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Probabilities of electoral outcomes: from three-candidate to four-candidate elections

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Abstract
The main purpose of this paper is to compute the theoretical likelihood of some electoral outcomes under the impartial anonymous culture in four-candidate elections by using the last versions of software like LattE or Normaliz. By comparison with the three-candidate case, our results allow to analyze the impact of the number of candidates on the occurrence of these voting outcomes.

Keywords Voting rules · Voting paradoxes · Condorcet efficiency · Condorcet loser · Manipulability

1 Introduction

A significant part of voting theory is concerned with the computation of the likelihood of various electoral outcomes, including voting paradoxes. The basic motivation for these studies is of course to determine whether these possible paradoxical events might actually pose real threats to election; a good illustration of this line of research is the book by Gehrlein (2006), entirely devoted to the famous Condorcet's paradox. Another possible motivation is to measure and compare the ability of alternative voting rules to meet some normative criteria, often based on majority principle (see, e.g., Gehrlein and Lepelley 2011, 2017).

In the literature, the most often used probabilistic model for computing the likelihood of these events is the IAC model, introduced by Gehrlein and Fishburn (1976), with IAC standing for Impartial Anonymous Culture. IAC condition assumes that every voting situation is equally likely to occur, a voting situation being defined as a distribution of the voters on the possible preferences. The IAC computations have recently made substantial progress using the connection between IAC, on one hand, and Ehrhart's theory on the other hand (see Huang and Chua 2000; Wilson and Pritchard 2007; Lepelley et al. 2008). However, with some notable exceptions

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(Gehrlein 2001; Schürmann 2013; Brandt et al. 2016; Diss and Doghmi 2016, Bubboloni et al. 2018; and very recently, Bruns et al. 2019; Brandt et al. 2019; Diss and Mahajne 2019; Diss et al. 2019), the results available in the literature only deal with three-candidate elections, not because it is the most interesting case but due to the difficulties arising when considering more than three candidates. The first goal of this paper is to present some further illustrations of the following observation, first suggested by Schürmann 2013, and Bruns et al. 2019: an appropriate use of the last versions of software like LattE (De Loera et al. 2004, 2013) or Normaliz (Bruns and Söger 2015; Bruns et al. 2019) now allows to obtain exact results for four-candidate elections. Our second (and correlated) objective is to study the impact of the number of candidates on the occurrence of various electoral outcomes by comparing the results obtained with four candidates with the ones previously derived for the three-candidate case.

We first provide a series of results on the likelihood of majority condition violations by some usual voting rules in four-candidate elections. Interestingly, some of these results have been obtained (independently) by Bruns et al. (2019), who use a method different from ours. As emphasized by these authors, it is a good test of the correctness of the algorithms involved.

The other results which we derive deal with the manipulability of two widely used voting procedures (plurality rule and plurality runoff), on one hand, and with the concordance of scoring rules to determine the winner on the other.

The remainder of the paper is organized as follows. The basic notions used in our study are introduced in Sect. 2. As our technical approach is partly original, Sect. 3 is devoted to methodological considerations. Sections 4, 5, 6 offer our results and Sect. 7 concludes.

2 Voting rules and electoral outcomes

We consider an election with four candidates (a, b, c and d) and n voters ($n \geq 2$). The 24 possible complete preferences that a voter could have on the four candidates are numbered as indicated in Fig. 1.

We suppose that voters' preferences are anonymous and we denote by n_i ($1 \leq i \leq 24$) the number of voters with preference R_i, so that n_1 voters rank a first, b second, c third and d fourth. A voting situation (of size n) reports the value of each n_i and can be represented by a 24-tuple $\left(n_1, \dots, n_{24}\right)$, such that:

$$\sum_{i=1}^{24} n_i = n \tag{1}$$

and

$$n_i \geq 0 \; (1 \leq i \leq 24). \tag{2}$$

We denote by $V(n)$ the set of all voting situations with n voters and by V the set of all voting situations. As mentioned above, the IAC model assumes that all possible voting situations are equally likely to occur. For a voting situation x in $V(n)$

and two different candidates w, w', we will denote by $P_x(w, w')$ the number of voters that prefer w to w'. For example, the numbers involved in the binary comparisons between a and b are:

$$P_x(a,b) = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_{13} + n_{14} + n_{17} + n_{19} + n_{20} + n_{23}$$

$$P_x(b,a) = n_7 + n_8 + n_9 + n_{10} + n_{11} + n_{12} + n_{15} + n_{16} + n_{18} + n_{21} + n_{22} + n_{24}.$$

A Condorcet Winner (CW) is a candidate who beats each other candidate in pairwise majority comparisons. In the same way, a Condorcet Loser (CL) is a candidate who loses against every other candidate in pairwise majority contests. To illustrate, candidate a is the CW if and only if the following inequalities are satisfied:

$$P_x(a,b) - P_x(b,a) > 0,\ P_x(a,c) - P_x(c,a) > 0 \text{ and } P_x(a,d) - P_x(d,a) > 0. \quad (3)$$

We will also make use of the notions of Absolute Condorcet Winner (ACW) and Absolute Condorcet Loser (ACL); a ACW is a candidate who is top ranked by more than half of the voters, and, similarly, a ACL is a candidate who is bottom ranked by more than half of the voters.

A voting rule is a mapping F associating with every voting situation x in V a (winning) candidate $F(x)$ in W. A "good" voting rule should select the CW when such a candidate exists (CW condition) and should not select the CL when such a candidate exists (CL condition). If a voting rule does not select the CW, it should at least select the ACW when such a candidate exists (ACW condition); similarly, a voting rule should not select the ACL when such a candidate exists (ACL condition). In this sense, the non-selection of the CW (ACW) or the selection of the CL (ACL) can be considered as voting paradoxes. All voting rules studied in this paper belong to the class of (simple) scoring rules or to the class of elimination scoring rules. We evaluate the conditional probability of electing the CW or the ACW (given that such candidates exist) and the conditional probability to select the CL or the ACL (given that such candidates exist) for the following voting rules:

- Plurality rule (PR): the widely used Plurality Rule selects the candidate with a majority of first preferences.
- Negative plurality rule (NPR): it selects the candidate who obtains the minimum of last place votes.
- Borda rule (BR): in a four-candidate election, each candidate gets 0 points for each last place vote received, 1 point for each third place vote, 2 points for each second place vote, and 3 points for each first place vote. The candidate with the largest total point wins the election.
- Plurality elimination rule (PER): it is an iterative procedure, in which, at each step, the candidate who obtained the minimum number of first place votes is eliminated. The last candidate non-eliminated is the winner.
- Negative plurality elimination rule (NPER): at each step of this iterative procedure, the candidate with the maximum number of last place votes is eliminated.
- Borda elimination rule (BER): at each step of this iterative procedure, the candidate with the minimum Borda score is eliminated.

abcd	(R_1)	*bacd*	(R_7)	*cabd*	(R_{13})	*dabc*	(R_{19})
abdc	(R_2)	*badc*	(R_8)	*cadb*	(R_{14})	*dacb*	(R_{20})
acbd	(R_3)	*bcad*	(R_9)	*cbad*	(R_{15})	*dbac*	(R_{21})
acdb	(R_4)	*bcda*	(R_{10})	*cbda*	(R_{16})	*dbca*	(R_{22})
adbc	(R_5)	*bdac*	(R_{11})	*cdab*	(R_{17})	*dcab*	(R_{23})
adcb	(R_6)	*bdca*	(R_{12})	*cdba*	(R_{18})	*dcba*	(R_{24})

Fig. 1 The possible complete preference rankings on four candidates

For the sake of simplicity, we will consider a truncated version of the three iterative procedures, in which in a first step, the two candidates obtaining the lowest scores are eliminated and the second (and final) step is a majority contest between the two remaining candidates (in this case, PER coincides with the so-called plurality runoff rule, often used in political elections). The particular versions of these elimination rules will be denoted by PRR (PR Runoff), NPRR (NPR Runoff), and BRR (BR Runoff). It is worth noticing that BRR is susceptible to elect a candidate different from the CW when such a candidate exists, in contrast to BER (the non-truncated version), which always selects the CW; BRR can even choose a candidate different from the ACW, as shown in the following example: consider an election with 4 candidates and 15 voters: 4 voters have preference R_1, 4 voters have preference R_2, and 7 voters have preference R_{22} (see Fig. 1); candidate a is ranked first by an absolute majority of voters, and the Borda scores of a, b, c, d are (respectively) 24, 30, 11, and 25; thus, c and a (the ACW) are eliminated at the first step of the procedure.

