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NTU core, TU core and strong equilibria of coalitional population games with infinitely many pure strategies

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Abstract

Inspired by Scarf (J Econ Theory 3: 169–181, 1971), Zhao (Int J Game Theory 28: 25–34, 1999), Sandholm (Population games and evolutionary dynamics. MIT Press, Cambridge, 2010) and Yang and Zhang (Optim Lett. https://doi.org/10.1007/s11590-018-1303-5, 2018), we introduce the model of coalitional population games with infinitely many pure strategies, and define the notions of NTU core and TU core for coalitional population games. We next prove the existence results for NTU cores and TU cores. Furthermore, as an extension of the NTU core, we introduce the notion of strong equilibria and prove the existence theorem of strong equilibria.

Keywords Coalitional population game · NTU core · TU core · Strong equilibria · Existence

1 Introduction

Inspired by Nash (1950a), the population game was established by Nash (1950b) and Sandholm (2010). In the classical population game, the following properties are assumed: (1) the number of populations is finite; (2) there are largely many agents in each population; (3) for each population, all members are homogenous and take the pure strategy from a finite set of actions; (4) the payoff of each agent depends on own

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pure strategy and all populations' states. Sandholm (2010) proved that every continuous population game has at least a Nash equilibrium. Following the representation of Sandholm (2010), Yang and Yang (2017) considered a population game with vector-valued payoffs, and obtained the existence and stability results for weakly Pareto–Nash equilibria of multiobjective population games. Recently, Yang and Zhang (2018) first defined the cooperative equilibria of population games in the model of Sandholm (2010) and analyzed the existence and essential stability of cooperative equilibria for population games with finitely many pure strategies.

The α-core is another major solution concept of normal-form games. It is a cooperative solution by considering the coalitional behavior of players. Due to the α-blocking concept, Aumann (1961) first introduced the notion of the α-core for normal-form games. The existence result of the α-core was then proved by Scarf (1971) in a normal-form game with continuous and quasiconcave payoffs. Later, the work of Scarf (1971) was extended to games with nonordered preferences (Kajii 1992), infinite-player games (Askoura 2011, 2017; Yang 2017, 2018), Bayesian games (Askoura et al. 2013; Noguchi 2014, 2018; Askoura 2015), normal-form games with discontinuous payoffs (Uyanik 2015) and strong Nash equilibrium problems (Nessah and Tian 2014). On the other hand, to answer the question of "how to split joint profits among firm", Zhao (1999) first introduced the notion of TU α-cores for normal-form games and proved its nonemptiness. Accordingly, a generalization of Zhao (1999) to discontinuous games was provided by Uyanik (2015).

Inspired by above literature, we shall present a model of coalitional population games by considering the coalitional behavior of populations. Accordingly, following Scarf (1971), Zhao (1999), and Yang and Zhang (2018), we introduce three cooperative solution concepts: strong equilibrium, NTU core and TU core for coalitional population games, and prove their existence results under some regular conditions.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts and results, and introduces new models, Sect. 3 provides the main results, and Sect. 4 concludes.

2 Preliminaries and models

2.1 Population games

We first recall the classical population game from Sandholm (2010).

Let $P = \{1, \ldots, p_0\}$ be the set of populations. For each population $p \in P$, $S_p = \{1, \ldots, n_p\}$ is the set of pure strategies, the mass of agents is $m_p > 0$ in the population p, and X_p denotes the set of population states (or strategy distributions) for population p, i.e.,

$$X_p = \left\{ x_p = (x_{p,1}, \ldots, x_{p,n_p}) \mid \sum_{i=1}^{n_p} x_{p,i} = m_p, \quad \forall x_{p,i} \geq 0 \right\}.$$

Let $X = \prod_{p \in P} X_p$ be the set of social states. For each $p \in P$, $F_{p,i} : \prod_{p \in P} R_+^{n_p} \longrightarrow R$ is the payoff function of every agent in the population p when the pure strategy $i \in S_p$ is selected. Let $F_p = (F_{p,i})_{i \in S_p}$ and $F = (F_p)_{p \in P}$. Therefore, a population game can be represented by a list:

$$\Gamma = (P, (m_p, S_p, X_p, F_p)_{p \in P}),$$

which is simplified by $\Gamma = F$.

