1^s - y_1^s \leq \left(\frac{d}{2((1 - d^2) + c_1^1)} \right)^{\frac{s+1}{2}} \left(\frac{d(1 + 2c_2^1)}{2(1 + c_2^1)} \right)^{\frac{s-1}{2}} K, \tag{1}$$

and for firm 2,

$$z_2^s - y_2^s \leq \left(\frac{d}{2((1 - d^2) + c_1^1)} \right)^{\frac{s-1}{2}} \left(\frac{d(1 + 2c_2^1)}{2(1 + c_2^1)} \right)^{\frac{s+1}{2}} K. \tag{2}$$

[4] Recall from the discussion following Lemma 1 that the regularity assumptions on the profit functions are satisfied for $d \in (-1, 1)$ in the duopoly case.

Then, for firm 1, we have that for such $s \geq 1$ odd,

$$z_1^{s+1} - y_1^{s+1} = BR_1(z_2^s) - BR_1(y_2^s) \leq q_1^\star(z_2^s) - q_1^\star(y_2^s) = \left(\frac{d}{2((1-d^2)+c_1^1)}\right)(z_2^s - y_2^s)$$

$$\leq \left(\frac{d}{2((1-d^2)+c_1^1)}\right)\left(\frac{d}{2((1-d^2)+c_1^1)}\right)^{\frac{s-1}{2}}\left(\frac{d(1+2c_2^1)}{2(1+c_2^1)}\right)^{\frac{s+1}{2}}K$$

$$= \left(\frac{d^2(1+2c_2^1)}{4((1-d^2)+c_1^1)(1+c_2^1)}\right)^{\frac{s+1}{2}}K.$$

Likewise, for firm 2, we have that for such $s \geq 1$ odd,

$$z_2^{s+1} - y_2^{s+1} = BR_2(y_1^s) - BR_2(z_1^s) \leq p_2^\star(y_1^s) - p_2^\star(z_1^s) = \left(\frac{d(1+2c_2^1)}{2(1+c_2^1)}\right)(z_1^s - y_1^s)$$

$$\leq \left(\frac{d(1+2c_2^1)}{2(1+c_2^1)}\right)\left(\frac{d}{2((1-d^2)+c_1^1)}\right)^{\frac{s+1}{2}}\left(\frac{d(1+2c_2^1)}{2(1+c_2^1)}\right)^{\frac{s-1}{2}}K$$

$$= \left(\frac{d^2(1+2c_2^1)}{4((1-d^2)+c_1^1)(1+c_2^1)}\right)^{\frac{s+1}{2}}K.$$

It is readily checked that one more application shows that the next odd terms satisfies Eqs. (1) and (2), respectively, completing the induction step. Therefore, as long as

$$\left(\frac{d^2(1+2c_2^1)}{4((1-d^2)+c_1^1)(1+c_2^1)}\right) < 1.$$

we have that the even terms of the sequences $(z^s)_{s=0}^\infty$ and $(y^s)_{s=0}^\infty$ converge as $s \to \infty$. Finally, because $(z^s)_{s=0}^\infty$ and $(y^s)_{s=0}^\infty$ are decreasing and increasing sequences, respectively, we have that for all $s \geq 1$ and $j \geq 1$,

$$z^{s+j} - y^{s+j} \leq z^s - y^s,$$

which implies that the entire sequences $(z^s)_{s=0}^\infty$ and $(y^s)_{s=0}^\infty$ converge. Because $z^s \to z$ and $y^s \to y$, it follows that $z = y$, and hence Proposition 1 can be applied. For $d < 0$, we now have that firm 1 is the strategic substitutes player, while firm 2 is the strategic complements player. The remainder of the proof follows similarly for this case. □

We now discuss the strengths of Theorem 1: first, Naimzada and Tramontana (2012) study the same Cournot–Bertrand duopoly specification, but in the case of constant marginal costs, so that $c_k^1 = 0$ for both firms as well as require that $d \geq 0$. They find that the unique equilibrium is globally stable under best-reply dynamics as long as $d < \frac{2}{\sqrt{5}}$. Observe that if we set $c_k^1 = 0$ for both firms, then our condition is exactly equivalent to theirs. Note, however, that we are not only able to conclude stability under best-reply dynamics, but under all adaptive dynamics, as well as guarantee that the unique equilibrium is serially undominated. Furthermore, we are able to account for more general cost structures by allowing for positive c_k^1 terms in both the substitute and complementary goods case. In particular, notice that if

$$c_1^1 > \frac{1 + 2c_2^1}{4(1 + c_2^1)},$$

so that the quantity competitor has sufficiently convex costs, then Theorem 1 holds for all $d \in [-1, 1]$.[5]

Section 4 extends the methodology used in Theorem 1 to the case of N-firm Cournot–Bertrand oligopolies.