Hence, the six voting rules that we consider here violate the CW condition. And, among these six rules:

(i) BR, NPR, BRR and NPRR (and only these rules) violate the ACW condition;
(ii) PR and NPR are the only rules violating the CL condition;
(iii) and PR is the only rule violating the ACL condition (see, *e.g.,* Lepelley 1989).

The results on the frequency of violation of each of the Condorcet (or majority) conditions which we have introduced will be presented in Sect. 4. In Sect. 5, we will tackle a completely different problem: we will compute the vulnerability of PR and PRR to strategic misrepresentation of preferences by coalitions of voters in four-alternative elections. Finally, in Sect. 6, we will evaluate the probability that all the scoring rules select the same winner when the number of candidates is equal to four.

In the remainder of this study, we will need to compute the scores of the candidates under each of the three scoring rules PR, NPR, and BR. For a scoring rule F, a candidate w, and a voting situation x, we will denote by $S_F(w, x)$ the score of w under F. We only write the scores of candidate a under each of the three rules (the other scores are easily obtained in the same way):

$$S_{\mathrm{PR}}(a,x) = (n_1 + n_2 + n_3 + n_4 + n_5 + n_6),$$
$$S_{\mathrm{NPR}}(a,x) = \left(n_1 + n_2 + n_3 + n_4 + n_5 + n_6\right) + \left(n_7 + n_8 + n_{13} + n_{14} + n_{19} + n_{20}\right) + \left(n_9 + n_{11} + n_{15} + n_{17} + n_{21} + n_{23}\right),$$
$$S_{\mathrm{BR}}(a,x) = 3\left(n_1 + n_2 + n_3 + n_4 + n_5 + n_6\right) + 2\left(n_7 + n_8 + n_{13} + n_{14} + n_{19} + n_{20}\right) + \left(n_9 + n_{11} + n_{15} + n_{17} + n_{21} + n_{23}\right).$$

Note that the score of candidate a under the Negative Plurality rule can be obtained more simply as $S_{\mathrm{NPR}}(a,x) = n - \left(n_{10} + n_{12} + n_{16} + n_{18} + n_{22} + n_{24}\right)$. We conclude this section by emphasizing that, as we only consider large electorates, the problem of tied elections can be disregarded: under each of the voting rules which we study, the probability of *ex aequo* tends to 0 when n tends to infinity (see, *e.g.*, Lepelley 1989).

3 Methodology

Under the IAC assumption, the voting events are often described by a parametric system of linear constraints with integer (or rational) coefficients on the variables n_i and the parameter n. For example, with n voters, the event "a is the CW" is characterized by the system formed by equality (1), the 24 sign inequalities in (2) and the three strict inequalities in (3). Thus, the frequency of a voting event E can be evaluated by computing the number of integer solutions of the parametric linear system describing E. It is now well known in voting theory, since Wilson and Pritchard (2007) and Lepelley et al. (2008), that the use of polytopes and quasi-polynomials is the most appropriate mathematical tool for such computations.

3.1 Integral points in parametric polytopes

A rational polytope P of dimension d is a bounded subset of $\mathbb{R}^d$ defined by a system of integer linear inequalities. P is said to be semi-open when some of these inequalities are strict. A parametric polytope of dimension d (with a single parameter n) is a sequence of d-dimensional rational polytopes P_n ($n \in \mathbb{N}$) of the form $P_n = \left\{x \in \mathbb{R}^d \,:\, Mx \geq bn + c\right\}$, where M is an $t \times d$ integer matrix and b and c two integer vectors with t components. When the constant term c is equal to the zero vector, P_n is denoted nP and corresponds to the dilatation, by the positive integer factor n, of the rational polytope P defined by $P = \left\{x \in \mathbb{R}^d \,:\, Mx \geq b\right\}$. In this case, Ehrhart's theorem (1962) tells us that the number of integer points (lattice points) in nP is a quasi-polynomial in n of degree d, i.e., a polynomial expression $f(n)$ in the parameter n where the coefficients are not constants, but periodic functions of n with integral period. Each coefficient can have its own period, but we can always write $f(n)$ in a form where the coefficients have a common period called the period of the quasi-polynomial (or the denominator of P) and defined as the least common multiple (lcm) of the periods of all coefficients. The leading coefficient of the quasi-polynomial f is the same for all congruence classes and is equal to the (relative) volume of P.

Clauss and Loechner (1998) extended Ehrhart's result to the general class of parametric polytopes P_n, showing that the number of lattice points in P_n can be described by a finite set of quasi-polynomials, each valid on a different subset of $\mathbb{N}$. Note that this implies that for n large enough, the number of lattice points in P_n is given by a single quasi-polynomial. Note also that this generalization makes possible to count the number of lattice points inside the dilatation of a semi-open polytope P. It suffices to use the rule "$\forall x \in \mathbb{Z}, x > 0 \Leftrightarrow x \geq 1$" to transform each strict inequality in the system describing nP into a non-strict inequality, and thus obtain a parametric polytope having the same number of lattice points than nP.

3.2 Limiting probabilities of voting events

Consider an election with n voters and m candidates. Let E be a voting event for which we want to calculate the probability under the IAC hypothesis. Let $V(n)$ be the set of all possible voting situations of size n and (E, n) the set of all elements of $V(n)$ in which E occurs. The probability of (E, n) is a function of n and is given by:

$$\Pr(E, n) = |(E, n)|/|V(n)|. \tag{4}$$

The expression of $|V(n)|$ is well known: with m candidates, it is given by $|V(n)| = \binom{n + m! - 1}{m! - 1}$. Hence, $|V(n)|$ is a polynomial of degree $m! - 1$ and the coefficient of the leading term is equal to $1/(m! - 1)$. In general, (E, n) is described by a parametric linear system $S(n)$ that defines a dilatation of a semi-open rational polytope P of dimension $m! - 1$. Thus, $|(E, n)|$ is equal to the number of lattice points inside nP and is given by the quasi-polynomial describing this number.

To compute $|(E, n)|$, we usually resort to (parametrized) Barvinok's algorithm (Barvinok, 1994). The software [Barvinok] (see Verdoolaege and Bruynooghe 2008) applies to any parametric polytope and can, therefore, deal with the case of interest for us, that of a dilated semi-open polytope. [Barvinok] performs very well for $m = 3$ and, since 2008, the use of this program has yielded many results giving the exact analytical representation for the frequency of various voting events. Note that in the case $m = 3$, there are only 6 variables and the quasi-polynomials describing $|(E, n)|$ are generally of degree 5. Unfortunately, with $m = 4$, there are 24 variables, the quasi-polynomials are of degree 23 and [Barvinok] does not allow to obtain the desired results. Other software packages such as LattE with its new version Latte integrale (see [latte]) and Normaliz (see [Normaliz]) allow, in some cases, to calculate quasi-polynomials corresponding to polytopes of dimension 23. However, we know that for $m = 4$, the periods of the quasi-polynomials can be very large and that the exact formulas for $Pr(E, n)$ can be far too heavy for meaningful analysis.

Therefore, in what follows, attention will be focused on the limiting case where the number of voters, n, tends to infinity.