Definition 2.1 (Sandholm 2010) A social state $x^* \in X$ is a Nash equilibrium of a population game F if

$$\left[x_{p,i}^* > 0 \Longrightarrow F_{p,i}(x^*) \geq F_{p,j}(x^*), \quad \forall j \in S_p\right], \quad \forall i \in S_p, \quad \forall p \in P.$$

Theorem 2.1 (Sandholm 2010)

(1) *Suppose that $F_{p,i}$ is continuous on X for each $i \in S_p$ and each $p \in P$. Then the population game F has a Nash equilibrium at least.*
(2) *A social state $x^* \in X$ is a Nash equilibrium of F if and only if*

$$x_p^* \bullet F_p(x^*) \geq y_p \bullet F_p(x^*), \quad \forall y_p \in X_p, \quad \forall p \in P,$$

where $y_p \bullet F_p(x) = \sum_{i=1}^{n_p} y_{p,i} F_{p,i}(x)$.

2.2 Cooperative equilibria of population games

Following the model of Sandholm (2010), Yang and Zhang (2018) first introduced the notion of cooperative equilibria for population games by assuming that there are coalitional behaviors of different populations.

For any $C \subseteq P$, Yang and Zhang (2018) defined the set of feasible states for the coalition C by

$$\widehat{X}_C = \left\{ y_C = (y_{p,1}, \ldots, y_{p,n_p})_{p \in C} \mid \sum_{p \in C} \sum_{i=1}^{n_p} y_{p,i} = \sum_{p \in C} m_p, \quad \forall y_{p,i} \geq 0 \right\}.$$

The definition of $\widehat{X}_C$ means that every pure strategy of $\bigcup_{p \in C} S_p$ can be selected by any agent of the population coalition C. A social state $x^* \in \widehat{X}_P$ is said to be a cooperative equilibrium of F if for any $C \subseteq P$, there exists no $y_C \in \widehat{X}_C$ such that

$$(y_C)_p \bullet F_p(x^*) > x_p^* \bullet F_p(x^*), \quad \forall p \in C.$$

Theorem 2.2 (Yang and Zhang 2018) *Suppose that $F_{p,i}$ is continuous for any $i \in S_p$ and any $p \in P$. Then F has a cooperative equilibrium at least.*

Remark 2.1 As shown in Example 2.1 of Yang and Zhang (2018), Nash equilibria and cooperative equilibria are two different concepts.

2.3 Coalitional population games with infinitely many pure strategies

Generally speaking, the coalitional behavior of each coalition does not exist independently but all agents are interrelated and interact on each other. Moreover, the realistic world is too complex to describe different actions by finitely many pure strategies. Thus, in this section, following above idea, we introduce the model of coalitional population games by assuming that the set of pure strategies is infinite for each agent and the feasible state set of every coalition is related to all populations' states.

Let $P = \{1, \ldots, p_0\}$ be the set of populations. For each population $p \in P$, the set S_p of pure strategies is a nonempty compact subset of a metric space S. Denote by Δ_p the space of measures μ_p defined on S_p such that $\mu_p(S_p) \leq \sum_{p \in P} m_p$. We say that $\{\mu^m \in \Delta_p\}$ converges to $\mu \in \Delta_p$ under the *weak** topology if

$$\int_{S_p} f d\mu^m \longrightarrow \int_{S_p} f d\mu$$

for any continuous function f defined on S_p. By adopting the proof technique analogous to Theorem 6.5 in Walters (2003), it is easy to prove the *weak** compactness of Δ_p. Thus, Δ_p is a nonempty convex *weak** compact subset of a locally convex Hausdorff topological vector space E_p.

