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Condorcet efficiency of the preference approval voting and the probability of selecting the Condorcet loser

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Abstract
Under approval voting (AV), each voter just distinguishes the candidates he approves of from those appearing as unacceptable. The preference approval voting (PAV) is a hybrid version of the approval voting first introduced by Brams and Sanver (in: Brams, Gehrlein, Roberts (eds) The mathematics of preference, choice and order. Springer, Berlin, pp 215–237, 2009). Under PAV, each voter ranks all the candidates and then indicates the ones he approves. In this paper, we provide an analytical representation of the limiting probability that PAV elects the Condorcet winner (resp. the Condorcet loser) when she exists in three-candidate elections. We perform our analysis by assuming the assumption of the Extended Impartial Culture. The aim is to measure at which extend PAV performs better than AV both on the propensity of electing the Condorcet winner and on that of the non-election of the Condorcet loser. For this purpose, we also provide an analytical representation of the limiting probability that AV elects the Condorcet winner (resp. the Condorcet loser) when she exists in three-candidate elections. Our representation of the limiting probability that AV elects the Condorcet winner is more general than that provided by Diss et al. (in: Laslier and Sanver (eds) Handbook on approval voting. Springer, Berlin, pp 255–283, 2010) and it leads to the same figures as the representation provided by Gehrlein and Lepelley (Group Decis Negot 24:243–269, 2015).

Keywords Approval voting · Ranking · Condorcet · Extended impartial culture · Probability

1 Introduction

Popularized by Brams and Fishburn (1978), the approval voting (AV) rule is a voting system under which each voter approves (any number of) candidates that he considers

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as acceptable and the winner is the most-approved candidate. This rule has made (and continues to be) the subject of numerous research works in political science, economics and computer science. To have a quick overview of these works, the reader may refer to the books of Brams and Fishburn (2007); Brams (2008) and to the *Handbook of Approval Voting* edited by Laslier and Sanver (2010). Under AV, there is no need to rank the candidates as under the scoring rules.[1] This absence of rankings gave rise to a controversy between Saari and van Newenhizen (1988a, b) and Brams et al. (1988a, b). Saari and van Newenhizen (1988b) blamed AV of hiding the real preferences of the voters which can be strict between the candidates approved by a voter. Brams and Sanver (2009) may have brought what appears as a possible response to this criticism by introducing the preference approval voting (PAV). Under PAV, each voter ranks all the candidates then indicates the ones he approves.[2] According to Brams and Sanver (2009), the winner under PAV is determined by two rules:

Rule 1 The PAV winner is the AV winner if[3]

i. no candidate receives a majority of approval votes (*i.e* approved by more than half of the electorate)
ii. exactly one candidate receives a majority of approval votes.

Rule 2 In the case that two or more candidates receive a majority of approval votes,

i. the PAV winner is the one among these candidates who is preferred by a majority to every other majority-approved candidate.
ii. In the case of a cycle among the majority-approved candidates, then the AV winner among them is the PAV winner.

Brams and Sanver (2009) noticed that it is Rule 2 that clearly differentiates PAV from AV. They pointed out that for some situations where a *Condorcet winner* exists, this candidate may not be a PAV winner under each of the subcases of Rule 1 and Rule 2. When she exists, a Condorcet winner is a candidate who defeats each of the other candidates in pairwise comparisons. We know that AV always elects the Condorcet winner when she exists given that voters' preferences are dichotomous (Ju 2010; Xu 2010). This is no more the case when the voters' true preferences are assumed to be strict orderings (Gehrlein and Lepelley 1998) or when indifference are allowed in the voters' true preferences (Diss et al. 2010; Gehrlein and Lepelley 2015). For large electorates and three candidates, Gehrlein and Lepelley (1998) found that AV has the same Condorcet efficiency (probability of electing the Condorcet winner when she exists) as both the Plurality rule and the Antiplurality rule.[4] Going from a more general framework, Diss et al. (2010) found that for large electorates and three candidates, AV performs better that both the Plurality rule and the Antiplurality rule

[1] A scoring rule is a voting rule under which voters give points to candidates according to the ranks they have in voter's preferences. The winner is the candidate with the highest total number of points.

[2] Brams and Sanver (2009) also introduced the Fallback Voting under which voters only rank the candidates they approve. In this paper, we are not concerned with this rule.

[3] Here, we have chose to split *Rule 1* into two. This will be helpful for our analysis.

[4] The Plurality rule is a scoring rule under which each voter votes only for (gives one point to) his top ranked candidate and the winner is the one with the highest total number first places; under the Antiplurality rule, the winner is the candidate with the fewest total number of last places.

on the Condorcet efficiency; they also found some scenarios under which the Borda rule performs better than AV. Their results were strongly reinforced by Gehrlein and Lepelley (2015).

To our knowledge, nothing is known about the Condorcet efficiency of PAV. One objective of this paper was thus to try to fill this void for voting situations with three candidates by focusing on the Condorcet efficiency of PAV when indifference are allowed as in Diss et al. (2010). So, we provide a representation of the limiting probability of the Condorcet efficiency of PAV. All the computations are done under the extended impartial culture assumption introduced by Diss et al. (2010); this assumption will be defined later. By definition, it is obvious that PAV performs better than AV on electing the Condorcet winner when she exists. It would be interesting to measure the extent of this dominance. In order to better reflect this, we first provide a more general representation of the limiting probability of the Condorcet efficiency of AV. Then, the representation provided by Diss et al. (2010) comes as a particular case of ours which appears as an alternative form of the representation provided by Gehrlein and Lepelley (2015).

When she exists, a Condorcet loser is a candidate who is defeated by each of the other candidates in pairwise comparisons. Gehrlein and Lepelley (1998) showed that with more than three candidates and under the impartial culture assumption, AV is more likely to elect the Condorcet loser than the Plurality rule. For three-candidate elections, they showed that AV has the same probability of electing the Condorcet loser as both the Plurality rule and the Antiplurality rule. This result is a bit challenged by a recent paper by Gehrlein et al. (2016). Using impartial anonymous culture-like assumptions[5] and considering a range of scenarios, Gehrlein et al. (2016) concluded that in three-candidate elections, AV is less likely to elect the Condorcet loser than both the Plurality rule and the Antiplurality rule. By definition, PAV is less likely to elect the Condorcet loser than AV. The second objective of this paper was to focus on the probability that PAV elects the Condorcet loser when she exists. We provide for AV and for PAV, analytical representations of the limiting probability of electing the Condorcet loser in three-candidate elections under the extended impartial culture assumption. By doing so, we will highlight at which extent PAV is less likely to elect the Condorcet loser than AV.

The rest of the paper is structured as follows: Sect. 2 is devoted to basic notations and definitions. Section 3 presents our results on the Condorcet efficiency. Section 4 deals with the probability of electing the Condorcet loser. Section 5 concludes.