We set the number of candidates to $m = 4$ and we denote by $Pr(E, \infty)$ the limit of $Pr(E, n)$ when n tends to infinity. From the above, $Pr(E, n)$ is the quotient of the quasi-polynomial $|(E, n)|$ by the polynomial $|V(n)|$. For $|V(n)|$, the coefficient of the leading term is equal to $1/23!$. For $|(E, n)|$, this coefficient is independent of n and is equal to the volume of the semi-open polytope P obtained by taking $n = 1$ in the linear system $S(n)$. Going to the limit in (4), we get:

$$Pr(E, \infty) = 23!\mathrm{Vol}(P). \tag{5}$$

It is obvious that the same reasoning can be applied for conditional probabilities. In this case, $Pr(E, n)$ is of the form $Pr(E, n) = \left|\left(E_1, n\right)\right| / \left|\left(E_2, n\right)\right|$, where (E_1, n) and $\left(E_2, n\right)$ are two voting events characterized by some linear systems $S(n)$ and $T(n)$ that define two dilated semi-open polytopes, nP and nQ. If P and Q are of the same dimension, we can write:

$$Pr(E, \infty) = \mathrm{Vol}(P)/\mathrm{Vol}(Q). \tag{6}$$

In general, algorithms that compute the volume of polytopes are not always efficient when, as in this paper, the number of variables is equal to 24. However, recent improvements in algorithms such as LattE, Normaliz or Convex (see [Convex]) have made it possible to obtain some results describing the probability of voting events with four candidates, requiring the calculation of the volumes of certain polytopes of dimension 23 (see Schürmann 2013; Bruns and Söger 2015; Brandt et al. 2016; Bruns et al. 2019). To compute the volumes involved in the calculations developed in the remainder of this paper, we will not use any algorithm of direct volume computation (with, however, some exceptions[1]). Instead, we will apply a (new) method based on Ehrhart theory and on the combined use of two software, LattE integrale and lrs (see [lrs]). The command (count-ehrhart-polynomial) in the first program allows to calculate in a reasonable time (from a few seconds to a few hours) the quasi-polynomial associated with a dilated polytope nP. With LattE integrale, this computation is possible only when P is an integral polytope (i.e. when all its vertices have integer coordinates). In this case, the quasi-polynomial has period equal to 1 and, hence, is simply a polynomial. The second program, lrs, allows to obtain (usually within seconds) the coordinates of all vertices of a rational polytope. Since in our calculations, P is in general a non-integral polytope, we proceed as follows to calculate $\mathrm{Vol}(P)$. We start by dilating P by a positive integer factor k, such that the obtained polytope kP is integral; for this, k must be a multiple of the period of P. Now, we know by Ehrhart theorem that the period of P is a divisor of the lcm of the denominators of the vertices of P. It suffices then to take k equal to this number that we can easily obtain by applying the lrs program. After this step, we apply LattE integrale to the integral polytope kP and we obtain the polynomial associated with the dilated polytope nkP. It is obvious that if A is the coefficient of the

[1] See Sect. 6 and the "Appendix" where we make use of the last version of Normaliz to deal with some particularly complicated computations.

leading term of this polynomial, then: $A = \mathrm{Vol}(kP) = k^{24}\mathrm{Vol}(P)$. Finally, we have: $\mathrm{Vol}(P) = A/k^{24}$.

4 Results on Condorcet conditions

For a total number of voters equal to n, let $(X - F, n)$ be the event "X is elected under F, given that X exists", with X in $\{\mathrm{CW}, \mathrm{CL}, \mathrm{ACW}, \mathrm{ACL}\}$ and F a voting rule in $\{\mathrm{PR}, \mathrm{NPR}, \mathrm{BR}, \mathrm{PRR}, \mathrm{NPRR}, \mathrm{BRR}\}$. We denote by $Pr(X - F, n)$ and $Pr(X - F, \infty)$ the IAC probability of $(X - F, n)$ and the limit of this probability when n tends to infinity. We know that $Pr(\mathrm{CL} - F, n) = 0$ (and thus $Pr(\mathrm{CL} - F, \infty) = 0$) for F in $\{\mathrm{BR}, \mathrm{PRR}, \mathrm{NPRR}, \mathrm{BRR}\}$, $Pr(\mathrm{ACW} - F, n) = 1$ for F in $\{\mathrm{PR}, \mathrm{PRR}\}$ and $Pr(\mathrm{ACL} - F, n) = 0$ for F in $\{\mathrm{BR}, \mathrm{NPR}, \mathrm{PRR}, \mathrm{NPRR}, \mathrm{BRR}\}$. We derive the other probabilities in the following subsections.

4.1 Condorcet winner election

We assume without loss of generality that a is the CW. We denote by (CW^a, n) the event "a is the CW" and by $\left(\mathrm{CW}^a_F, n\right)$ the event "a is the CW and a is selected under F". It is easy to see that under IAC:

$$Pr(\mathrm{CW} - F, n) = \frac{\left|\left(\mathrm{CW}^a_F, n\right)\right|}{|(\mathrm{CW}^a, n)|}. \tag{7}$$

The voting situations x associated with the event (CW^a, n) are characterized by the following parametric linear system:

$$T(n)\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, i = 1, \ldots, 24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \end{cases}$$

Let Q_1 be the (semi-open) polytope defined by the system $T(1)$. Applying the method described in Sect. 3, we obtain:

$$\mathrm{Vol}\left(Q_1\right) = \frac{101 \times 23!}{12457630654408572272640000}.$$

4.1.1 Voting rules PR, NPR, and BR

For a voting rule F in $\{\text{PR}, \text{NPR}, \text{BR}\}$, the voting situations x associated with the event $\left(\text{CW}_F^a, n\right)$ are characterized by the parametric linear system, $S^F(n)$, consisting of the constraints in $T(n)$ and the following three inequalities:

$$S_F(a,x) - S_F(b,x) > 0,\ S_F(a,x) - S_F(c,x) > 0,\ S_F(a,x) - S_F(d,x) > 0.$$

Let P_1^F be the (semi-open) polytope defined by the system $S^F(1)$. Taking the limit in (7) and using formula (6), we obtain the limit of the probability $Pr(\text{CW} - F, n)$ as:

$$Pr(\text{CW} - F, \infty) = \frac{\text{Vol}\left(P_1^F\right)}{\text{Vol}\left(Q_1\right)}.$$

We have already calculated $\text{Vol}\left(Q_1\right)$. To calculate $\text{Vol}\left(P_1^F\right)$ for PR, NPR and BR, we replace successively, in system $S^F(1)$, the voting rule F by PR, NPR, and BR (by referring to the scores defined in Sect. 2), and then, we use the calculation method based on the LattE and lrs algorithms. Finally, we obtain:

$$Pr(\text{CW} - \text{PR}, \infty) = \frac{10658098255011916449318509}{14352135440302080000000000} \approx 74.26\%$$

$$Pr(\text{CW} - \text{NPR}, \infty) = \frac{2431999845589783615}{4408976007260798976} \approx 55.16\%$$

$$Pr(\text{CW} - \text{BR}, \infty) = \frac{828894710496058365982223276647}{952076453898607919942860800000} \approx 87.06\%.$$

Our result for $Pr(\text{CW} - \text{PR}, \infty)$ is in accordance with the value obtained by Schürmann (2013) and (more recently) by Bruns et al. (2019).

4.1.2 Runoff voting rules

Let F be a voting rule in $\{\text{PR}, \text{NPR}, \text{BR}\}$ and FR be the runoff voting rule using F, so that FR belongs to $\{\text{PRR}, \text{NPRR}, \text{BRR}\}$. Let $\left(CW1_F^a, n\right)$ and $\left(CW2_F^a, n\right)$ be the events defined, respectively, by "a is the CW and is ranked first under F" and "a is the CW and obtains the second score under F". As the CW always wins the second round, these two events describe the two possible configurations for the occurrence of the event $\left(\text{CW}_{FR}^a, n\right)$ and we can then write: $\left(\text{CW}_{FR}^a, n\right) = \left(CW1_F^a, n\right) \cup \left(CW2_F^a, n\right)$. The voting situations associated with $\left(CW1_F^a, n\right)$ are the same as those associated with $\left(\text{CW}_F^a, n\right)$, and are, therefore, characterized by the system $S^F(n)$. To characterize $\left(CW2_F^a, n\right)$, we must distinguish three cases according to the identity of the candidate ranked first under F (b, c or d). Since these three cases are symmetrical, we have $\left|\left(CW2_F^a, n\right)\right| = 3|(E, n)|$, where (E, n) is the set of voting situations belonging to $\left(CW2_F^a, n\right)$ and satisfying the additional condition that b is ranked first by the scoring rule F. This set is characterized by the parametric linear system, $Z^F(n)$, formed by the five constraints in $T(n)$ and the three additional inequalities $S_F(b,x) - S_F(a,x) > 0$, $S_F(a,x) - S_F(c,x) > 0$, and

$S_F(a,x) - S_F(d,x) > 0$. Let K_1^F be the (semi-open) polytope defined by the system $Z^F(1)$. Using formula (6), we obtain the limit of the probability $Pr(\text{CW} - FR, n)$ as:

$$Pr(\text{CW} - FR, \infty) = \frac{\text{Vol}\left(P_1^F\right) + 3\text{Vol}\left(K_1^F\right)}{\text{Vol}\left(Q_1\right)}.$$

We substitute successively, in system $Z^F(1)$, the voting rule F by PR, NPR, and BR, and we use the calculation method based on LattE and lrs algorithms. We obtain:

$$Pr(\text{CW} - \text{PRR}, \infty) = \frac{19627224002877404784030049}{21528203160453120000000000} \approx 91.16\%$$

$$Pr(\text{CW} - \text{NPRR}, \infty) = \frac{18192354603646054002780049}{21528203160453120000000000} \approx 84.50\%$$

$$Pr(\text{CW} - \text{BRR}, \infty) = \frac{5578946122366746282083602696 9}{56004497288153407055462400000} \approx 99.66\%.$$

Note that our result for PRR is in accordance with Bruns et al. (2019), who have obtained this probability (91.16%) using Normaliz.