Denote

$$\Delta_C = \prod_{i \in C} \Delta_i, \ \Delta_{-C} = \prod_{i \notin C} \Delta_i, \quad \forall C \subseteq P.$$

For a social state, $\mu = (\mu_1, \ldots, \mu_{p_0}) \in \Delta_P$ and a point $x \in S_p$, $\mu_p(x)$ represents the mass of members who select the pure strategy $x \in S_p$. A coalition of populations is a subset C of P. For each $C \subseteq P$, $G_C : \Delta_P \rightrightarrows \Delta_C$ is the feasible state correspondence of the coalition C. For each $p \in P$, $F_p(x, \bullet) : \Delta_P \longrightarrow R$ is the payoff function of every agent in the population p when the pure strategy x is taken. Our coalitional population game, therefore, can be represented by a list:

$$\Gamma = (P, (m_p, S_p, \Delta_p)_{p \in P}, (F_p(x, \bullet))_{x \in S_p, p \in P}, (G_C)_{C \subseteq P}).$$

Definition 2.2 A social state $\mu^* \in \Delta_P$ is said to be in the NTU core of a coalitional population game Γ if $\mu^* \in G_P(\mu^*)$ and for any $C \subseteq P$, there exists no $(\mu_p)_{p \in C} \in G_C(\mu^*)$ such that

$$\int_{x \in S_p} F_p(x, \mu^*) d\mu_p > \int_{x \in S_p} F_p(x, \mu^*) d\mu_p^*, \quad \forall p \in C.$$

Definition 2.3 A social state $(\mu^*, \sigma) \in \Delta_P \times R^{p_0}$ is said to be in the TU core of a coalitional population game Γ if $\mu^* \in G_P(\mu^*)$ and

(1) $$\sum_{p\in P}\sigma_p=\sum_{p\in P}\int_{x\in S_p}F_p(x,\mu^*)d\mu_p^*=\max_{\mu\in G_P(\mu^*)}\sum_{p\in P}\int_{x\in S_p}F_p(x,\mu^*)d\mu_p;$$

(2) for any $C\subseteq P$, there exists no $(\nu_p)_{p\in C}\in G_C(\mu^*)$ such that

$$\sum_{p\in C}\int_{x\in S_p}F_p(x,\mu^*)d\nu_p>\sum_{p\in C}\sigma_p.$$

Remark 2.2 We say that a social state is in the NTU core of coalitional population games if there exists no feasible state of a population coalition which makes every population of this coalition better. However, the TU core is defined by the social state and the allocation of outcomes. We say that a social state and an allocation of outcomes is in the TU core if the sum of all populations' payoffs is the maximum and there exists no population coalition which can change the allocation of the outcome.

Remark 2.3 When $S_p=\{1,\dots,n_p\}$ is a finite set for each $p\in P$, we can represent a population game:

$$\Gamma=(P,(m_p,S_p,X_p,F_p)_{p\in P})$$

by the model of coalitional population games in Sect. 2.3. Given

$$\Gamma=(P,(m_p,S_p,X_p,F_p)_{p\in P}),$$

we construct a coalitional population game

$$\Gamma'=(P,(m_p,S_p,\Delta_p)_{p\in P},(\widetilde{F}_p(x,\bullet))_{x\in S_p,p\in P},(G_C)_{C\subseteq P}),$$

as follows:

(i) P and $(m_p,\Delta_p)_{p\in P}$ are defined as Sect. 2.3;
(ii) For any $C\subseteq P$,

$$G_C(\mu)=\left\{(\mu'_p)_{p\in C}\in\Delta_C\mid\sum_{p\in C}\sum_{i=1}^{n_p}\mu'_p(i)=\sum_{p\in C}m_p\right\}=\widehat{X}_C;$$

(iii) For any $p\in P$ and any $i\in S_p$,

$$\widetilde{F}_p(i,\mu)=F_{p,i}\left(\left(\mu_p(1),\dots,\mu_p(n_p)\right)_{p\in P}\right),\ \forall\mu\in\Delta_P.$$

It is obvious that $\mu^*\in\Delta_P$ is in the NTU core of Γ' if and only if

$$\left(\mu_p^*(1),\dots,\mu_p^*(n_p)\right)_{p\in P}\in\widehat{X}_P$$

is a cooperative equilibrium of Γ.

2.4 Social coalitional equilibrium existence lemma

To prove the nonemptiness of NTU cores for coalitional population games, we recall a special case of social coalitional equilibrium existence lemma in Ichiishi (1981).