2 Preliminaries

2.1 Preferences in three-candidate elections

Let N be a set of n voters ($n \geq 2$) and $A = \{a, b, c\}$ a set of three candidates. We assume that voters rank all the candidates, indifference is allowed and they indicate

[5] We will say more on this assumption later.

Table 1 The 19 possible preference types with three candidates

Class I	$\underline{a} \succ b \succ c$	p_1	**Class II**	$\underline{a} \succ \underline{b} \succ c$	p_7
	$\underline{a} \succ c \succ b$	p_2		$\underline{a} \succ \underline{c} \succ b$	p_8
	$\underline{b} \succ a \succ c$	p_3		$\underline{b} \succ \underline{a} \succ c$	p_9
	$\underline{b} \succ c \succ a$	p_4		$\underline{b} \succ \underline{c} \succ a$	p_{10}
	$\underline{c} \succ a \succ b$	p_5		$\underline{c} \succ \underline{a} \succ b$	p_{11}
	$\underline{c} \succ b \succ a$	p_6		$\underline{c} \succ \underline{b} \succ a$	p_{12}
Class III	$\underline{a} \sim \underline{b} \succ c$	p_{13}	**Class IV**	$\underline{a} \succ b \sim c$	p_{16}
	$\underline{a} \sim \underline{c} \succ b$	p_{14}		$\underline{b} \succ a \sim c$	p_{17}
	$\underline{b} \sim \underline{c} \succ a$	p_{15}		$\underline{c} \succ a \sim b$	p_{18}
Class V	$\underline{a} \sim \underline{b} \sim \underline{c}$	p_{19}			

Table 2 The AV score of the candidates

$S(a)$	$=$	$n_1 + n_2 + n_7 + n_8 + n_9 + n_{11} + n_{13} + n_{14} + n_{16} + n_{19}$
$S(b)$	$=$	$n_3 + n_4 + n_7 + n_9 + n_{10} + n_{12} + n_{13} + n_{15} + n_{17} + n_{19}$
$S(c)$	$=$	$n_5 + n_6 + n_8 + n_{10} + n_{11} + n_{12} + n_{14} + n_{15} + n_{18} + n_{19}$

which candidates they approve by underlining the names of the candidates.[6] So, there are 19 possible types of preferences. Following Diss et al. (2010), these 19 types of preferences can be partitioned into five classes of preferences:

Class I this class is made of voters with strict rankings and who only approve their top ranked candidates. These voters are labeled 1–6 in Table 1.
Class II voters in this class also have strict ranking and they approve their top two ranked candidates. These types of voters are labeled 7–12 in Table 1.
Class III in this class, voters are indifferent between their two preferred candidates or do not consider the difference significant enough to reveal their true strict preference. These types of voters are labeled 13–15 in Table 1.
Class IV in this class, voters rank one candidate strictly above the two other between whom they are indifferent. These types of voters are labeled 16–17 in Table 1.
Class V voters of this class are indifferent between the three candidates; thus they approve all the three candidates(type 19).

If we denote by n_t the number of voter of type t, a voting situation is an 19-tuple $\tilde{n} = (n_1, n_2, \ldots, n_t, \ldots, n_{19})$ that indicates the total number n_t of voters casting each type of preferences such that $\sum_{t=1}^{19} n_t = n$. A *voting profile* identifies the specific ranking that each voter has on the candidates. We denote by p_t the probability that a voter chooses the preference type t such that $\sum_{t=1}^{19} p_t = 1$ where $p_t = \frac{n_t}{n}$. In Table 2, $S(a)$ denotes the AV score of candidate a given the labels of Table 1. We denote by p the preference profile of voters which identifies the specific linear ranking that each voter has on the candidates.

Given $a, b \in A$, we denote by n_{ab} the total number of voters who strictly prefer a to b. If $n_{ab} > n_{ba}$, we say that a majority dominates candidate b; or equivalently,

[6] It is assumed that voters vote sincerely. So, we are not concerned with strategic behaviors.

a beats b in a pairwise majority voting. In such a case, we will simply write $a\mathbf{M}b$. Candidate a is said to be the *Condorcet winner* (resp. the Condorcet loser) if for all $b \in A\backslash\{a\}$, $a\mathbf{M}b$ (resp. $b\mathbf{M}a$). If for a given voting situation we get $a\mathbf{M}b$, $b\mathbf{M}c$ and $c\mathbf{M}a$, this describes a majority cycle.

2.2 PAV, the Condorcet winner and the Condorcet loser

By the definition of PAV, it is obvious that with three candidates, if there is a Condorcet winner who belongs to the subset of majority-approved candidates, she is always elected if Rule 2i applies while rule 2ii will never apply. So, Rule 1i and 1ii can fail to elect the Condorcet winner. It is also obvious that if there is a Condorcet loser in a three-candidate election, she cannot be elected under Rule 2i and 2ii; so, PAV may elect the Condorcet loser only when Rule 1i or 1ii applies.

In order to motivate the paper, let us take the following two voting profiles[7] each with 9 voters V_i $(i = 1 \ldots 9)$ in order to illustrate that in three-candidate elections, PAV can fail to select the Condorcet winner when she exists (under Rules 1i, 1ii and 2i) and that it can select the Condorcet loser (under Rules 1i and 1ii).

Profile 1		
V_1:$\underline{a} \succ c \succ b$	V_2:$\underline{a} \succ c \succ b$	V_3:$\underline{b} \succ c \succ a$
V_4:$\underline{b} \succ c \succ a$	V_5:$\underline{c} \succ a \succ b$	V_6:$\underline{c} \succ a \succ b$
V_7:$\underline{b} \succ a \succ c$	V_8:$\underline{c} \succ b \succ a$	V_9:$\underline{a} \succ \underline{b} \succ c$

Profile 2		
V_1:$\underline{a} \succ c \succ b$	V_2:$\underline{b} \succ a \succ c$	V_3:$\underline{b} \succ c \succ a$
V_4:$\underline{b} \succ c \succ a$	V_5:$\underline{c} \succ a \succ b$	V_6:$\underline{c} \succ a \succ b$
V_7:$\underline{c} \succ a \succ b$	V_8: $\underline{a} \succ \underline{b} \succ c$	V_9:$\underline{c} \succ \underline{b} \succ a$

Under both profiles, the reader can check that c is the Condorcet winner and b is the Condorcet loser. Under the first profile, we get $S(a) = S(c) = 3$ and $S(b) = 4$; no candidate gets the majority of the approvals (5 votes), according to Rule 1i, b is the winner since she is the AV winner. Thus, PAV under Rule 1i fails to select the Condorcet winner but selects the Condorcet loser. Under the second profile, b is the unique majority-approved candidate with 5 votes; Rule 1ii applies and b is the PAV winner: PAV under Rule 1i fails to elect the Condorcet winner but can select the Condorcet loser.