4.2 Condorcet Loser election

As already mentioned, among the six rules studied, only PR and NPR are susceptible to elect the CL, when such a candidate exists. We assume without loss of generality that candidate a is the CL and we denote by (CL^a, n) the event "a is the CL" and by $\left(\text{CL}_F^a, n\right)$, for F in $\{\text{PR}, \text{NPR}\}$, the event "$a$ is the CL and a is selected under F". It is easy to show that:

$$Pr(\text{CL} - F, n) = \frac{\left|\left(\text{CL}_F^a, n\right)\right|}{|(\text{CL}^a, n)|}.$$

The voting situations x associated with the event (CL^a, n) are characterized by the following parametric linear system:

$$L(n)\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, i = 1, \ldots, 24 \\ P_x(b,a) - P_x(a,b) > 0 \\ P_x(c,a) - P_x(a,c) > 0 \\ P_x(d,a) - P_x(a,d) > 0 \end{cases}$$

The voting situations x associated with the event $\left(\mathrm{CL}_F^a, n\right)$ are characterized by the parametric linear system, $M^F(n)$, consisting of the constraints in $L(n)$ and the following three inequalities:

$$S_F(a,x) - S_F(b,x) > 0,\ S_F(a,x) - S_F(c,x) > 0,\ S_F(a,x) - S_F(d,x) > 0.$$

To get the limit values of $Pr(\mathrm{CL} - F, n)$, it is enough to compute the volume of the semi-open polytope defined by $L(1)$ and the volume of the semi-open polytope defined by $M^F(1)$ for F in $\{\mathrm{PR}, \mathrm{NPR}\}$. After calculation, we obtain:

$$Pr(\mathrm{CL} - \mathrm{PR}, \infty) = \frac{325451674835828550681491}{14352135440302080000000000} \approx 2.27\%$$

$$Pr(\mathrm{CL} - \mathrm{NPR}, \infty) = \frac{104898234852130241}{4408976007260798976} \approx 2.38\%.$$

The same results have been obtained by Bruns et al. (2019) via Normaliz.

4.3 Absolute Condorcet winner election and Absolute Condorcet loser election

We assume without loss of generality that candidate a is the ACW and we denote by (ACW^a, n) the event "a is the ACW". The voting situations associated with this event are characterized by the parametric linear system obtained from $T(n)$ when the three inequalities in (3) are replaced by the following single condition: $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 > n/2$. Let $Q_1^{'}$ be the polytope associated with this new system. Computing $\mathrm{Vol}\left(Q_1^{'}\right)$ and applying (5), we get:

$$Pr((\mathrm{ACW}^a, \infty) = \frac{5569}{1048576}.$$

This implies that the probability of having a ACW is equal to $4 \times Pr((\mathrm{ACW}^a, \infty) = \frac{5569}{262144} \approx 2.12\%$. We know from Lepelley (1989) that the corresponding probability for the three-candidate case is $\frac{9}{16} \approx 56.25\%$: consequently, moving from three to four candidates dramatically decreases the percentage of voting situations with a ACW.

Suppose, however, that such a candidate exists. What is the probability for this candidate to be selected? Proceeding as in Sect. 4.1, but replacing, everywhere in the calculations concerning NPR, BR, NPRR, and BRR, the inequalities describing the event (CW^a, n) with the one describing the event (ACW^a, n), we obtain:

$$Pr(\mathrm{ACW} - \mathrm{NPR}, \infty) = \frac{6712690981925}{10775556292608} \approx 62.30\%$$

$$Pr(\mathrm{ACW} - \mathrm{BR}, \infty) = \frac{362167801256100095003885 29}{36278317087318348922880000} \approx 99.83\%$$

Table 1 CW election, CL election, ACW election, and ACL election

Events	3 candidates (%)	4 candidates (%)
CW – PR	88.15	74.26*
CW – NPR	62.96	55.16
CW – BR	91.11	87.06
CW – PRR	96.85	91.16*
CW – NPRR	97.04	84.50
CW – BRR	100	99.61
CL – PR	2.96	2.27*
CL – NPR	3.15	2.38*
ACW – NPR	60.76	62.30
ACW – BR	96.32	99.83
ACW – NPRR	97.53	90.84
ACW – BRR	100	99.99
ACL – PR	2.47	0.45

$$Pr(\text{ACW} - \text{NPRR}, \infty) = \frac{396415547534699}{436410029850624} \approx 90.84\%$$

$$Pr(\text{ACW} - \text{BRR}, \infty) = \frac{18139154487212563566077 6587}{18139158543659174461440 0000} \approx 99.99\%.$$

Consider now the election of the ACL. Let (ACL^a, n) be the event "a is the ACL". The voting situations associated with this event are characterized by the system obtained from $L(n)$ when the last three inequalities are replaced with the single condition $n_{10} + n_{12} + n_{16} + n_{18} + n_{22} + n_{24} > n/2$.

By a symmetry argument, the volume associated with this new system is equal to $\text{Vol}\left(Q_1'\right)$, so we have $Pr(\text{ACL}^a, \infty) = Pr((\text{ACW}^a, \infty)$. When an *Absolute Condorcet Loser* exists, the only voting rule (among the six rules that we consider) susceptible to elect such a candidate is the Plurality Rule. Proceeding as in subsection 4.2, but replacing everywhere in the calculations concerning PR the inequalities describing (CL^a, n) with the one describing (ACL^a, n), we obtain:

$$Pr(\text{ACL} - \text{PR}, \infty) = \frac{3950740911499}{872820059701248} \approx 0.45\%.$$

4.4 Summary of the results on Condorcet conditions

Table 1 summarizes our four-candidate results on the ability of various voting rules to fulfill Condorcet conditions and compares these results to known results obtained in the literature for the three-candidate case (see Lepelley 1989; Gehrlein and Lepelley 2011 and Diss et al. 2018). The four-candidate results with an asterix* have been independently obtained by Schürmann (2013) and Bruns et al. (2019) (Table 1).

Some interesting conclusions emerge from this comparison. First, it turns out that the probability of electing the CW, given that such a candidate exists, decreases when the number of candidates moves from three to four for each of the voting rules which we have considered. Note, however, that the decreasing rate is lower for BR (4.4%) than for PR (16%), NPR (12.4%), PRR (5.9%), and NPRR (12.9%). The ability to electing the CW (or Condorcet efficiency) of BR is now closer to the two-stage PRR value, and it is higher than the NPRR value. These results reinforce the conclusion recently obtained by Gehrlein et al. (2018) that the expected benefit that would be gained from using two-stage voting rules like PRR or NPRR instead of BR is quite small.

Second, we find that the probability of electing the CL (the so-called Strong Borda Paradox) decreases from three to four candidates for PR and NPR, as well, thus (slightly) increasing the ability of these two voting rules to fulfill the CL condition.

Third, our results show that the impact of the number of candidates on the Absolute Condorcet Winner election depends on the voting rule under consideration: when moving from three to four candidates, the probability of electing the ACW increases for NPR and BR, but decreases for NPRR (and, of course, for BRR, which satisfies the ACW condition in the three-candidate case). It is worth noticing that, in the four-candidate case, the BR probability is close to 100%: in this case, the possible non- election of the ACW should not be considered as a significant flaw of the Borda rule. In addition, it turns out that the truncated version of the Borda Elimination Rule which we consider here has only a very marginal impact on the ability of this rule to elect the ACW.

Finally, we obtain that the likelihood of the ACL election under PR is divided by 5.5 when we move from three to four candidates: such an event becomes very unlikely when four candidates are in contention.