A society is defined by a list:

$$\tau = (N, (X_i)_{i\in N}, (G_C)_{C\in\mathcal{N}}, (u_{C,i})_{i\in C\in\mathcal{N}}),$$

where $N = \{1, \ldots, n\}$ is the set of agents and $\mathcal{N}$ is the set of all nonempty subsets of N. For each $i \in N$, X_i is the set of actions for player i. Let

$$X_C = \prod_{i\in C} X_i, X_{-C} = \prod_{i\notin C} X_i, \quad \forall C \in \mathcal{N}.$$

For each $C \in \mathcal{N}$, $G_C : X_N \rightrightarrows X_C$ is the feasible strategy correspondence of the coalition C, and $u_{C,i} : Graph(G_C) \longrightarrow R$ is the payoff function of the player i in the coalition C.

Definition 2.4 A point $x^* \in X_N$ is a social coalitional equilibrium of a society τ if $x^* \in G_N(x^*)$ and for any $C \in \mathcal{N}$, there exists no $y_C \in G_C(x^*)$ such that

$$u_{C,i}(x^*, y_C) > u_{N,i}(x^*, x^*), \forall i \in C.$$

Theorem 2.3 (Ichiishi 1981) *Suppose that a society τ satisfies the following conditions:*

(i) *for each $i \in N$, X_i is a nonempty convex compact subset of a locally convex Hausdorff topological vector space;*
(ii) *for each $C \in \mathcal{N}$, G_C is continuous with nonempty values;*
(iii) *for each $C \in \mathcal{N}$ and each $i \in C$, $u_{C,i}$ is continuous on $Graph(G_C)$;*
(iv) *for any balanced family β of subsets of N, any $\omega \in R^n$ and any $x \in X$, if*

$$y_C \in \{z_C \in G_C(x) \mid u_{C,i}(x, z_C) \geq \omega_i, \quad \forall i \in C\}, \quad \forall C \in \beta,$$

then there exists $y' \in G_N(x)$ such that $u_{N,i}(x, y') \geq \omega_i, \quad \forall i \in N$;
(v) *for any $\omega \in R^n$ and any $x \in X$, the set*

$$\{z \in G_N(x) \mid u_{N,i}(x, z) \geq \omega_i, \quad \forall i \in N\}$$

is convex.

Then there exists at least a social coalitional equilibrium of τ.

3 NTU and TU cores

We make the following assumptions to obtain existence results for NTU cores and TU cores.

(A-1) For any $p \in P$, F_p is continuous on $S_p \times \Delta_P$.
(A-2) For any balanced family β of subsets of P with balancing weights $\{\lambda_C \mid C \in \beta\}$ and any $\mu \in \Delta_P$, if $v_C \in G_C(\mu)$ for any $C \in \beta$, then $\bar{v} \in G_P(\mu)$, where

$$\bar{v}_p = \sum_{C \in \beta, p \in C} \lambda_C (v_C)_p, \quad \forall p \in P.$$

(A-3) For any $C \subseteq P$, G_C is continuous with nonempty compact values and for any $\mu \in \Delta_P$, $G_P(\mu)$ is convex in Δ_P.

We next prove the following lemma.

Lemma 3.1 *Under the assumption (A-1),*

$$(\mu, \nu_p) \longrightarrow \int_{x \in S_p} F_p(x, \mu) d\nu_p$$

is continuous on $\Delta_P \times \Delta_p$ for each $p \in P$.

Proof Let $(\mu^m, \nu_p^m) \in \Delta_P \times \Delta_p$ and $(\mu^m, \nu_p^m) \longrightarrow (\mu, \nu_p)$ under the $weak^*$ topology. By (A-1), we have

$$\begin{aligned} &\left| \int_{x \in S_p} F_p(x, \mu^m) d\nu_p^m - \int_{x \in S_p} F_p(x, \mu) d\nu_p \right| \\ &\leq \left| \int_{x \in S_p} F_p(x, \mu^m) d\nu_p^m - \int_{x \in S_p} F_p(x, \mu) d\nu_p^m \right| \\ &\quad + \left| \int_{x \in S_p} F_p(x, \mu) d\nu_p^m - \int_{x \in S_p} F_p(x, \mu) d\nu_p \right| \longrightarrow 0. \end{aligned}$$