To get a profile under which Rule 2i applies and that PAV fails to select the Condorcet winner, the reader only needs to add the following groups of voters to Profile 1: 3 voters with $\underline{a} \succ \underline{b} \succ c$, 2 voters with $\underline{a} \succ c \succ b$, 3 voters with $\underline{c} \succ \underline{a} \succ b$ and 4 voters with $\underline{c} \succ \underline{b} \succ a$.

[7] Other examples are provided in Brams (2008); Brams and Sanver (2009).

The profiles we just used illustrate that under some voting situations, PAV can fail to elect the Condorcet winner when she exists and that it can elect the Condorcet loser when she exists. These two behaviors of PAV are just rare oddities or are a common occurrence? The aim of this paper was then to provide an answer to this question. So, we compute the Condorcet efficiency of PAV and its probability of electing the Condorcet loser for voting situations with three candidates. Before starting this task, we need to define a probability model for this.

2.3 The probability model: the extended impartial culture assumption

The impartial culture (IC) assumption, first introduced in the social choice literature by Gehrlein and Fishburn (1976), is one of the hypothesis used in the social choice literature when computing the likelihood of voting events. Under IC, it is assumed that each voter chooses her (strict) preference according to a uniform probability distribution. When only strict rankings are allowed with m candidates, IC gives a probability $\frac{1}{m!}$ for each of the $m!$ rankings to be chosen independently. The likelihood of a given voting situation $\tilde{n} = (n_1, n_2, \ldots, n_t, \ldots, n_{m!})$ is

$$Prob\,(\tilde{n} = (n_1, n_2, \ldots, n_t, \ldots, n_{m!})) = \frac{n!}{\prod_{t=1}^{m!} n_t!} \times (m!)^{-n}$$

For more details about the IC assumption, see among others Gehrlein and Fishburn (1976); Berg and Lepelley (1994); Gehrlein and Lepelley (2010, 2017); Gehrlein and Fishburn (1980b). According to Gehrlein (1979), one can derive the likelihood of most voting events under the IC assumption using existing results on the representations of quadrivariate normal rules as suggested by Plackett (1954).[8] Gehrlein–Fishburn's technique usually needs a good knowledge of the existing formulas in statistics for the representation of quadrivariate positive orthants (Abrahamson 1964; David and Mallows 1961; Gehrlein 1979).

When indifference is allowed, the Impartial Weak Ordering Culture (IWOC) was introduced by Gehrlein and Fishburn (1980a) as an extension of IC. The reader may refer to Gehrlein and Lepelley (1998); Gehrlein and Valognes (2001); Lepelley and Martin (2001); Merlin and Valognes (2004); Gehrlein and Lepelley (2015) for a non exhaustive review of theoretical works taken under IWOC-like assumptions. Recently, Diss et al. (2010) provided an extension of IC that allows the possibility that voters could have dichotomous preferences with complete indifference between two of the candidates and also the possibility of a complete indifference between all three candidates: the Extended Impartial Culture (EIC) assumption. Let us describe how it works. Consider the 5 classes of preferences described in Table 1 and let us denote by k_1 the probability that a voter's preference belongs to Class **I**; by k_2 the probability that a voter's preference belongs to Class **II**; by k_3 the probability that a voter's preference belongs to Class **III**; by k_4 the probability that a voter's preference belongs to Class

[8] Assume $(X_1, X_2, \ldots, X_n)$ a vector of n random variables with a nonsingular multivariate normal distribution. Plackett (1954) evaluated the probability $P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n)$; he ended with a reduction formula of this probability based on the numerical quadrature for $n = 3, 4$.

IV and by k_5 the probability that a voter's preference belongs to Class **V** such that $k_1 + k_2 + k_3 + k_4 + k_5 = 1$. Under EIC, it is assumed that the rankings within a class are equally likely: $p_t = \frac{k_1}{6}$ for $t = 1, 2, \ldots, 6$, $p_t = \frac{k_2}{6}$ for $t = 7, 8, \ldots, 12$, $p_t = \frac{k_3}{3}$ for $t = 13, 14, 15$, $p_t = \frac{k_4}{3}$ for $t = 16, 17, 18$ and $p_{19} = k_5$.

Diss et al. (2010) used EIC to analyze the Condorcet efficiency of AV and that of all the extended scoring rules. They also provided the limiting probability that a Condorcet winner exists as follows:[9]

$$P^{\infty}_{\text{Con}} = \frac{3}{4} + \frac{3}{2\pi} \arcsin\left(\frac{k_1 + k_2 + k_3 + k_4}{3k_1 + 3k_2 + 2k_3 + 2k_4}\right)$$

Given that $k_1 + k_2 + k_3 + k_4 + k_5 = 1$, we can rewrite P^{∞}_{Con}:

$$\begin{aligned} P^{\infty}_{\text{Con}}(k_{34}, k_5) &= \frac{3}{4} + \frac{3}{2\pi} \arcsin\left(\frac{1 - k_5}{3 - k_3 - k_4 - 3k_5}\right) \\ &= \frac{3}{4} + \frac{3}{2\pi} \arcsin\left(\frac{1 - k_5}{3 - k_{34} - 3k_5}\right) \text{ with } k_{34} = k_3 + k_4 \end{aligned}$$

3 Probability that PAV elects the Condorcet winner

Diss et al. (2010) compute the Condorcet efficiency of AV under EIC assumption by assuming that $p_{19} = 0$. They made this assumption because the preference type of Class **V** has no impact on the outcome under AV; this is not the case under PAV where type 19 can really matter. Gehrlein and Lepelley (2015) provided a representation of the limiting probability of the Condorcet efficiency of AV by assuming that $p_{19} \geq 0$. In this paper, we provide an alternative form of this representation, and then we move to that of PAV.

Given the voting situation $\tilde{n}$ on $A = \{a, b, c\}$, assume without loss of generality that candidate a is the Condorcet winner; this means that $a\mathbf{M}b$ and $a\mathbf{M}b$. Using the labels of Table 1, these conditions are, respectively, equivalent to Eqs. 1 and 2.

$$\begin{aligned} &n_1 + n_2 - n_3 - n_4 + n_5 - n_6 + n_7 + n_8 - n_9 \\ &\quad -n_{10} + n_{11} - n_{12} + n_{14} - n_{15} + n_{16} - n_{17} > 0 \end{aligned} \tag{1}$$

$$\begin{aligned} &n_1 + n_2 + n_3 - n_4 - n_5 - n_6 + n_7 + n_8 + n_9 - n_{10} \\ &\quad -n_{11} - n_{12} + n_{13} - n_{15} + n_{16} - n_{18} > 0 \end{aligned} \tag{2}$$

Candidate a being also the AV winner means that $S(a) > S(b)$ and $S(a) > S(c)$ which are, respectively, equivalent to Eqs. 3 and 4.

$$n_1 + n_2 - n_3 - n_4 + n_8 - n_{10} + n_{11} - n_{12} + n_{14} - n_{15} + n_{16} - n_{17} > 0 \tag{3}$$

$$n_1 + n_2 - n_5 - n_6 + n_7 + n_9 - n_{10} - n_{12} + n_{13} - n_{15} + n_{16} - n_{18} > 0 \tag{4}$$

[9] Notice that P^{∞}_{Con} is also the probability that a Condorcet loser exists.