5 Results on coalitional manipulability

5.1 Coalitional manipulability of plurality rule

A strategic manipulation of a voting rule occurs in an election when some voters express insincere preferences to obtain a final winner that they prefer to the candidate that would have been elected if they had voted in a sincere way. To illustrate, consider the following voting situation (with 30 voters and 4 candidates), supposed to correspond to the sincere preferences: $n_1 = 12, n_7 = 10, n_{15} = 8, n_i = 0$ for all $i \notin \{1, 7, 15\}$ (the numbering of the preferences is the one given in Fig. 1). Under PR and sincere voting, a is the winner (with 12 votes for a, 10 votes for b, 8 votes for c, and 0 votes for d. If (at least) three of the eight electors who rank c in the first position vote for their second choice (b), then b is elected, and the voters who vote in an insincere way are better off, since they prefer b to a (the "sincere" winner). Such a voting situation is said to be instable: a coalition of voters, by misrepresenting their preferences, may secure an outcome that they all prefer to the result of sincere voting.

It makes sense to evaluate the coalitional manipulability of a voting rule by calculating the proportion of instable voting situations when the voting rule under consideration is used. We consider first the plurality rule. Let x be a voting situation where candidate a is elected under PR:

$$S_{PR}(a,x) - S_{PR}(b,x) > 0,\ S_{PR}(a,x) - S_{PR}(c,x) > 0,\ S_{PR}(a,x) - S_{PR}(d,x) > 0. \quad (8)$$

According to Lepelley and Mbih (1987), PR is not vulnerable to strategic manipulation by a coalition of voters at this voting situation if, in addition, $S_{PR}(a,x)$ is higher than the number of voters preferring b to a, the number of voters preferring c to a, and the number of voters preferring d to a, that is:

$$S_{PR}(a,x) - P_x(b,a) > 0,\ S_{PR}(a,x) - P_x(c,a) > 0 \text{ and } S_{PR}(a,x) - P_x(d,a) > 0. \quad (9)$$

Let P_n be the (semi-open) parametric polytope defined by the system formed by equality (1), the 24 sign inequalities in (2), the three inequalities in (8), and the three inequalities in (9). Applying formula (5) and multiplying by 4 (the number of candidates), we obtain that the probability for PR to be vulnerable to misrepresentation of preferences by coalitions of voters, denoted by $Pr(\text{Manip} - \text{PR}, \infty)$, is given as: $Pr(\text{Manip} - \text{PR}, \infty) = 1 - 4 \times 23!\text{Vol}(P_1)$. Evaluating $\text{Vol}(P_1)$ by the method described in Sect. 3, we obtain for the four-candidate case:

$$Pr(\text{Manip} - \text{PR}, \infty) = 1 - \frac{1938509031230593}{15116544000000000} \approx 87.28\%.$$

Lepelley and Mbih (1987) have shown that, in the three-candidate case, the vulnerability of PR to strategic manipulation by coalitions of voters for large electorates is equal to $7/24$, i.e., 29.17%. We conclude that moving from three candidates to four candidates very significantly increases the PR vulnerability to strategic manipulation

5.2 Coalitional manipulability of plurality rule with runoff

Do we obtain a similar conclusion for PRR ? We know from Lepelley (1989) that the vulnerability of PRR to strategic manipulation for large electorate in three-candidate elections is equal to $1/9$, i.e., 11.11%. The aim of the current subsection is to investigate what happens when a further candidate is added.

Our computations will be based on the two following propositions.

Proposition 1 (Lepelley 1989) *If a voting situation is such that either there is no* CW *or a* CW *exists and is not the* PRR *winner, then this voting situation is instable for* PRR.

This first proposition is valid regardless of the number of candidates. The second one only deals with four-candidate elections and needs some additional notation: $F_x(a)$ is the number of voters in x who rank a in first position (this is simply $S_{\text{PR}}(a, x)$), $F_x^{ab}(c)$ is the number of voters in x who rank c in first position and prefer a to b, $F_x^{ab}(d)$ is the number of voters in x who rank d in first position and prefer a to b, and y will denote the voting situation obtained from x after manipulation by a coalition of voters.

Proposition 2 *Consider a four-candidate election and a voting situation x in $V(n)$ in which candidate a is both the* CW *and the* PRR *winner. Then, x is instable under* PRR *if and only if there are two candidates, say b and c, different from a, such that*
(i) $P_x(b,a) > F_x(a)$, (ii) $P_x(b,a) > F_x^{ab}(d)$, (iii) $P_x(b,a) + F_x^{ab}(c) > 2F_x(a)$, (iv) $P_x(b,a) + F_x^{ab}(c) > 2F_x^{ab}(d)$, (v) $P_x(b,c) > n/2$.

Proof[2]
Necessity Suppose that x is instable for PRR. It means that there exists a candidate different from a, say b, and a voting situation y derived from x, such that $\text{PRR}(y) = b$, and in which the manipulating voters belong to the set of voters preferring b to a in x. This implies that the scores of a and b in y are such that: (α) $S_{\text{PR}}(a, y) = F_x(a)$ and (β) $S_{\text{PR}}(b, y) \leq P_x(b, a)$. As a is the Condorcet Winner in x, (β) implies that b cannot win in the first stage in y. Thus, there exists a candidate different from a and b, say c, who goes to the second stage with b in y and is beaten by b in this second stage. The only possible strategies for the manipulating voters being to rank b or c in the first position; it follows that: (γ) $S_{\text{PR}}(b, y) + S_{\text{PRR}}(c, y) \leq P_x(b, a) + F_x^{ab}(c)$ and (δ) $S_{\text{PR}}(d, y) \geq F_x^{ab}(d)$.

Condition (i) is necessary, because, if it does not hold, by (α) and (β), we would have $S_{\text{PR}}(b, y) < S_{\text{PR}}(a, y)$ and this implies that b is either eliminated in the first stage or confronted to a in the second stage; in both cases, it contradicts the fact that b and c are together in the second stage in y. Similarly, (ii) has to hold: if not, by (β) and (δ), we would have $S_{\text{PR}}(b, y) < S_{\text{PR}}(d, y)$, which would imply that b is either eliminated in the first round or confronted to d in the second stage, contradicting the presence of b and c in the second stage in y.

Condition (iii) is also necessary: if not, using (β) and (γ), we would have $S_{\text{PR}}(b, y) < S_{\text{PR}}(a, y)$ or $S_{\text{PR}}(c, y) < S_{\text{PRR}}(a, y)$. This would imply that either b or c (or both of them) would be eliminated in the first stage in y (contradicting the presence of b et c in the second stage). A similar argument using (γ) and (δ) instead of (β) and (γ) shows that (iv) is necessary, as well.

[2] Recall that we only consider large electorates; consequently, we ignore here the cases where two candidates obtain the same score: for instance, if the score of a is not strictly higher than the score of b, it means that the score of c is strictly lower than the score of c.

Finally, condition (v) is necessary, because, to be the winner in y, b has to beat c in the second stage by a majority of votes.

Sufficiency Assume there exist two candidates, say b and c, different from a, such that conditions (i)-(v) hold. Let $r = \max\left\{F_x(a), F_x^{ab}(d)\right\} + 1$ and $s = P_x(b,a) - r$; by (iii) and (iv), we have $s \geq 0$. Let y be the voting situation resulting from x where voters preferring b to a (all or part of them) strategically vote to have b ranked first exactly r times and c ranked first exactly s times (it is possible by (i) and (ii)). Thus, we have: $S_{\text{PR}}(a,y) = F_x(a)$, $S_{\text{PR}}(b,y) = r$, $S_{\text{PR}}(c,y) = s + F_x^{ab}(c)$, and $S_{\text{PR}}(d,y) = F_x^{ab}(d)$. It is then easy to see that $S_{\text{PR}}(b,y) > S_{\text{PR}}(a,y)$ and $S_{\text{PR}}(b,y) > S_{\text{PR}}(d,y)$ (by definition of r), and $S_{\text{PR}}(c,y) > S_{\text{PR}}(a,y)$ and $S_{\text{PR}}(c,y) > S_{\text{PR}}(d,y)$ [by definition of r and s, and by (iii) and (iv)]. Consequently, b and c are selected for the second stage in y and b beats c in the second stage, by (v). Hence, $\text{PRR}(y) = b$, showing that x is instable for PRR. □

Let E_1 denote the event "there is no CW", E_2 the event "a is the CW and is not selected under PRR", and E_3 the event "a is the CW, is selected under PRR and the voting situation is instable for PRR". It follows from Proposition 1 that the probability for PRR to be vulnerable to misrepresentation of preferences by coalitions of voters can be written as (we assume large electorates):

$$Pr(\text{Manip} - \text{PRR}, \infty) = Pr\left(E_1, \infty\right) + 4(\Pr\left(E_2, \infty\right) + Pr\left(E_3, \infty\right). \tag{10}$$