4 N-firm Cournot–Bertrand oligopoly

We now extend the methodology used in the the duopoly case to study the stability and dominance solvability of equilibria in N-firm Cournot–Bertrand oligopolies, where $n, m \geq 1$ and $n + m > 2$, as well as study the impact of the addition of both types of firms to the market. Here, we will assume that firms have constant marginal costs which, because these will play no role in our results, will be assumed to equal 0 for the sake of notation. Then, after solving demand for the strategic variable of each firm, we have that the best responses for each quantity competitor and price competitor $\forall a_{-k} \in \mathbb{R}^{N-1}$ in the case of unconstrained strategy sets are given, respectively, by

$$q_k^\star(a_{-k}) = \frac{\alpha(1-d) + d\Big(\sum_{i=m+1}^{N} p_i - (1-d)\sum_{i=1, i\neq k}^{m} q_i\Big)}{2(1-d)(1+nd)},$$

$$p_k^\star(a_{-k}) = \frac{\alpha(1-d) + d\Big(\sum_{i=m+1, i\neq k}^{N} p_i - (1-d)\sum_{i=1}^{m} q_i\Big)}{2(1+(n-2)d)}.$$

Notice that the denominators of both best response functions are positive under our regularity assumption from Lemma 1.[6] Hence, the slope of the best response functions with respect to each opponent's strategic variable solely depends on the sign of d. As in the duopoly case, we will assume that for each firm $k \in I$, $A_k = [\underline{K}_k, \bar{K}_k]$, for some arbitrarily large and small $\bar{K}_k$ and $\underline{K}_k$, respectively. By concavity of profit functions, best responses can then be expressed in a manner similar to the duopoly case.

[5] This can be seen by setting $d = 1$ ($d = -1$) and noting that $\frac{d^2(1+2c_2^1)}{4((1-d^2)+c_1^1)(1+c_2^1)}$ is increasing (decreasing) in d for $d \geq 0$ ($d < 0$).

[6] This observation follows immediately for the quantity players when $d > -\frac{1}{N-m} = -\frac{1}{n}$. For the price competitors, notice that the term $1 + (n-2)d$ is equal to $1 - d > 0$ for all $d \in (-1, 1)$ for $n = 1$ and 1 for $n = 2$. For all $n > 2$, we have that $-\frac{1}{n-2} < -\frac{1}{n}$ and hence $1 + (n-2)d > 0$ for all $d > -\frac{1}{N-m}$ in this case as well.

One crucial observation is that this game is *not* a GSH, and therefore, Proposition 1 cannot be applied as in the duopoly case. To see this, notice that both price and quantity competitors best respond in a monotone decreasing way to other quantity competitors, but in a monotone increasing way to other price competitors. That is, firms do not respond in a monotonic way to an increase in the decisions of all other firms, and hence the definition of a GSH is not satisfied. Our next result shows that Cournot–Bertrand oligopolies can be "monotonically embedded" into a GSH so that Proposition 1 can still be applied. We adapt to our setting the definition of a monotonic embedding given in Barthel and Hoffmann (2019):

Definition 2 Let $\Gamma = \{I, (A_k, \pi_k)_{k\in I}\}$ be any game, where each $A_k \subset \mathbb{R}$ is a closed interval. We call $\widetilde{\Gamma} = \{I, (\widetilde{A}_k, \widetilde{\pi}_k, f_k)_{k\in I}\}$ a **monotonic embedding** of Γ if the following conditions hold for each firm $k \in I$:

1. $\widetilde{A}_k$ is a set of actions for firm k, which is a closed interval in $\mathbb{R}$.
2. $\widetilde{\pi}_k : \widetilde{A} \to \mathbb{R}$ is continuous, and satisfies either increasing or decreasing differences in $(\widetilde{a}_k; \widetilde{a}_{-k})$.
3. $f_k : A_k \to \widetilde{A}_k$ is a homeomorphism, and either strictly increasing or strictly decreasing.
4. For each $a \in A$, we have that $\widetilde{\pi}_k(f(a)) = \pi_k(a)$ (where $f(a) \equiv (f_k(a_k)_{k\in I})$).

To understand this definition, notice that (1) and (2) above first establish that the game $\widetilde{\Gamma}$ is a GSH. The last two requirements then define what it means for some game Γ to be embedded monotonically into $\widetilde{\Gamma}$.