So, a voting situation under which AV elects the Condorcet winner is fully described by Eqs. 1–4. In order to get a representation of the Condorcet efficiency of AV, we follow the same technique as Gehrlein and Fishburn (1978a). So, considering each of Eqs. 1–4, we define the following four discrete variables:

$$X_1 = \begin{array}{rl} 1 & : p_1 + p_2 + p_5 + p_7 + p_8 + p_{11} + p_{14} + p_{16} \\ -1 & : p_3 - p_4 + p_6 + p_9 + p_{10} + p_{12} p_{15} + p_{17} \\ 0 & : p_{13} + p_{18} + p_{19} \end{array}$$

$$X_2 = \begin{array}{rl} 1 & : p_1 + p_2 + p_3 + p_7 + p_8 + p_9 + p_{13} + p_{16} \\ -1 & : p_4 + p_5 + p_6 + p_{10} + p_{11} + p_{12} + p_{15} + p_{18} \\ 0 & : p_{14} + p_{17} + p_{19} \end{array}$$

$$X_3 = \begin{array}{rl} 1 & : p_1 + p_2 + p_8 + p_{11} + p_{14} + p_{16} \\ -1 & : p_3 + p_4 + p_{10} + p_{12} + p_{15} + p_{17} \\ 0 & : p_5 + p_6 + p_7 + p_9 + p_{13} + p_{18} + p_{19} \end{array}$$

$$X_4 = \begin{array}{rl} 1 & : p_1 + p_2 + p_7 + p_9 + p_{13} + p_{16} \\ -1 & : p_5 + p_6 + p_{10} + p_{12} + p_{15} + p_{18} \\ 0 & : p_3 + p_4 + p_8 + p_{11} + p_{14} + p_{17} + p_{19} \end{array}$$

where p_i is the probability that a voter who is randomly selected from the electorate is associated with the ith ranking of Table 1; $X_1 > 0$ indicates that a is preferred to b and $X_1 < 0$ indicates the reverse; $X_1 = 0$ indicates that there are as many voters who prefer a to b than those who prefer b to a. Similarly, $X_2 > 0$ indicates that a is preferred to c. X_3 and X_4, respectively, represent $S(a) - S(b)$ and $S(a) - S(c)$ which are, respectively, the differences in scores between a and b, then between a and c. Eqs. 1–4 fully describe a situation under which AV elects the Condorcet winner when the average value $\overline{X}_j$ of each of the X_j (for $j = 1, 2, 3, 4$) is positive. According the Gehrlein and Fishburn (1978a, b), the probability of such a situation is equal to the joint probability $\overline{X}_1 > 0$, $\overline{X}_2 > 0$, $\overline{X}_3 > 0$ and $\overline{X}_4 > 0$; when $n \to \infty$, it is equivalent to the quadrivariate normal positive orthant probability $\Phi(R_4)$ such that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ and R_4 is a correlation matrix between the variables X_j. The expectation value of X_j is $E(X_j) = 0$, the variances ($V(X_j) = E(X_j^2)$) and covariances ($\text{Cov}(X_j, X_k) = E(X_j X_k)$) are:

$$V(X_1) = V(X_2) = \frac{3k_1 + 3k_2 + 2k_3 + 2k_4}{3} = \frac{3 - 3k_5 - k_{34}}{3}$$
$$V(X_3) = V(X_4) = \frac{2(k_1 + k_2 + k_3 + k_4)}{3} = \frac{2(1 - k_5)}{3}$$
$$\text{Cov}(X_1, X_2) = \text{Cov}(X_1, X_4) = \text{Cov}(X_2, X_3) = \text{Cov}(X_3, X_4) = \frac{1 - k_5}{3}$$
$$\text{Cov}(X_1, X_3) = \text{Cov}(X_2, X_4) = 2\text{Cov}(X_1, X_2)$$

We derive the correlation matrix R_4 where the components r_{jk} are $r_{jk} = r_{kj} = \frac{\mathrm{Cov}(X_j, X_k)}{\sqrt{V(X_j)V(X_k)}}$:

$$R_4 = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} \\ & 1 & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{pmatrix}$$

Gehrlein (1979) has developed a general representation of the orthant probabilities for obtaining numerical values of $\Phi(R_4)$ as a function of a series of bounded integrals over a single variable. Given r_{jk} the correlation terms in the matrix R_4, this general representation is defined as follows:

$$\begin{aligned}
&\Phi(R_4) \\
&= f(r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) \\
&= \frac{1}{16} + \frac{\arcsin(r_{12}) + \arcsin(r_{13}) + \arcsin(r_{23})}{8\pi} \\
&+ \frac{r_{14}}{4\pi^2}\left[\int_0^1 \frac{\arccos\left(\frac{r_{24}r_{34}z^2 - r_{13}r_{14}r_{24}z^2 + r_{12}r_{13} + r_{14}^2 r_{23}z^2 - r_{12}r_{14}r_{34}z^2 - r_{23}}{\sqrt{(1-r_{14}^2z^2-r_{13}^2-r_{34}^2z^2+2r_{13}r_{14}r_{34}z^2)(1-r_{24}^2z^2-r_{12}^2-r_{14}^2z^2+2r_{12}r_{14}r_{24}z^2)}}\right)}{\sqrt{1-r_{14}^2z^2}}\mathrm{d}z\right] \\
&+ \frac{r_{24}}{4\pi^2}\left[\int_0^1 \frac{\arccos\left(\frac{r_{14}r_{34}z^2 - r_{14}r_{23}r_{24}z^2 + r_{12}r_{23} + r_{24}^2 r_{13}z^2 - r_{12}r_{24}r_{34}z^2 - r_{13}}{\sqrt{(1-r_{24}^2z^2-r_{23}^2-r_{34}^2z^2+2r_{23}r_{24}r_{34}z^2)(1-r_{24}^2z^2-r_{12}^2-r_{14}^2z^2+2r_{12}r_{14}r_{24}z^2)}}\right)}{\sqrt{1-r_{24}^2z^2}}\mathrm{d}z\right] \\
&+ \frac{r_{34}}{4\pi^2}\left[\int_0^1 \frac{\arccos\left(\frac{r_{14}r_{24}z^2 - r_{14}r_{23}r_{34}z^2 + r_{13}r_{23} + r_{34}^2 r_{12}z^2 - r_{13}r_{24}r_{34}z^2 - r_{12}}{\sqrt{(1-r_{24}^2z^2-r_{23}^2-r_{34}^2z^2+2r_{23}r_{24}r_{34}z^2)(1-r_{14}^2z^2-r_{13}^2-r_{34}^2z^2+2r_{13}r_{14}r_{34}z^2)}}\right)}{\sqrt{1-r_{34}^2z^2}}\mathrm{d}z\right]
\end{aligned}$$