We know from Gehrlein (2001) that, in four-candidate elections:

$$\Pr\left(E_1, \infty\right) = \frac{331}{2048}.$$

We easily deduce from the above-computed Condorcet Efficiency of PRR for four candidates (see Sect. 4.1.2) that:

$$\begin{aligned}\Pr\left(E_2, \infty\right) &= \frac{1717}{8192}\left(1 - \frac{1962722400287740478403004 9}{21528203160453120000000000}\right)\\ &= \frac{1900979157575715215969951}{10271347716390912000000000},\end{aligned}$$

and we have used Proposition 2 to obtain the following fraction for $\Pr\left(E_3, \infty\right)$:

$$\frac{1087728064806496337719968633307455328929405251956556660146836615246691931}{2888468385284284682471525385156207812319890365847961600000000000000000000}.$$

The computations are tedious and are detailed in "Appendix". Using (10), we finally obtain the following result for $Pr(\text{Manip} - \text{PRR}, \infty)$:

$$\frac{2789407566080353053037581459785742662134938536492206505121233415246691931}{7221170963210711706178813462890519530799725914619904000000000000000000000}$$

i.e., $Pr(\text{Manip} - \text{PRR}, \infty) \approx 38.63\%$. Consequently, the vulnerability of PRR to coalitional manipulation is multiplied by a factor higher than 3.4 when a fourth candidate is introduced! The manipulability of PRR remains, however, significantly lower than the one of PR.

6 Concordance of all scoring rules

In four-candidate elections, a scoring rule can be defined by a 4-tuple $(1, \lambda, \mu, 0)$, with $1 \geq \lambda \geq \mu \geq 0$. Candidates get 1 point for each first position in voters' rankings, λ points for each second position, μ points for each third position, and 0 points for each last position. We obtain PR by taking $\lambda = \mu = 0$, NPR by taking $\lambda = \mu = 1$, and BR by taking $\lambda = 2/3$ and $\mu = 1/3$. We wish to compute the probability that all the scoring rules agree, i.e., select the same winner, in four-candidate elections. This calculation is of interest, since it allows to know, a contrario, the proportion of voting situations for which the choice of a specific scoring rule is susceptible to impact the determination of the winner. In three-candidate elections, the result is known: Gehrlein (2002) shows that, in this case, the probability that all scoring rules give the same winner is equal to $113/216 = 0.5231$; thus, the proportion of voting situations where the choice of a particular voting rule really matters is about 48%. We would like to know how these figures are modified when we consider four-candidate elections. We know from Moulin (1988) that, in four-candidate elections, all the scoring rules will select the same winner if and only if the three "elementary" scoring rules $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, and $(1, 1, 1, 0)$ lead to the choice of the same winner. The first and the third elementary scoring rules are simply PR and NPR; we denote the second elementary rule by IR (the "intermediate" rule). The voting situations x (of size n) at which the event "All the scoring rules select candidate a" occurs are characterized by the system formed by (1), (2), and the following nine inequalities:

$$S_{\mathrm{PR}}(a,x) - S_{\mathrm{PR}}(b,x) > 0,\ S_{\mathrm{PR}}(a,x) - S_{\mathrm{PR}}(c,x) > 0,\ S_{\mathrm{PR}}(a,x) - S_{\mathrm{PR}}(d,x) > 0$$

$$S_{\mathrm{IR}}(a,x) - S_{\mathrm{IR}}(b,x) > 0,\ S_{\mathrm{IR}}(a,x) - S_{\mathrm{IR}}(c,x) > 0,\ S_{\mathrm{IR}}(a,x) - S_{\mathrm{IR}}(d,x) > 0$$

$$S_{\mathrm{NPR}}(a,x) - S_{\mathrm{NPR}}(b,x) > 0,\ S_{\mathrm{NPR}}(a,x) - S_{\mathrm{NPR}}(c,x) > 0,\ S_{\mathrm{NPR}}(a,x) - S_{\mathrm{NPR}}(d,x) > 0.$$

Let P_n be the (semi-open) parametric polytope defined by this system. Our method failed to compute the volume of P_1; but we have been able to obtain the desired result using the latest version of Normaliz, based on a new computation technique called "Descent" (see Bruns and Ichim 2018). The numerator and the denominator of the fraction which we obtain are very high:

$$\frac{9349139401127690533566796418557025794950223592401117880473766953518003491604967}{368671423306032432425264913277226467431071673877195749185288083487258517314423930373537792000000000000000}.$$

Using formula (5), multiplying by 4 (the number of candidates), and evaluating this fraction, we obtain the following probability for the event SW: "All the scoring rules give the same winner":

$$Pr(\mathrm{SW}, \infty) = 0.2622325388.$$

We conclude that the probability that the choice of the voting rule impacts the winner determination increases from 48% to about 74% when the number of candidates moves from three to four. We note also that our result is consistent with the probability obtained by Bruns and Ichim (2018) for the concordance of the following four voting rules: PR, NPR, BR, and MR (Majority Rule): they found that the probability that these voting rules select the same winner is about 31%.

Another interesting result given in Gehrlein (2002) for three-candidate elections is the probability that all the scoring rules select the CW. Gehrlein obtains $3437/6912 = 0.4973$. Let SW = CW denote this event. Adding (3) to (1), (2) and the above inequalities, we obtain via Normaliz and using (5) that the probability of having candidate a as both the CW and the winner of all the scoring rules is given as:

$$\frac{568055338354786205174773927167883538897629861665210587445140156808948928563283325950753}{921911839232355698843682814423426078543096938554105823055571802053911951441920000000000}.$$

Multiplying by 4, we have:

$$Pr(\text{SW} = \text{CW}, \infty) = 0.2464683993.$$

Hence, as in the case of three-candidate elections, the addition of the restriction that the common winner of the scoring rules is also a CW has little impact on the probability that all the scoring rules select the same winner.

7 Conclusion

We have derived in this paper some exact results for the likelihood of various electoral outcomes and voting paradoxes under the IAC assumption in four-candidate elections. These computations have made possible a first investigation (based on exact results rather than on estimates obtained from simulations) of the impact of the number of candidates on the occurrence of these voting outcomes. Among other results, we showed that the non-election of the Absolute Condorcet Winner under the Borda rule and the election of the Absolute Condorcet Loser under the plurality rule are not a big concern when the number of candidates is equal to four. By contrast, the introduction of a fourth candidate significantly increases (1) the manipulability of the plurality and plurality with runoff rules, and (2) the significance of voting rule selection.

From a technical point of view, the major part of our calculations have been done thanks to an original method, based on a combination of the software packages LattE and lrs. It seems, however, that the latest version of Normaliz is, at the present time, the most efficient software tool to obtain the IAC probabilities of electoral outcomes when more than three alternatives are in contention, as suggested by the recent paper of Bruns and Ichim (2018) and illustrated by the computations which we have conducted in Sect. 6 and in “Appendix”.

Appendix: Computation of Pr (E_3, ∞)

Let (E_3, n) be the set of all voting situations, of size n, in which E_3 occurs. Since a must be first or second in the first stage of the sincere vote, by symmetry, we can write:

$$\left|(E_3, n)\right| = 3(|(G, n)| + |(H, n)|), \tag{11}$$

where (G, n) is the set of the voting situations in $\left(E_3, n\right)$ for which a is first and b is second in the first stage, and (H, n) the set of the voting situations in $\left(E_3, n\right)$ for which a is second and b is first in the first stage. Considering all possibilities for the choice of the candidate who wins after manipulation and the candidate who goes with him to the second stage, we obtain:

$$(G,n) = \left(G^{bc},n\right) \cup \left(G^{cb},n\right) \cup \left(G^{bd},n\right) \cup \left(G^{db},n\right) \cup \left(G^{cd},n\right) \cup \left(G^{dc},n\right) \quad (12)$$

$$(H,n) = \left(H^{bc},n\right) \cup \left(H^{cb},n\right) \cup \left(H^{bd},n\right) \cup \left(H^{db},n\right) \cup \left(H^{cd},n\right) \cup \left(H^{dc},n\right). \quad (13)$$

Here, for α, β in $\{b, c, d\}$ and $\alpha \neq \beta$, the notation $\left(G^{\alpha\beta}, n\right)$ (resp. $\left(H^{\alpha\beta}, n\right)$) denotes the subset of (G, n) (resp. (H, n)) of voting situations where α and β go to the second stage after manipulation (in favor of α) and α beats β by a majority of votes. For simplicity, in what follows, these subsets will be denoted by $G^{\alpha\beta}$ (resp. $H^{\alpha\beta}$).