Proposition 2 below shows that Cournot–Bertrand oligopolies Γ can be monotonically embedded into a GSH $\widetilde{\Gamma}$. The importance of this embedding is that Proposition 1 can then be applied to Γ by studying $\widetilde{\Gamma}$.

Proposition 2 *Let Γ be the N-firm Cournot–Bertrand oligopoly described above. Then there exists a monotonic embedding $\widetilde{\Gamma}$ of Γ. Furthermore, Γ is dominance solvable if and only if $\widetilde{\Gamma}$ is dominance solvable. Also, if $a \in A$ is the dominance solvable strategy of Γ, it is also globally stable under every adaptive dynamic in Γ.*

Proof See Appendix. □

Thus, we see that the Cournot–Bertrand oligopoly Γ can be evaluated as a GSH, where each price competitor is a "strategic complements (substitutes)" player, and each quantity competitor is a "strategic substitutes (complements)" player for $d \geq (<) 0$. Then, conditions which imply dominance solvability in the transformed game $\widetilde{\Gamma}$ will also guarantee dominance solvability in Γ, where the resulting dominance solvable equilibrium is guaranteed to be stable under adaptive dynamics. To analyze the transformation described in Proposition 2, let $I_P \subset I$ and $I_Q \subset I$ denote the price and quantity competing firms, respectively. Solving for unconstrained optimal responses gives, for each price competitor k:

$$\tilde{p}_k^{\star}(\tilde{a}_{-k}) = \frac{\alpha(1-d) + d\Big(\sum_{i\in I_P, i\neq k}\tilde{p}_i + (1-d)\sum_{i\in I_Q}\tilde{q}_i\Big)}{2(1+(n-2)d)}, \quad \forall \tilde{a}_{-k}\in\mathbb{R}^{N-1},$$

and for each quantity competitor k,

$$\tilde{q}_k^{\star}(\tilde{a}_{-k}) = \frac{-\alpha(1-d) - d\Big(\sum_{i\in I_P}\tilde{p}_i + (1-d)\sum_{i\in I_Q, i\neq k}\tilde{q}_i\Big)}{2(1-d)(1+nd)}, \quad \forall \tilde{a}_{-k}\in\mathbb{R}^{N-1}.$$

Note that in the embedding $\widetilde{\Gamma}$ of the Cournot–Bertrand oligopoly Γ, we have for price competitors that $\widetilde{A}_k = A_k = [\underline{K}_k, \bar{K}_k]$, while for quantity competitors, $\widetilde{A}_k = -A_k = [-\bar{K}_k, -\underline{K}_k]$. Thus, in $\widetilde{\Gamma}$, best responses for quantity competitors and price competitors can be written, respectively, as

$$BR_k(\tilde{a}_{-k}) = \begin{cases} -\underline{K}_k, & \tilde{q}_k^{\star}(\tilde{a}_{-k}) > -\underline{K}_k \\ \tilde{q}_k^{\star}(\tilde{a}_{-k}), & \text{otherwise}, \quad \forall \tilde{a}_{-k}\in\tilde{A}_{-k}. \\ -\bar{K}_k, & \tilde{p}_k^{\star}(\tilde{a}_{-k}) < -\bar{K}_k \end{cases}$$

$$BR_k(\tilde{a}_{-k}) = \begin{cases} \bar{K}_k, & \tilde{p}_k^{\star}(\tilde{a}_{-k}) > \bar{K}_k \\ \tilde{p}_k^{\star}(\tilde{a}_{-k}), & \text{otherwise}, \quad \forall \tilde{a}_{-k}\in\tilde{A}_{-k}. \\ \underline{K}_k, & \tilde{p}_k^{\star}(\tilde{a}_{-k}) < \underline{K}_k \end{cases}$$

We now come to our second main result, which extends the methodology used in the duopoly case to the oligopoly case.

Theorem 2 *Let Γ be the N-firm Cournot–Bertrand oligopoly described above, and suppose that $d\in(-\frac{1}{N-m}, 1)$. Then Γ has a unique, dominance solvable equilibrium which is globally stable under all adaptive dynamics as long as*[7]

$$\left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)}\right) < 1,$$

where

1. $\gamma_1(n,m,d) = \max\{d(n+(1-d)(m-1)), -d((n-1)+(1-d)m)\}$,
2. $\gamma_2(n,m,d) = \min\{(1-d)(1+nd), 1+(n-2)d\}$.

Proof By Proposition 2, it is enough to show that this condition implies that the monotonic embedding $\widetilde{\Gamma}$ is dominance solvable. To do this, we will follow the same iterative procedure described in Sect. 3. We will first show the $d\geq 0$ case. The proof for $d<0$ follows similarly by simply adjusting for the fact that in this case, price (quantity) competitors assume the role of the strategic substitutes (complements) players.