Let us define the following quantities:

$$\begin{aligned}
N_1(k_{34}, k_5, z) &= \frac{1}{2}(z+1)(z-1)(1-k_5-k_{34})(3-3k_5-k_{34})^{-\frac{3}{2}}\left(2(1-k_5)\right)^{\frac{1}{2}}; \\
N_2(k_{34}, k_5, z) &= -\frac{1}{4}\left((k_{34}+7k_5-7)z^2 - 4k_{34} - 10k_5 + 10\right)(3-3k_5-k_{34})^{-\frac{3}{2}}\left(2(1-k_5)\right)^{\frac{1}{2}}; \\
D_1(k_{34}, k_5, z) &= \frac{1}{4}(z-2)(z+2)(1-k_5-k_{34}) \\
&\quad (3k_5+k_{34}-3)^{-1}; \\
D_2(k_{34}, k_5, z) &= \frac{\left(16-32k_5-12k_{34}+16k_5^2+12k_5k_{34}+2k_{34}^2+(22k_5+5k_{34}-5k_5k_{34}-11k_5^2-11)z^2\right)}{2(3k_5+k_{34}-3)^2}; \\
D_3(k_{34}, k_5, z) &= -\frac{1}{4}\left((k_{34}+7k_5-7)z^2 - 4k_{34} - 10k_5 + 10\right)(3k_5+k_{34}-3)^{-1};
\end{aligned}$$

$$\mu(k_{34}, k_5, z) = 1 + z^2(1 - k_5)(2(3k_5 + k_{34} - 3))^{-1};$$
$$\nu(k_{34}, k_5, z) = 1 + 2z^2(1 - k_5)(3k_5 + k_{34} - 3)^{-1}$$

We derive Theorem 1 which gives the representation of the limiting Condorcet efficiency of AV.

Theorem 1 *With three candidates and an infinite number of voters, the Condorcet efficiency of AV is given by:*

$$
\begin{aligned}
CE^{\infty}_{AV}(k_{34}, k_5) &= 3\left(\frac{\Phi(R_4)}{P^{\infty}_{\text{Con}}}\right)\\
&= \left(3\left(\pi + 2\arccos\left(\frac{k_5 - 1}{3k_5 + k_{34} - 3}\right)\right)\right)^{-1}\\
&\times\left\{\frac{3\pi}{4} + \frac{3}{2}\arcsin\left(\frac{1 - k_5}{3 - 3k_5 - k_{34}}\right)\right.\\
&+ \frac{3}{2}\arcsin\left(\sqrt{\frac{2(1 - k_5)}{3 - 3k_5 - k_{34}}}\right) + \frac{3}{2}\arcsin\left(\frac{1}{2}\sqrt{\frac{2(1 - k_5)}{3 - 3k_5 - k_{34}}}\right)\\
&+ \frac{3}{2\pi}\sqrt{\frac{2(1 - k_5)}{3 - 3k_5 - k_{34}}}\left(\int_0^1 \frac{\arccos\left(\frac{N_1(k_{34},k_5,z)}{\sqrt{D_1(k_{34},k_5,z)\times D_2(k_{34},k_5,z)}}\right)}{\sqrt{\mu(k_{34}, k_5, z)}}\mathrm{d}z\right)\\
&+ \frac{3}{\pi}\sqrt{\frac{2(1 - k_5)}{3 - 3k_5 - k_{34}}}\left(\int_0^1 \frac{\arccos\left(\frac{N_2(k_{34},k_5,z)}{\sqrt{D_2(k_{34},k_5,z)\times D_3(k_{34},k_5,z)}}\right)}{\sqrt{\nu(k_{34}, k_5, z)}}\mathrm{d}z\right)\\
&\left.+ \frac{3}{4}\left(\int_0^1 \frac{1}{\sqrt{1 - \frac{z^2}{4}}}\mathrm{d}z\right)\right\}
\end{aligned}
$$

Notice that the formula of Theorem 1 is a conditional probability; we multiply the numerator by three in order to annihilate the coefficient 3 that had already been used in the calculation of P^{∞}_{Con}. This remark holds for all the other probabilities of the paper.

In Table 3, we report some computed values of $CE^{\infty}_{AV}(k_{34}, k_5)$. It comes out that given the proportion of voters of each of the types 3, 4 and 5, AV always elects the Condorcet winner in more than 75% of the cases. In this table, one can notice that for a given value of one parameter, the probability tends to increase with the other parameter. Notice that for $k_5 = 0$, we recover the same figures as Diss et al. (2010); for well-defined values of k_{34} and k_5, we get the same figures obtained by Gehrlein and Lepelley (2015).

Let us turn to the Condorcet efficiency of PAV. Assume without loss of generality that a is the PAV winner on $A = \{a, b, c\}$. By definition, PAV elects the Condorcet winner when she exists if one of the following cases holds:

Table 3 Some values of the probabilities $CE_{AV}^{\infty}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.75720	0.75720	0.75720	0.75720	0.75720	0.75723	0.75723	0.75720	0.75720	0.75720	–
0.1	0.76626	0.76725	0.76860	0.77031	0.77271	0.77604	0.78132	0.79074	0.81249	–	–
0.2	0.77604	0.77835	0.78135	0.78525	0.79074	0.79890	0.81249	0.84033	–	–	–
0.3	0.78684	0.79074	0.79578	0.80265	0.81252	0.82827	0.85815	–	–	–	–
0.4	0.79890	0.80469	0.81249	0.82353	0.84036	0.87078	–	–	–	–	–
0.5	0.81252	0.82095	0.83262	0.85005	0.88044	–	–	–	–	–	–
0.6	0.82824	0.84036	0.85812	0.88818	–	–	–	–	–	–	–
0.7	0.84705	0.86490	0.89448	–	–	–	–	–	–	–	–
0.8	0.87081	0.89979	–	–	–	–	–	–	–	–	–
0.9	0.90435	–	–	–	–	–	–	–	–	–	–
1	1	–	–	–	–	–	–	–	–	–	–

(i) when no candidate receive a majority of approvals, the Condorcet winner is the AV winner;
(ii) the Condorcet winner is among the majority approved candidates.