This completes the proof. □

Theorem 3.1 *Under assumptions (A-1)–(A-3), the NTU core of a coalitional population game Γ is nonempty.*

Proof We construct a society

$$\tau = (P, (\Delta_p)_{p \in P}, (G_C)_{C \subseteq P}, (u_{C,p})_{p \in C \subseteq P})$$

from a coalitional population game

$$\Gamma = (P, (m_p, S_p, \Delta_p)_{p \in P}, (F_p(x, \bullet))_{x \in S_p, p \in P}, (G_C)_{C \subseteq P}),$$

where $u_{C,p}$ is defined by

$$u_{C,p}(\mu, \nu_C) = \int_{x \in S_p} F_p(x, \mu) d(\nu_C)_p, \quad \forall p \in C \subseteq P, \quad \forall (\mu, \nu_C) \in \Delta_P \times \Delta_C.$$

Obviously, the conditions (i)–(iii) of Theorem 2.3 hold.

Let a balanced family β of subsets of P with balancing weights $\{\lambda_C \mid C \in \beta\}$ and $\omega \in R^{p_0}$, $\mu \in \Delta_P$. If

$$\nu_C \in \{\nu'_C \in G_C(\mu) \mid u_{C,p}(\mu, \nu'_C) \geq \omega_p, \quad \forall p \in C\}\ \forall C \in \beta,$$

then, by (A-2), we have $\bar{\nu} \in G_P(\mu)$, where

$$\bar{\nu}_p = \sum_{C \in \beta, p \in C} \lambda_C (\nu_C)_p, \quad \forall p \in P,$$

and

$$\int_{x \in S_p} F_p(x, \mu) d(\nu_C)_p \geq \omega_p, \quad \forall p \in C, \quad \forall C \in \beta.$$

For any fixed $q \in P$, since

$$\nu_p \longrightarrow \int_{x \in S_p} F_p(x, \mu) d\nu_p$$

is linear for any $p \in P$, it follows that

$$\begin{aligned} u_{P,q}(\mu, \bar{\nu}) &= \int_{x \in S_q} F_q(x, \mu) d\bar{\nu}_q \\ &= \int_{x \in S_q} F_q(x, \mu) d\left(\sum_{C \in \beta, q \in C} \lambda_C (\nu_C)_q\right) \\ &= \sum_{C \in \beta, q \in C} \lambda_C \int_{x \in S_q} F_q(x, \mu) d(\nu_C)_q \\ &\geq \sum_{C \in \beta, q \in C} \lambda_C \omega_q \\ &= \omega_q. \end{aligned}$$

It shows that

$$\bar{\nu} \in \{\nu' \in G_P(\mu) \mid u_p(\mu, \nu') \geq \omega_p, \quad \forall p \in P\}.$$

Therefore, the condition (iv) of Theorem 2.3 holds.

Let $\mu \in \Delta_P$ and $\omega \in R^{p_0}$. By (A-3), it is obvious that the set

$$\begin{aligned} &\{\nu \in G_P(\mu) \mid u_{P,p}(\mu, \nu) \geq \omega_p, \quad \forall p \in P\} \\ &\quad = \bigcap_{p \in P} \left\{\nu \in G_P(\mu) \mid \int_{x \in S_p} F_p(x, \mu) d\nu_p \geq \omega_p\right\} \end{aligned}$$

is convex in $\varDelta_P$. Therefore, the condition (v) of Theorem 2.3 holds.