To that end, define $K = \max_{k\in I}\{\bar{K}_k - \underline{K}_k\}$. Then, for each $k\in I_P$, we have

[7] It is straightforward to show that for $d<0$, $\gamma_1(n,m,d) = -d(n-1) - d(1-d)m$ and $\gamma_2(n,m,d) = (1-d)(1+nd)$, and for $d\geq 0$, $\gamma_1(n,m,d) = dn + d(1-d)(m-1)$.

$$\begin{aligned} z_k^1 - y_k^1 &= BR_k((\bar{K}_i)_{i\neq k}) - BR_k((\underline{K}_i)_{i\neq k}) \leq q_k^\star((\bar{K}_i)_{i\neq k}) - q_k^\star((\underline{K}_i)_{i\neq k}) \\ &= \frac{d}{2(1+(n-2)d)} \sum_{i\in I_P, i\neq k} (\bar{K}_i - \underline{K}_i) + \frac{d(1-d)}{2(1+(n-2)d)} \sum_{i\in I_Q} (\bar{K}_i - \underline{K}_i) \\ &\leq \frac{d(n-1) + d(1-d)m}{2(1+(n-2)d)} K. \end{aligned}$$

Notice that $\gamma_2(n,m,d) \leq 1 + (n-2)d$ by definition, and that for $d \geq 0$, $d(n-1) + d(1-d)m \leq dn + d(1-d)(m-1) \leq \gamma_1(n,m,d)$. Hence,

$$z_k^1 - y_k^1 \leq \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right) K.$$

We can likewise show that for all $k \in I_Q$,

$$z_k^1 - y_k^1 \leq \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right) K.$$

Suppose by way of induction that for each $k \in I$, and some $s \geq 1$,

$$z_k^s - y_k^s \leq \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right)^s K.$$

Then, for each $k \in I_P$, we have

$$\begin{aligned} z_k^{s+1} - y_k^{s+1} &= BR_k((z_i^{s+1})_{i\neq k}) - BR_k((y_i^{s+1})_{i\neq k}) \leq q_k^\star((z_i^{s+1})_{i\neq k}) - q_k^\star((y_i^{s+1})_{j\neq k}) \\ &= \frac{d}{2(1+(n-2)d)} \sum_{i\in I_P, i\neq k} (z_i^s - y_i^s) + \frac{d(1-d)}{2(1+(n-2)d)} \sum_{i\in I_Q} (z_i^s - y_i^s) \\ &\leq \frac{d(n-1)}{2(1+(n-2)d)} \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right)^s K + \frac{d(1-d)m}{2(1+(n-2)d)} \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right)^s K \\ &= \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right)^s \left(\frac{\gamma_2(n,m,d)}{2\gamma_1(n,m,d)} \right) K = \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right)^{s+1} K, \end{aligned}$$

where the first inequality follows from the induction hypothesis, and the second to last equality once again follows from the fact that $\gamma_2(n,m,d) \leq 1 + (n-2)d$ by definition, and that for $d \geq 0$, $d(n-1) + d(1-d)m \leq dn + d(1-d)(m-1) \leq \gamma_1(n,m,d)$. Because a similar argument holds for all $k \in I_Q$, we then have that for all $k \in I$, and $s \geq 1$,

$$z_k^s - y_k^s \leq \left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right)^s K.$$

Hence, as long as

$$\left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)} \right) < 1,$$

we have that the iterative procedure converges, giving the result. □

This theorem shows that existence and stability can be addressed simply from the underlying parameters of the game: n, m, and d. To the best of the authors' knowledge, this is the first condition to address the general N-firm Cournot–Bertrand oligopoly. As an example of an application of Theorem 2, note that in the case of $n = 2$ and $m = 1$ and $d \geq 0$, we have that $\gamma_2(n,m,d) = 1$ for all $d < \frac{1}{2}$, so that Theorem 2 is satisfied as long as $d < 1$, which is of course true for all $d < \frac{1}{2}$. For $d \geq \frac{1}{2}$, we have that $\gamma_2(n,m,d) = (1-d)(1+2d)$, and hence Theorem 2 is satisfied as long as

$$\frac{d}{(1-d)(1+2d)} < 1,$$

which is true for all $d < \frac{1}{\sqrt{2}}$. Hence, we can conclude that in the case of $n = 2$ and $m = 1$, existence and stability can be guaranteed for all $d \in [0, \frac{1}{\sqrt{2}})$. In fact, is is straightforward to verify that for $m = 1$ and arbitrary $n \geq 1$, Theorem 2 is satisfied for all $d < \frac{1}{2}$. Notice, however, that Theorem 2 does not generalize Theorem 1 in the case when $m = n = 1$.