These two cases are disjoints and they fully describe the situations under which PAV always selects the Condorcet winner when she exists. So, given the voting situation $\tilde{n}$ on $A = \{a, b, c\}$, we get

Case (i) when no candidate receive a majority of approvals

$$\begin{cases} a\mathbf{M}b \\ a\mathbf{M}b \\ S(b) < S(a) \\ S(c) < S(a) \\ S(a) < \frac{n}{2} \\ S(b) < \frac{n}{2} \\ S(c) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} a\mathbf{M}b \\ a\mathbf{M}b \\ S(b) < S(a) \\ S(c) < S(a) \\ S(a) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ab} - n_{ba} > 0 \\ n_{ac} - n_{ca} > 0 \\ S(a) - S(b) > 0 \\ S(a) - S(c) > 0 \\ -2S(a) + n > 0 \end{cases}$$

The first four inequalities of the final system, respectively, correspond to Eqs. 1–4. We derive the last inequality as follows:

$$\begin{aligned} &-n_1 - n_2 + n_3 + n_4 + n_5 + n_6 - n_7 - n_8 - n_9 + n_{10} - n_{11} + n_{12} \\ &\quad -n_{13} - n_{14} + n_{15} - n_{16} + n_{17} + n_{18} - n_{19} > 0 \end{aligned} \tag{5}$$

So, a voting situation under which no candidate is majority-approved and that PAV elects the Condorcet winner is fully described by Eqs. 1–5. We proceed as in the proof of Theorem 1 by defining for each equation, a discrete variable. As we have already defined the discrete variables X_1, X_2, X_3 and X_4 for Eqs. 1–4, it remains for us to define the discrete variable X_5 associated with Eq. 5.

$$X_5 = \begin{array}{rl} 1 : & p_3 + p_4 + p_5 + p_6 + p_{10} + p_{12} + p_{15} + p_{17} + p_{18} \\ -1 : & p_1 + p_2 + p_7 + p_8 + p_9 + p_{11} + p_{13} + p_{14} + p_{16} + p_{19} \end{array}$$

Following Gehrlein and Fishburn (1978a, b), the probability of such a situation is equal to the joint probability $\overline{X}_1 > 0$, $\overline{X}_2 > 0$, $\overline{X}_3 > 0$, $\overline{X}_4 > 0$ and $\overline{X}_5 > 0$; when $n \to \infty$, it is equivalent to the quadrivariate normal positive orthant probability $\Phi(R_5)$ such that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ and R_5 is a correlation matrix between the variables X_j. It remains for us to compute the following variances and covariances: $V(X_5)$ and $\text{Cov}(X_j, X_5)$ $(j = 1, 2, 3, 4)$; we find that

$$\begin{aligned} V(X_5) &= 1 \\ \text{Cov}(X_1, X_5) &= \text{Cov}(X_2, X_5) = \text{Cov}(X_3, X_5) = \text{Cov}(X_4, X_5) = -2\text{Cov}(X_1, X_2) \\ \text{Cov}(X_1, X_2) &= \frac{1 - k_5}{3} \end{aligned}$$

We derive the correlation matrix R_5:

$$R_5 = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & -\frac{2\sqrt{3}}{3}\left(\frac{(1-k_5)\sqrt{3-3k_5-k_{34}}}{3-3k_5-k_{34}}\right) & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} \\ & 1 & -\frac{2\sqrt{3}}{3}\left(\frac{(1-k_5)\sqrt{3-3k_5-k_{34}}}{3-3k_5-k_{34}}\right) & \sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} & \sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} \\ & & 1 & -\sqrt{\frac{2(1-k_5)}{3}} & -\sqrt{\frac{2(1-k_5)}{3}} \\ & & & 1 & \frac{1}{2} \\ & & & & 1 \end{pmatrix}$$

Based on the Boole's Theorem, Gehrlein (2017, 2014) developed a general representation of the orthant probabilities for obtaining numerical values of $\Phi(R_5)$ as a linear combination of $\Phi(R_4)$ values, where matrices R_4 are obtained from correlation terms in R_5.

$$\begin{aligned}\Phi(R_5) = \frac{1}{2}\Big(& f(r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) - f(r_{12}, r_{13}, -r_{15}, r_{23}, -r_{25}, -r_{35}) \\ &+ f(r_{12}, -r_{14}, -r_{15}, -r_{24}, -r_{25}, r_{45}) \\ &- f(-r_{13}, -r_{14}, -r_{15}, r_{34}, r_{35}, r_{45}) + f(r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45})\Big)\end{aligned}$$

Then, the probability of electing the Condorcet winner in this case is equal to $\frac{3\Phi(R_5)}{P^{\infty}_{\text{Con}}}$. For the sake of space, we do not report the expression of $\Phi(R_5)$ here; this will also be the case throughout the paper for all the orthant probabilities associated with a 5×5-correlation matrix.

Case (ii) the Condorcet winner is among the majority approved candidates.

$$\begin{cases} a\mathbf{M}b \\ a\mathbf{M}b \\ S(a) > \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ab} - n_{ba} > 0 \\ n_{ac} - n_{ca} > 0 \\ 2S(a) - n > 0 \end{cases}$$

The first two inequalities of this system, respectively, correspond to Eqs. 1 and 2; the last inequality is derived from Eq. 5. We then derive R_3 the matrix associated with the final system we obtain:

$$R_3 = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & \frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} \\ & 1 & \frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} \\ & & 1 \end{pmatrix}$$

Following David and Mallows (1961), we derive $\Phi(R_3)$ the corresponding positive-orthant probability.

$$\Phi(R_3) = \frac{1}{8} + \frac{1}{4\pi}\left[\arcsin\left(\frac{1-k_5}{3-3k_5-k_{34}}\right) + 2\arcsin\left(\frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}}\right)\right]$$

In this case, the probability of electing the Condorcet winner is equal to $\frac{3\Phi(R_3)}{P^{\infty}_{\text{Con}}}$.

From the above, we derive Theorem 2 which gives the representation of the limiting Condorcet efficiency of PAV.

Theorem 2 *With three candidates and an infinite number of voters, the Condorcet efficiency of PAV is given by:*

$$\text{CE}^{\infty}_{\text{PAV}}(k_{34}, k_5) = 3\left(\frac{\Phi(R_5) + \Phi(R_3)}{P^{\infty}_{\text{Con}}}\right)$$

In Appendix, we provide an external link to the full formula of $\text{CE}^{\infty}_{\text{PAV}}(k_{34}, k_5)$. In Table 4 we display some values of $\text{CE}^{\infty}_{\text{PAV}}(k_{34}, k_5)$.

In Table 4, we notice that given a value of k_5, the probability tends to increase with k_{34} and vice versa; PAV always elects the Condorcet winner in more than 88% of the cases. When comparing the figures of Table 4 to those of Table 3, it clearly comes out that PAV performs on the election of the of the Condorcet winner when she exists better than AV for each pair (k_5, k_{34}; moreover, the Condorcet-efficiency of PAV is, for each pair (k_5, k_{34}, at least 10% higher than that of AV.