Using Proposition 2 and deleting the redundant inequalities, it follows that the voting situations x in G^{bc}, G^{cb}, and G^{cd} are characterized by the following parametric linear systems:

$$(S_n^{bc})\begin{cases} n_1 + \ldots + n_{24} = n \\ n_1 + \cdots + n_{24} = n \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(a) - F_x(b) > 0 \\ F_x(b) - F_x(c) > 0 \\ F_x(b) - F_x(d) > 0 \\ P_x(b,a) - F_x(a) > 0 \\ P_x(b,a) + F_x^{ab}(c) - 2F_x(a) > 0 \\ P_x(b,c) - P_x(c,b) > 0 \end{cases} \quad (S_n^{cb})\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, i = 1, \ldots, 24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(a) - F_x(b) > 0 \\ F_x(b) - F_x(c) > 0 \\ F_x(b) - F_x(d) > 0 \\ P_x(c,a) - F_x(a) > 0 \\ P_x(c,a) + F_x^{ac}(b) - 2F_x(a) > 0 \\ P_x(c,b) - P_x(b,c) > 0 \end{cases}$$

$$(S_n^{cd})\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, i = 1, \ldots, 24 \\ P_x(a,b) - P_x(b,a) > 0 P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(a) - F_x(b) > 0 \\ F_x(b) - F_x(c) > 0 \\ F_x(b) - F_x(d) > 0 \\ P_x(c,a) - F_x(a) > 0 \\ P_x(c,a) + F_x^{ac}(d) - 2F_x(a) > 0 \\ P_x(c,d) - P_x(d,c) > 0 \end{cases}.$$

By symmetry between candidates c and d, the systems characterizing G^{bd}, G^{db}, and G^{dc} are obtained by permuting c and d in S_n^{bc}, S_n^{cb} and S_n^{cd}, respectively; therefore, we have $\left|G^{bd}\right| = \left|G^{bc}\right|$, $\left|G^{db}\right| = \left|G^{cb}\right|$, and $\left|G^{dc}\right| = \left|G^{cd}\right|$.

Now, we use (12) and we apply the inclusion–exclusion principle to calculate $|(G, n)|$. For the 15 pairwise intersections, it is obvious that $G^{bc} \cap G^{cb}$, $G^{bd} \cap G^{db}$, and $G^{cd} \cap G^{dc}$ are empty, and that by symmetry, we have $\left|G^{cb} \cap G^{bd}\right| = \left|G^{bc} \cap G^{db}\right|$, $\left|G^{bd} \cap G^{dc}\right| = \left|G^{bc} \cap G^{cd}\right|$, $\left|G^{bd} \cap G^{cd}\right| = \left|G^{bc} \cap G^{dc}\right|$, $\left|G^{db} \cap G^{dc}\right| = \left|G^{cb} \cap G^{cd}\right|$, and $\left|G^{db} \cap G^{cd}\right| = \left|G^{cb} \cap G^{dc}\right|$. Of the 20 triple intersections, the only ones that are (possibly) non-empty are the 8 that are obtained by choosing one and only one element in each of the three sets $\left\{G^{bc}, G^{cb}\right\}$, $\left\{G^{bd}, G^{db}\right\}$ and $\left\{G^{cd}, G^{dc}\right\}$; and by symmetry we have $\left|G^{bc} \cap G^{bd} \cap G^{cd}\right| = \left|G^{bc} \cap G^{bd} \cap G^{dc}\right|$, $\left|G^{bc} \cap G^{db} \cap G^{cd}\right| = \left|G^{cb} \cap G^{bd} \cap G^{dc}\right|$, $\left|G^{cb} \cap G^{bd} \cap G^{cd}\right| = \left|G^{bc} \cap G^{db} \cap G^{dc}\right|$, and $\left|G^{cb} \cap G^{db} \cap G^{cd}\right| = \left|G^{cb} \cap G^{db} \cap G^{dc}\right|$. Finally, all intersections of 4, 5, or 6 subsets $G^{\alpha\beta}$ (α, β in $\{b, c, d\}$ and $\alpha \neq \beta$) are empty, because each of them is included in (at least) one of the three empty intersections, $G^{bc} \cap G^{cb}$, $G^{bd} \cap G^{db}$, and $G^{cd} \cap G^{dc}$ (to form an intersection of 4, 5 or 6 subsets $G^{\alpha\beta}$, it is necessary to choose the two elements of at least one of the sets $\left\{G^{bc}, G^{cb}\right\}$, $\left\{G^{bd}, G^{db}\right\}$ and $\left\{G^{cd}, G^{dc}\right\}$).

We can now write the formula giving the cardinality of (G, n):

$$
\begin{aligned}
|(G, n)| = {} & 2\left(\left|G^{bc}\right| + \left|G^{cb}\right| + \left|G^{cd}\right|\right) - \left(\left|G^{bc} \cap G^{bd}\right| + \left|G^{cb} \cap G^{db}\right|\right. \\
& + 2\left|G^{bc} \cap G^{db}\right| + 2\left|G^{bc} \cap G^{cd}\right| + 2\left|G^{cb} \cap G^{cd}\right| \\
& \left. + 2\left|G^{cb} \cap G^{dc}\right|\right) + 2\left|G^{bc} \cap G^{dc}\right| \\
& + 2\left(\left|G^{bc} \cap G^{bd} \cap G^{cd}\right| + \left|G^{bc} \cap G^{db} \cap G^{dc}\right|\right. \\
& \left. + \left|G^{cb} \cap G^{db} \cap G^{dc}\right| + \left|G^{cb} \cap G^{bd} \cap G^{dc}\right|\right)
\end{aligned}
\quad . \tag{14}
$$

To obtain $Pr(G, \infty)$, we replace each cardinality that appears in the second member of (14) by the volume of the associated polytope (for example, the polytope associated with G^{bc} is the one described by the system S_1^{bc}), and then, we divide by the volume associated with the total number of voting situations (*i.e., by* $1/23!$). Using the method based on LattE and Lrs (and Normaliz for the triple intersections), we get the following results:

	Volume of the associated polytope
G^{bc}	215799651022148336618223418954725782642961476613018207489968576368915 34517195601 $\mid$ 1696627485451304216801706693769459989090225696942760631 389397188608000000000000000
G^{cb}	5573287256158164544822279100674154863906571658254932679572826363232135917 $\mid$ 538163337264415891621828147363607120928617085478067328201523200000000000000
G^{cd}	690763588675926208892200201275635242424317217116771810711456400097168 422586192409 $\mid$ 1085841590688834698753092284012454393017744446043366804089214200709120000000000000000