Our last result allows us to address how stability is affected by the entrance of either a quantity or price competing firm. For each $n, m \geq 1$, we let $P(n, m)$ denote those "permissible" degrees of product substitutability d such that the condition in Theorem 1 is satisfied in the duopoly case, or the condition in Theorem 2 is satisfied for the general oligopoly case.

We now show that with the addition of a quantity competitor, the set $P(n, m)$ becomes smaller, and that the opposite is true after the addition of a price competitor for $d \leq \frac{1}{2}$ when the products are substitutes, while the addition of either type of players negatively affects the size of $P(n, m)$ in the case of when goods are complements. To make comparisons involving the duopoly case, when referring to the condition in Theorem 1, we will assume that each firm has constant marginal costs, so that the $c_k^1 = 0$ for each firm, $k = 1, 2$. When considering cases of $d < 0$, we will also assume the regularity condition stated in Lemma 1.

Theorem 3 *Let Γ be an N–firm Cournot–Bertrand oligopoly, and let $n, m \geq 1$ be given. Then,*

1. *$P(n, m)$ is an interval containing* 0.
2. $P(n, m+1) \subset P(n, m)$.
3. (a) *For* $d \in [0, \frac{1}{2})$, $P(n, m) \subset P(n+1, m)$.
   (b) *For* $d \in (-\frac{1}{N-m}, 0)$, $P(n+1, m) \subset P(n, m)$.

Proof See Appendix. □

5 Conclusion

This paper studies the existence and stability of Nash equilibria in N-firm Cournot–Bertrand oligopolies. Our approach differs from those in the existing literature in that we first show that such games can be viewed as GSH, where each firm monotonically responds to opponents' actions. By framing this problem in a monotone framework, we derive conditions on product substitutability $d \in (-1, 1)$ which guarantees the existence of a unique, dominance solvable equilibrium under all adaptive dynamics. In the duopoly case, our condition coincides with the bounds in Naimzada and Tramontana (2012), who study stability of Nash equilibrium under best response dynamics. However, our condition implies stability not only under best response dynamics, but under all adaptive dynamics. Moreover, contrary to other studies in the literature that focus exclusively on substitutes, our result allows for both complementary and substitute goods. We then generalize these results to the N-firm mixed oligopoly case. Finally, we explore how our stability condition changes as additional price or quantity competitors enter the market.

Appendix

Proof of Lemma 1

Proof To prove (1.), we will first express demand in terms of each firm's strategic variable. Let $I_Q \subset I$ denote the set of all quantity competitors and $I_P \subset I$ be the set of all price competitors. Then, for each quantity competitor $k \in I_Q$,

$$p_k = A - bq_k - c\sum_{j\in I_Q, j\neq k} q_j + f\sum_{j\in I_P} p_j,$$

where

$$A = \frac{(1-d)\alpha}{1+(N-m-1)d}, \qquad b = \frac{(1-d)(1+(N-m)d)}{1+(N-m-1)d},$$
$$c = \frac{d(1-d)}{1+(N-m-1)d}, \qquad f = \frac{d}{1+(N-m-1)d},$$

and for each $k \in I_P$,

$$q_k = \hat{A} - \hat{b}p_k - \hat{c}\sum_{j\in I_Q} q_j + \hat{f}\sum_{j\in I_P, j\neq k} p_j$$

where

$$\hat{A} = \frac{(1-d)\alpha}{1+(N-m-1)d}, \qquad \hat{b} = \frac{1+(N-m-2)d}{(1-d)(1+(N-m-1)d)},$$
$$\hat{c} = \frac{d(1-d)}{(1-d)(1+(N-m-1)d)}, \qquad \hat{f} = \frac{d}{(1-d)(1+(N-m-1)d)}.$$

Then each quantity competitor k's profit function is given by

$$\pi_k = p_k \cdot q_k = \left(A - bq_k - c \sum_{j \in I_Q, j \neq k} q_j + f \sum_{j \in I_P} p_j \right) q_k, \tag{3}$$

while each price competitor k has the profit function

$$\pi_k = p_k \cdot q_k = p_k \left(\hat{A} - \hat{b} p_k - \hat{c} \sum_{j \in I_Q} q_j + \hat{f} \sum_{j \in I_P, j \neq k} p_j \right). \tag{4}$$