4 Probability that AV and PAV elect the Condorcet loser

When the Condorcet loser exists and is elected, this is well documented in the literature as the strong Borda paradox. A large literature is devoted to the occurrence of the strong Borda paradox and its variations for the family of scoring rules both for single-winner and multi-winner elections under various probability assumptions. For an overview of these works, the reader may refer without being exhaustive, to the recent papers by Diss and Gehrlein (2012); Diss and Tlidi (2018); Diss et al. (2018); Kamwa (2019); Gehrlein and Lepelley (2010b) and Kamwa and Valognes (2017). Only a little attention has been paid to how often AV could elect the Condorcet loser when she exists; to our knowledge, there is no work dealing with PAV on this.

Lepelley (1993) showed under an extension of IC assumption that if preferences are single-peaked, the election of the Condorcet loser is much less frequent with AV than with the Plurality rule. More recently, Gehrlein et al. (2016) built a framework to compare AV and the Plurality rule and they found under impartial anonymous culture-like assumptions[10] that AV is much less susceptible to elect the Condorcet loser than the Plurality rule. Notice that Gehrlein et al. (2016) investigated different scenarios on voters' preferences included the one assumed in this paper. In this section, we first reconsider the likelihood of AV to elect the Condorcet loser when she exists under the EIC assumption in three-candidate election.

Given the voting situation $\tilde{n}$ on $A = \{a, b, c\}$, assume that candidate a is the Condorcet loser and she is the AV winner; this means that $b\mathbf{M}a$, $c\mathbf{M}a$, $S(a) > S(b)$

[10] Under impartial anonymous culture-like assumptions, voting situations are assumed to be equally likely.

Table 4 Some values of the probabilities $\mathrm{CE}^{\infty}_{\mathrm{PAV}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.90229	0.89834	0.89543	0.89309	0.89109	0.88931	0.88767	0.88608	0.88443	0.88256	–
0.1	0.90815	0.90464	0.90233	0.90083	0.90001	0.89995	0.90098	0.90415	0.91341	–	–
0.2	0.91436	0.91137	0.90981	0.90935	0.91004	0.91231	0.91737	0.92932	–	–	–
0.3	0.92095	0.91859	0.91795	0.91884	0.9216	0.92736	0.93979	–	–	–	–
0.4	0.92799	0.92642	0.92696	0.92968	0.9355	0.94759	–	–	–	–	–
0.5	0.93557	0.93501	0.93717	0.94259	0.95391	–	–	–	–	–	–
0.6	0.94381	0.94463	0.94916	0.95941	–	–	–	–	–	–	–
0.7	0.95288	0.95576	0.96452	–	–	–	–	–	–	–	–
0.8	0.96316	0.96967	–	–	–	–	–	–	–	–	–
0.9	0.97555	–	–	–	–	–	–	–	–	–	–
1	1	–	–	–	–	–	–	–	–	–	–

and $S(a) > S(c)$. The inequalities associated with each of these conditions can easily be derived from the inequalities defined above. Using the same technique as before, we derive the correlation matrix $\overline{R}_4$ as follows:

$$\overline{R}_4 = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & -\sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} & -\sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} \\ & 1 & -\sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} & -\sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{pmatrix}$$

and following Gehrlein (1979, 2017), we derive Theorem 3.

Theorem 3 *With three candidates and an infinite number of voters, the probability that AV elects the Condorcet loser under the EIC assumption is given by*

$$\begin{aligned} CL_{AV}^{\infty}(k_{34}, k_5) &= 3\left(\frac{\Phi(\overline{R}_4)}{P_{\text{Con}}^{\infty}}\right) \\ &= \left(3\left(\pi + 2\arccos\left(\frac{k_5-1}{3k_5+k_{34}-3}\right)\right)\right)^{-1} \\ &\quad \times\left\{\frac{3\pi}{4} + \frac{3}{2}\arcsin\left(\frac{1-k_5}{3-3k_5-k_{34}}\right)\right. \\ &\quad + \frac{3}{2}\arcsin\left(\sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}}\right) + \frac{3}{2}\arcsin\left(\frac{1}{2}\sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}}\right) \\ &\quad - \frac{3}{2\pi}\sqrt{\frac{2(1-k_5)}{(3-3k_5-k_{34})}}\left(\int_0^1 \frac{\arccos\left(\frac{N_1(k_{34},k_5,z)}{\sqrt{D_1(k_{34},k_5,z)\times D_2(k_{34},k_5,z)}}\right)}{\sqrt{\mu(k_{34},k_5,z)}}\mathrm{d}z\right) \\ &\quad - \frac{3}{\pi}\sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}}\left(\int_0^1 \frac{\arccos\left(\frac{N_2(k_{34},k_5,z)}{\sqrt{D_2(k_{34},k_5,z)\times D_3(k_{34},k_5,z)}}\right)}{\sqrt{\nu(k_{34},k_5,z)}}\mathrm{d}z\right) \\ &\quad \left. + \frac{3}{4}\left(\int_0^1 \frac{1}{\sqrt{1-\frac{z^2}{4}}}\mathrm{d}z\right)\right\} \end{aligned}$$

Table 5 reports some values of $CL_{AV}^{\infty}(k_{34}, k_5)$. In this table, one can notice that given the value of one of the parameters, the probability tends to decrease with the other parameter. We notice that the probability is maximized (at 3.709%) when there is no voters who are indifferent between their two preferred candidates ($k_{34} = 0$).

Let us now turn to the probability that PAV elects the Condorcet loser in order to envisage the comparison with AV. By definition, there are two cases under which the Condorcet loser may be elected when she exists. These two cases are disjoints and they fully describe the situations under which PAV selects the Condorcet loser when she exists. So, given the voting situation $\tilde{n}$ on $A = \{a, b, c\}$, by assuming that candidate a is the Condorcet loser, we get

Table 5 Some values of the probabilities $\mathrm{CL}_{\mathrm{AV}}^{\infty}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	0.03709	–
0.1	0.03393	0.03355	0.03311	0.03253	0.03174	0.03063	0.02896	0.02609	0.02013	–	–
0.2	0.03063	0.02989	0.02896	0.02775	0.02609	0.02374	0.02013	0.01379	–	–	–
0.3	0.02724	0.02609	0.02464	0.02272	0.02013	0.01637	0.01051	–	–	–	–
0.4	0.02374	0.02216	0.02013	0.01746	0.01379	0.00847	–	–	–	–	–
0.5	0.02013	0.01805	0.01541	0.01193	0.0071	–	–	–	–	–	–
0.6	0.01637	0.01379	0.01051	0.00612	–	–	–	–	–	–	–
0.7	0.01249	0.00938	0.00538	–	–	–	–	–	–	–	–
0.8	0.00847	0.00475	–	–	–	–	–	–	–	–	–
0.9	0.00436	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–	–	–