<table>
<tr><th></th><th>Volume of the associated polytope</th></tr>
<tr><td>$G^{bc} \cap G^{bd}$</td><td>437495870930736649855127528255082994516834134015800580312416390677735
5886621592418407 │ 15783821593121799418221188750549477369596125747406
42364397296025600000000000000000000000000</td></tr>
<tr><td>$G^{cb} \cap G^{db}$</td><td>371668307604755922004052912809417026311491652333420360381433590516107
912320558408681833214907876615097308537557287 │ 149865786232755160714
6037134925671445888954729659551879517970519909883201282194356264123 3
5975849000960000000000000000</td></tr>
<tr><td>$G^{bc} \cap G^{db}$</td><td>737287950976919106080326758029796495041775821126099995591553611069010
59950083499782664269443828800493654344962886188347693 │ 3197695068270
2112545122688614706446500385377035566310493107178621284727313905373 6
3262184316736774630182748160000000000000000000</td></tr>
<tr><td>$G^{bc} \cap G^{cd}$</td><td>112867869763858840602083748038141897630087342186785541038697068472975
371445096758295189908113319595397077959109125414764762490 9 │ 57302695
62340218568085985799755395212869059564773482840364806408934223134651
842954965834295592300137287484702720000000000000000000000</td></tr>
<tr><td>$G^{bc} \cap G^{dc}$</td><td>179120884374630246903781735660307774495056889927342505569670205616577
977805908161982115664774078815947037152735182335 7 │ 719655505489690 28
175152703219130742831587606118251681254452944366072591325570972987 80
32037356026902609920000000000000000000</td></tr>
<tr><td>$G^{cb} \cap G^{cd}$</td><td>156090914075338229952814649806280933859900502452413621807273601730707
932075993177383819 │ 10417322251460387616025984575362655063933442993 2
8823960502215376896000000000000000000000000</td></tr>
<tr><td>$G^{cb} \cap G^{dc}$</td><td>169871965226875995373638321448221960851557358072284253875160535563392
94374740663419980423405997963504376939242487320875885987 │ 1169442767
824534401650201183623550043442665217300710783747919675292698598908 53
937856445597869230615046683361280000000000000000000000</td></tr>
<tr><td>$G^{bc} \cap G^{bd} \cap G^{cd}$</td><td>674324033423332239489570724345486219235182962634052751321547924919 29
778259061543933321813727983064851831186699397550031850240974214990 3
│ 800337852832421947317129795594046017972207282304842286498696861 52
63879196741786982723337601601409510923501840629760000000000000000000
000000</td></tr>
<tr><td>$G^{bc} \cap G^{db} \cap G^{dc}$</td><td>662240849477329919480950576254097552097952637481468319635456332226 17
96216124815329855721189005229633888195819632671107118857573386123 7 │
102607417029797685553478178922487769461182144644933875467932011045 2
101015342843422855427405148736019370814126161920000000000000000000 00
00000</td></tr>
<tr><td>$G^{cb} \cap G^{db} \cap G^{dc}$</td><td>172499040886591309105147664280974842140465399609634373368829431585429
2717960912418441073266813169686832716096326322172396618251014 49 │ 337
46747041340105722597815634820568468427252834815492859245643706891499
070982369279272785293304692828016318873600000000000000000000000000</td></tr>
<tr><td>$G^{cb} \cap G^{bd} \cap G^{dc}$</td><td>643392533957704692205634948961666959707134893736156487067300619621472
3723 │ 14770885442042769376198830238432027684092123485420931514368000
000000000000000</td></tr>
</table>

After calculation, we obtain:

$$Pr(G, \infty) = \frac{52683297709949532142119507583496364663740732115091118336072352908066132417}{134874482602274422408582167690411633024678174847653642240000000000000000000000}.$$

To compute $Pr(H,\infty)$, we use (13) and we proceed in exactly the same way as for $Pr(G,\infty)$. Using Proposition 2 and deleting the redundant inequalities, we obtain the systems characterizing the voting situations x in H^{bc}, H^{cb}, and H^{cd}:

$$(T_n^{bc})\begin{cases} n_1 + \vdots + n_{24} = n \\ n_i \geq 0, i = 1,\dots,24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(b) - F_x(a) > 0 \\ F_x(a) - F_x(c) > 0 \\ F_x(a) - F_x(d) > 0 \\ P_x(b,a) - F_x(a) > 0 \\ P_x(b,a) + F_x^{ab}(c) - 2F_x(a) > 0 \\ P_x(b,c) - P_x(c,b) > 0 \end{cases} \quad (T_n^{cb})\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, i = 1,\dots,24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(b) - F_x(a) > 0 \\ F_x(a) - F_x(c) > 0 \\ F_x(a) - F_x(d) > 0 \\ P_x(c,a) - F_x(a) > 0 \\ P_x(c,a) + F_x^{ac}(b) - 2F_x(a) > 0 \\ P_x(c,b) - P_x(b,\mathrm{c}) > 0 \end{cases}$$

$$(T_n^{cd})\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, i = 1,\dots,24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(b) - F_x(a) > 0 \\ F_x(a) - F_x(c) > 0 \\ F_x(a) - F_x(d) > 0 \\ P_x(c,a) - F_x(a) > 0 \\ P_x(c,a) - F_x(b) > 0 \\ P_x(c,a) + F_x^{ac}(d) - 2F_x(a) > 0 \\ P_x(c,a) + F_x^{ac}(d) - 2F_x(b) > 0 \\ P_x(c,d) - P_x(d,\mathrm{c}) > 0 \end{cases}.$$

Here again, c and d being symmetrical, we obtain the same symmetries as before. Therefore, the formula describing the cardinality of (H,n) is exactly the same as (14), except that $|(G,n)|$ is replaced with $|(H,n)|$ and each $G^{\alpha\beta}$ (α, β in $\{b,c,d\}$ and $\alpha \neq \beta$) is replaced with $H^{\alpha\beta}$. By applying the method based on LattE and Lrs (and Normaliz for the triple intersections), we calculate the volumes of all the polytopes associated with the cardinalities involved in the second member of this formula. We then get:

<table>
<tr><th></th><th>Volume of the associated polytope</th></tr>
<tr><td>H^{bc}</td><td>1122570228285484416840414329038859444597 | 2203251766912513538732851200000000000000</td></tr>
<tr><td>H^{cb}</td><td>4815102613831086008681845331774767357 | 27540647086406419234160640000000000000000</td></tr>
<tr><td>H^{cd}</td><td>4859814977353382934768278656966765458531281829079037659495505447570096549131 | 2636152533698047281461986677988445611214432454458846070397992960000000000000000</td></tr>
<tr><td>$H^{bc} \cap H^{bd}$</td><td>6833482604299574922319228350257359109218343485126709565221 | 228793803992037731981659487629131907072000000000000000000000</td></tr>
<tr><td>$H^{cb} \cap H^{db}$</td><td>5049881685008345434475345496190114508046016393 | 10408519742515465059176289612595200000000000000000</td></tr>
<tr><td>$H^{bc} \cap H^{db}$</td><td>6758275249404490718040224145645192752456662124728171 63 | 831977469061955389811512540956047966208000000000000000000</td></tr>
<tr><td>$H^{bc} \cap H^{cd}$</td><td>54527549003500598877896693999359561653981068895014160097048506522520901040183861295097633826231569261 | 46107870190868020671174591850139263031166900356375104588254062448121512540343640260608000000000000000000</td></tr>
<tr><td>$H^{bc} \cap H^{dc}$</td><td>83080911883375546239440849322953287214079280529216373903274929398296166932089628546748 89 | 625378758656943442800240993567369483685580835156744735053071894059449057280000000000000000</td></tr>
<tr><td>$H^{cb} \cap H^{cd}$</td><td>536467444833359046580624594631977457827077563033371231007512557540128954421668859753948816150778793 | 1129430486744758491847310206009682124024272495502035679704440095241071735752097792000000000000000000000</td></tr>
<tr><td>$H^{cb} \cap H^{dc}$</td><td>8829802817815659461561114981596342439514389717111976228094675237975903263366333445667 | 245102394143422866078871641609786197799561369843913280444080695300587520000000000000000000</td></tr>
<tr><td>$H^{bc} \cap H^{bd} \cap H^{cd}$</td><td>65263604686786698610375890520344701268577792413313148398262300416175661525684405508957618850237853907565180581178634 13 | 72165997365031664080063249917879416916137943365917614875948003883000675346353714633933165503730155520000000000000000000000</td></tr>
<tr><td>$H^{bc} \cap H^{db} \cap H^{dc}$</td><td>98776097969549172971313133688622284918617353746834301920109098021322418636943121874636070643923251688514001489561 7 | 3299771255831351809787985821576562273257336230723256281479103972702362841625684253952133767888896000000000000000000000</td></tr>
<tr><td>$H^{cb} \cap H^{db} \cap H^{dc}$</td><td>540439807752805447252150446083538533286573675570475751453358522271869885502709580493393253727163452978773660538565 7 | 381830673889056423704038359353859348762634620983691083999724888269844843102400606528746907427143680000000000000000000000</td></tr>
<tr><td>$H^{cb} \cap H^{bd} \cap H^{dc}$</td><td>1417670314436584854903726386188928307415017721234018789316018452780768519881668622810626073602390164587360506928 37 | 76366134777811284740807671870771869752526924196738216799944977653968968620480121305749381485428736000000000000000000</td></tr>
</table>

After calculation, we obtain:

$$Pr(H,\infty) = \frac{86196191235167272312652407600525591350591906553065213523273899326331417319}{9968983496689848612808247177117381571389256401783095296000000000000000000000}.$$

Finally, going to the limit in (11), we have:

$$Pr\left(E_3, \infty\right) = 3(|(G, \infty)| + |(H, \infty)|)$$
$$= \frac{1087728064806496337719968633307455328929405251956556660146836615246691931}{288846838528428468247152538515620781231989036584796160000000000000000000000}.$$

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