Hence, for all $k \in I_Q$

$$\frac{\partial^2 \pi_k}{\partial q_k^2} = -2b = -2 \frac{(1-d)(1+(N-m)d)}{1+(N-m-1)d},$$

while for all $k \in I_P$

$$\frac{\partial^2 \pi_k}{\partial p_k^2} = -2\hat{b} = -2 \frac{1+(N-m-2)d}{(1-d)(1+(N-m-1)d)}.$$

Notice that for $N-m=1$, the terms $1+(N-m)d$, $1+(N-m-1)d$ and $1+(N-m-2)d$ are all positive for $d \in (-1,1)$. For $N-m=2$, $1+(N-m)d > 0$ for $d > -0.5 = -\frac{1}{N-m}$, while the other two terms are positive for all $d \in (-1,1)$. Finally, since for $N-m > 2$ we have that $-\frac{1}{N-m} > -\frac{1}{N-m-1} > -\frac{1}{N-m-2}$, we can conclude that $\frac{\partial^2 \pi_k}{\partial q_k^2} < 0$ for all quantity competitors k and $\frac{\partial^2 \pi_k}{\partial p_k^2} < 0$ for all price competitors k for $d > -\frac{1}{N-m}$ for all $N \geq 2$ and $m \geq 1$.

To show Part (2.), notice that the demand functions satisfy the Law of Demand with a positive intercept when $A, \hat{A}, b, \hat{b} > 0$, which by the same argument as above holds for $-\frac{1}{N-m} < d < 1$.

□

Proof of Proposition 2

Proof We first show the existence of a monotone embedding. Define $\widetilde{\Gamma}$ in the following way: For each price competitor k, let $\widetilde{A}_k = A_k$, and let $f_k : A_k \to \widetilde{A}_k$ be defined as $f_k(a_k) = a_k$. For each quantity competitor k, let $\widetilde{A}_k = -A_k$, and let $f_k : A_k \to \widetilde{A}_k$ be defined as $f_k(a_k) = -a_k$. It follows immediately that each each f_k is a homeomorphism and either strictly increasing or decreasing. Thus, because $f : A \to \widetilde{A}$ is a bijection, we can describe each $\widetilde{a} \in \widetilde{A}$ as $\widetilde{a} = f(a)$ for the appropriate $a \in A$.

To define the $\widetilde{\pi}_k : \widetilde{A} \to \mathbb{R}$, let us make the convention that for each $a \in \mathbb{R}^N$, $a = (p, q)$, where $p \in \mathbb{R}^n$ are those strategies from price competitors, and $q \in \mathbb{R}^m$ are those strategies from quantity competitors. Then, for each firm k define for each $a = (p, q) \in \mathbb{R}^N$,

$$\widetilde{\pi}_k(a) = \pi_k(p, -q).$$

It then follows that for all $a = (p, q) \in A$,

$$\widetilde{\pi}_k(f(p, q)) = \pi_k(f(p), -f(q)) = \pi_k(p, q),$$

and that the $\widetilde{\pi}_k$ are continuous on $\widetilde{A}$. Specifically, for the profit functions π_k given in equations (4) and (5) above, we have that for each price competitor k

$$\widetilde{\pi}_k = \widetilde{p}_k\left(\hat{A} - \hat{b}\widetilde{p}_k + \hat{c}\sum_{j\in I_Q}\widetilde{q}_j + \hat{f}\sum_{j\in I_P, j\neq k}\widetilde{p}_j\right)$$

and for each quantity competitor k

$$\widetilde{\pi}_k = \left(-A - b\widetilde{q}_k - c\sum_{j\in I_Q, j\neq k}\widetilde{q}_j - f\sum_{j\in I_P}\widetilde{p}_j\right)\widetilde{q}_k.$$

Then, for price competitors k and each $j \neq k$

$$\frac{\partial^2\widetilde{\pi}_k}{\partial\widetilde{p}_k\partial\widetilde{q}_j} = \hat{c} = \frac{d(1-d)}{(1-d)(1+(N-m-1)d)} \quad \text{and}$$
$$\frac{\partial^2\widetilde{\pi}_k}{\partial\widetilde{p}_k\partial\widetilde{p}_j} = \hat{f} = \frac{d}{(1-d)(1+(N-m-1)d)},$$

while for each quantity competitor k and each $j \neq k$

$$\frac{\partial^2\widetilde{\pi}_k}{\partial\widetilde{q}_k\partial\widetilde{q}_j} = -c = \frac{-d(1-d)}{1+(N-m-1)d} \quad \text{and}$$
$$\frac{\partial^2\widetilde{\pi}_k}{\partial\widetilde{q}_k\partial\widetilde{p}_j} = -f = \frac{-d}{1+(N-m-1)d}.$$