Case 1 if no candidate is majority-approved

$$\begin{cases} b\mathbf{M}a \\ c\mathbf{M}a \\ S(b) < S(a) \\ S(c) < S(a) \\ S(a) < \frac{n}{2} \\ S(b) < \frac{n}{2} \\ S(c) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} b\mathbf{M}a \\ c\mathbf{M}a \\ S(b) < S(a) \\ S(c) < S(a) \\ S(a) < \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ba} - n_{ab} > 0 \\ n_{ca} - n_{ac} > 0 \\ S(a) - S(b) > 0 \\ S(a) - S(c) > 0 \\ -2S(a) + n > 0 \end{cases}$$

Following the same technique as before, we derive the correlation matrix $\widehat{R}_5$:

$$\widehat{R}_5 = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & -\sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} & -\sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} & \frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} \\ & 1 & -\sqrt{\frac{1-k_5}{2(3-3k_5-k_{34})}} & -\sqrt{\frac{2(1-k_5)}{3-3k_5-k_{34}}} & \frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} \\ & & 1 & \frac{1}{2} & -\frac{\sqrt{6(1-k_5)}}{3} \\ & & & 1 & -\frac{\sqrt{6(1-k_5)}}{3} \\ & & & & 1 \end{pmatrix}$$

Case 2 if only the Condorcet loser receives a majority of approvals

$$\begin{cases} b\mathbf{M}a \\ c\mathbf{M}a \\ S(b) < \frac{n}{2} \\ S(c) < \frac{n}{2} \\ S(a) > \frac{n}{2} \end{cases} \Rightarrow \begin{cases} n_{ba} - n_{ab} > 0 \\ n_{ca} - n_{ac} > 0 \\ 2S(a) - n > 0 \\ -2S(b) + n > 0 \\ -2S(c) + n > 0 \end{cases}$$

Following the same technique as before, we derive the correlation matrix $\widetilde{R}_5$:

$$\widetilde{R}_5 = \begin{pmatrix} 1 & \frac{1-k_5}{3-3k_5-k_{34}} & -\frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} & 0 & -\frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} \\ & 1 & 0 & -\frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} & -\frac{2(1-k_5)}{\sqrt{3(3-3k_5-k_{34})}} \\ & & 1 & \frac{-3+4k_5}{3} & \frac{3-4k_5}{3} \\ & & & 1 & \frac{3-4k_5}{3} \\ & & & & 1 \end{pmatrix}$$

We then derive Theorem 4 which gives the representation of the limiting Condorcet efficiency of PAV.

Theorem 4 *With three candidates and an infinite number of voters, the probability that PAV elects the Condorcet loser when she exists is given by:*

$$\mathrm{CL}_{\mathrm{PAV}}^{\infty}(k_{34}, k_5) = 3\left(\frac{\Phi(\widehat{R}_5) + \Phi(\widetilde{R}_5)}{P_{\mathrm{Con}}^{\infty}}\right)$$

Table 6 Some values of the probabilities $\mathrm{CL}^{\infty}_{\mathrm{PAV}}(k_{34}, k_5)$

$k_5 \rightarrow$ $k_{34} \downarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.01655	0.01701	0.01729	0.01750	0.01766	0.01780	0.01792	0.01804	0.01815	0.01828	–
0.1	0.01515	0.01540	0.01545	0.01535	0.01513	0.01473	0.01402	0.01272	0.00988	–	–
0.2	0.01370	0.01374	0.01354	0.01313	0.01246	0.01144	0.00977	0.00675	–	–	–
0.3	0.01221	0.01203	0.01154	0.01078	0.00965	0.00791	0.00512	–	–	–	–
0.4	0.01067	0.01024	0.00946	0.00832	0.00665	0.00413	–	–	–	–	–
0.5	0.00907	0.00839	0.00729	0.00572	0.00345	–	–	–	–	–	–
0.6	0.00742	0.00645	0.00500	0.00296	–	–	–	–	–	–	–
0.7	0.00570	0.00442	0.00259	–	–	–	–	–	–	–	–
0.8	0.00392	0.00229	–	–	–	–	–	–	–	–	–
0.9	0.00202	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–	–	–

In Appendix, we provide an external link to the full formula of $CL^{\infty}_{PAV}(k_{34}, k_5)$. Table 6 displays some values of $CL^{\infty}_{PAV}(k_{34}, k_5)$ which teach us that PAV elects the Condorcet loser in less than 1.9%. We also notice that given the value of one of the parameters, the probability tends to decrease with the other parameter. More, the probability tends to be maximized when there is no voters who are indifferent between their two preferred candidates ($k_{34} = 0$).

The comparison between the figures of Table 6 to those of Table 5 highlights as to what extent PAV is less likely to elect the Condorcet loser than AV: the propensity of electing the Condorcet loser under AV is quite the double of that under PAV for all couple of values (k_5, k_{34}).

5 Conclusion

In this paper, we focused on the Preference Approval Voting (PAV) which is a voting rule combining approval and preferences. This rule was first introduced by Brams and Sanver (2009). As nothing is known on the propensity of this rule of electing the Condorcet winner/loser when she exists, the main objective of the paper was to fill this void. Under the extended impartial culture (EIC) assumption, we have provided for three-candidate elections, representation of the limiting probability that PAV elects the Condorcet winner when she exists. In addition to the analysis of Diss et al. (2010) and Gehrlein and Lepelley (2015), we have provided another representation of the limiting Condorcet efficiency of Approval Voting (AV) under EIC. By definition, PAV is built to be more Condorcet-efficient than AV. Our analysis has helped us to highlight as to what extent PAV performs better than AV on the Condorcet criterion. It comes out that AV always elects the Condorcet winner in more than 75% of the cases, while PAV does in more than 88% of the cases and that the efficiency of PAV is, in all cases, at least 10% higher than that of AV.

We also focused, in three-candidate elections, on the probability of electing the Condorcet loser when she exists. As part of our analysis, we have the representation of the limiting probability for AV and also for PAV. For both rules, it comes out that the probability is maximized when there are no voters who are indifferent between their two preferred candidates. By definition, we know that PAV is less susceptible to elect the Condorcet loser than AV in all the cases. We noted that in all the cases, the propensity of electing the Condorcet loser under AV is quite the double of that under PAV.

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Appendix

For space constraints, here are the links to:

– The full representation of $CE^{\infty}_{PAV}(k_{34}, k_5)$, the limiting probability that PAV elects the Condorcet winner when she exists.

https://www.dropbox.com/s/bt0lxbpi39700hm/Full%20CE_PAV.pdf?dl=0

– The full representation of $CL^{\infty}_{PAV}(k_{34}, k_5)$, the limiting probability that PAV elects the Condorcet loser when she exists.

https://www.dropbox.com/s/dc6e5kw3ji6gc27/Full%20CL_PAV.pdf?dl=0

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