Then, for each $\widetilde{a} \in \widetilde{A}$, we have that for each price competitor k, and each $j \neq k$,

$$\frac{\partial^2\widetilde{\pi}_k}{\partial\widetilde{a}_j\partial\widetilde{a}_k} \geq (\leq) 0,$$

so that increasing (decreasing) differences is satisfied for $d \geq (\leq) 0$. For each quantity competitor k, and each $j \neq k$, we have

$$\frac{\partial^2\widetilde{\pi}_k}{\partial\widetilde{a}_j\partial\widetilde{a}_k} \leq (\geq) 0,$$

so that decreasing (increasing) differences is satisfied for $d \geq (\leq) 0$, completing the proof.

The fact that if $\widetilde{\Gamma}$ is a monotone embedding of Γ implies that $\widetilde{\Gamma}$ is dominance solvable if and only if Γ is, and $a \in A$ is the dominance solvable strategy of Γ

implies that it is globally stable under every adaptive dynamic in Γ, is proven in Barthel and Hoffmann (2019). □

Proof of Theorem 3

Proof For the case of a duopoly, when $n = m = 1$, it is immediate that the condition

$$\frac{d^2}{4(1-d^2)} < 1 \tag{5}$$

from Theorem 1 is satisfied at $d = 0$, and because this term is monotone in d, it follows that $P(1,\ 1)$ is an interval. For the case of more than two firms, first note that in the case of $d < 0$,

$$\left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)}\right) = \left(\frac{-d((n-1)+(1-d)m)}{2(1-d)(1+nd)}\right),$$

which is equal to 0 at $d = 0$. It is straightforward to check that the first derivative of this expression with respect to d is negative as long as the regularity condition $d > -\frac{1}{N-m}$ of Lemma 1 is satisfied, and hence part 1 is satisfied for $d < 0$.

For $d \geq 0$, notice that

$$\gamma_1(n,m,d) = d(n + (1-d)(m-1)).$$

It follows that

$$\left(\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)}\right) = \left(\frac{d(n+(1-d)(m-1))}{2\gamma_2(n,m,d)}\right)$$

is equal to 0 at $d = 0$. Observe that $\gamma_2(n,m,d)$ is equal to $1 + (n-2)d$ whenever $nd < 1$, and equal to $(1-d)(1+nd)$ otherwise. To complete the proof of part 1, it is then sufficient to show that $\frac{\gamma_1(n,m,d)}{2\gamma_2(n,m,d)}$ is monotone increasing in d on [0, 1] in both cases. First, it is straightforward to check that the first derivative of

$$\frac{d(n+(1-d)(m-1))}{2(1-d)(1+nd)}$$

with respect to d is non-negative for all n and d. It, therefore, only remains to show that

$$\frac{d(n+(1-d)(m-1))}{2(1+(n-2)d)}$$

is increasing whenever $nd < 1$. Notice that the numerator of the first derivative of this expression with respect to d can be written as

$$2(n + (m-1)(1 - 2d - d^2(n-2))),$$

and hence the derivative is non-negative as long as

$$1 - 2d - d^2(n-2) \geq 0.$$

If $n = 1$, this expression reduces to $(d-1)^2$, which is always non-negative. Now suppose that $n \geq 2$. Then $nd < 1$ implies that

$$1 - 2d - d^2(n-2) \geq 2d^2 - 3d + 1,$$

where the right hand side is non-negative for $d < \frac{1}{2}$. However, this is true, since $nd < 1$ and $n \geq 2$ implies $d < \frac{1}{2}$, proving part 1.

For part 2, we first show $P(1,2) \subset P(1,1)$. Notice that for $d \geq 0$, we have that $d \in P(1,1) \subset P(1,2)$ if

$$\frac{d^2}{4(1-d^2)} \leq \frac{d + d(1-d)}{2(1-d)},$$

which is readily verified. For $d < 0$, we have that $d \in P(1,1) \subset P(1,2)$ if

$$\frac{d^2}{4(1-d^2)} \leq \frac{-d}{1+d},$$

which is once again readily verified. The general oligopoly case for part 2 follows immediately, as $\gamma_2(n,m,d)$ does not depend on m, and $\gamma_1(n,m,d)$ is increasing in m for all $d \in [-1,1]$.

For part 3, consider first the case when $d \geq 0$. First, it is straightforward to verify that we first show $[0,\frac{1}{2}) \subset P(1,1)$ and $[0,\frac{1}{2}) \subset P(2,1)$, and hence it is trivially true that $P(1,1) \subset P(2,1)$ for $d < \frac{1}{2}$. For the general oligopoly case, first notice that

$$\frac{dn + d(1-d)(m-1)}{2(1-d)(1+nd)} < 1$$

holds if and only if

$$d(1-d)(m-1) < 2(1-d) + nd(1-2d).$$

Thus, when $d < \frac{1}{2}$, if this inequality holds for n, it continues to hold for $n+1$. Also,

$$\frac{dn + d(1-d)(m-1)}{2(1+(n-2)d)} < 1,$$

holds if and only if

$$d(1-d)(m-1) < 2 + d(n-4).$$

Hence, if this expression holds for n, it continues to hold for $n+1$.

To complete the argument, notice that because $\gamma_2(n,m,d)$ is the minimum of $(1-d)(1+nd)$ and $1+(n-2)d$, then if

$$\frac{dn + d(1-d)(m-1)}{2\gamma_2(n,m,d)} < 1$$

holds, it continues to hold if we replace $\gamma_2(n,m,d)$ with either $(1-d)(1+nd)$ or $1+(n-2)d$. By the above arguments, it follows that the inequality still holds for both $(1-d)(1+nd)$ and $1+(n-2)d$ as we increase n, and hence for $\gamma_2(n,m,d)$ as well.

Now consider the case when $d<0$. Once again, we have that $d \in P(1,1)$ as long as equation 3 is satisfied, while $d \in P(2,1)$ if and only if

$$\frac{-2d+d^2}{2(1-d)(1+2d)} < 1$$

It is straightforward to verify that for $d<0$,

$$\frac{d^2}{4(1-d^2)} \leq \frac{-2d+d^2}{2(1-d)(1+2d)}$$

Hence, we have that $P(2,1) \subset P(1,1)$. For the general oligopoly case, it is readily checked that the expression

$$\frac{-d(n-1)-d(1-d)m}{2(1-d)(1+nd)}$$

is increasing in n, and hence it follows that $P(n+1,m) \subset P(n,m)$. □

References

Amir, R. (1996). Cournot oligopoly and the theory of supermodular games. *Games and Economic Behavior*, *15*(2), 132–148.

Amir, R., Erickson, P., & Jin, J. (2017). On the microeconomic foundations of linear demand for differentiated products. *Journal of Economic Theory*, *169*, 641–665.

Amir, R., & Evstigneev, I. V. (2018). A new look at the classical Bertrand duopoly. *Games and Economic Behavior*, *109*, 99–103.

Askar, S. (2014). On Cournot–Bertrand competition with differentiated products. *Annals of Operations Research*, *223*(1), 81–93.

Barthel, A. C., & Hoffmann, E. (2019). Rationalizability and learning in games with strategic heterogeneity. *Economic Theory*, *67*(3), 565–587.

Bertrand, J. (1883). Révue de la Théorie Mathématique de la Richesse Sociale et des Recherches sur les Principles Mathématiques de la Théorie des Richesses. *Journal des Savants*, *67*, 499–508.

Bramoullé, Y., Kranton, R., & D'Amours, M. (2014). Strategic interaction and networks. *American Economic Review*, *104*(3), 898–930.

Cournot, A. A. (1838). Recherches sur les principes mathématiques de la théorie des richesses par Augustin Cournot. chez L. Hachette

Matsumoto, A., Szidarovszky, F. (2011). Mixed Cournot–Bertrand competition in n-firm differentiated oligopolies. Unpublished manuscript, Department of Economics, Chuo University, Tokyo.

Milgrom, P., & Shannon, C. (1994). Monotone comparative statics. *Econometrica: Journal of the Econometric Society*, *62*(1), 157–180.

Naimzada, A., & Tramontana, F. (2012). Dynamic properties of a Cournot–Bertrand duopoly game with differentiated products. *Economic Modelling*, *29*(4), 1436–1439.

Singh, N., & Vives, X. (1984). Price and quantity competition in a differentiated duopoly. *The RAND Journal of Economics*, *15*(4), 546–554.

Tremblay, C. H., & Tremblay, V. J. (2011). The Cournot–Bertrand model and the degree of product differentiation. *Economics Letters*, *111*(3), 233–235.

Tremblay, V. J., Tremblay, C. H., & Isariyawongse, K. (2013). Cournot and Bertrand competition when advertising rotates demand: The case of Honda and Scion. *International Journal of the Economics of Business*, *20*(1), 125–141.

Tremblay, C. H., Tremblay, M. J., & Tremblay, V. J. (2011). A general Cournot–Bertrand model with homogeneous goods. *Theoretical Economics Letters*, *1*(2), 38.

Vives, X. (2001). Oligopoly pricing: Old ideas and new tools. MIT Press