By Theorem 2.3, there exists $\mu^* \in \varDelta_P$ such that $\mu^* \in G_P(\mu^*)$ and for any $C \subseteq P$, there exists no $(\nu_p)_{p\in C} \in G_C(\mu^*)$ for which

$$\begin{aligned}\int_{x\in S_p} F_p(x,\mu^*)d\nu_p &= u_{C,p}(\mu^*,(\nu_p)_{p\in C})\\ &> u_{P,p}(\mu^*,\mu^*)\\ &= \int_{x\in S_p} F_p(x,\mu^*)d\mu_p^*, \quad \forall p \in C.\end{aligned}$$

It shows that the NTU core of $\varGamma$ is nonempty. □

Remark 3.1 By Remark 2.3, if S_p is a finite set for each $p \in P$, we can derive Theorem 2.2 of Yang and Zhang (2018) by our Theorem 3.1. Note that the proof technique of Theorem 2.2 in Yang and Zhang (2018) cannot be applied to our Theorem 3.1, since $(X_p)_{p\in P}$ and $(\widehat{X}_C)_{C\subseteq P}$ are some subsets of finite dimensional spaces in Yang and Zhang (2018), but $\varDelta_p$ is a subset of a locally convex Hausdorff topological space for each $p \in P$.

We next generalized the work of Zhao (1999) in coalitional population games.

Theorem 3.2 *Under the assumptions (A-1)–(A-3), the TU core of a coalitional population game is nonempty.*

Proof Define a correspondence $T : \varDelta_P \rightrightarrows \varDelta_P$ by

$$T(\mu) = \left\{\mu' \in G_P(\mu) \mid \sum_{p\in P}\int_{x\in S_p} F_p(x,\mu)d\mu'_p = \max_{\nu\in G_P(\mu)}\sum_{p\in P}\int_{x\in S_p} F_p(x,\mu)d\nu_p\right\}.$$

By Lemma 3.1, and (A-1) and (A-3), it is easy to verify that T is upper semicontinuous with nonempty convex compact values. By Fan–Glicksberg fixed point theorem, there exists $\mu^* \in \varDelta_P$ such that $\mu^* \in T(\mu^*)$.

Now, we construct an TU game $V : 2^P \longrightarrow R$ by

$$V(P) = \sum_{p\in P}\int_{x\in S_p} F_p(x,\mu^*)d\mu_p^* = \max_{\nu\in G_P(\mu^*)}\sum_{p\in P}\int_{x\in S_p} F_p(x,\mu^*)d\nu_p$$

$$V(C) = \max_{\nu_C\in G_C(\mu^*)}\sum_{p\in C}\int_{x\in S_p} F_p(x,\mu^*)d(\nu_C)_p,\ \forall C \subseteq P.$$

We next prove the TU game $\{P,(V(C))_{C\subseteq P}\}$ is balanced, that is, for any balanced family β of subsets of P with balancing weights $\{\lambda_C \mid C \subseteq P\}$, we have

$$\sum_{C\in\beta}\lambda_C V(C) \leq V(P).$$

By Lemma 3.1, for any $C \in \beta$, there exists $\nu_C \in G_C(\mu^*)$ such that

$$V(C) = \sum_{p\in C} \int_{x\in S_p} F_p(x, \mu^*) d(\nu_C)_p.$$

By (A-2), we have $\bar{\nu} \in G_P(\mu^*)$, where

$$\bar{\nu}_p = \sum_{C\in\beta, p\in C} \lambda_C (\nu_C)_p, \ \forall p \in P.$$

Then, we have

$$\begin{aligned}
\sum_{C\in\beta} \lambda_C V(C) &= \sum_{C\in\beta} \sum_{p\in C} \lambda_C \int_{x\in S_p} F_p(x, \mu^*) d(\nu_C)_p \\
&= \sum_{p\in P} \sum_{C\in\beta, p\in C} \lambda_C \int_{x\in S_p} F_p(x, \mu^*) d(\nu_C)_p \\
&= \sum_{p\in P} \int_{x\in S_p} F_p(x, \mu^*) d\left(\sum_{C\in\beta, p\in C} \lambda_C (\nu_C)_p\right) \\
&= \sum_{p\in P} \int_{x\in S_p} F_p(x, \mu^*) d\bar{\nu}_p \\
&\le \max_{\nu'\in G_p(\mu^*)} \sum_{p\in P} \int_{x\in S_p} F_p(x, \mu^*) d\nu'_p \\
&= V(P).
\end{aligned}$$

It shows that the TU game $(P, (V(C))_{C\subseteq P})$ is balanced. Therefore, there exists $(\sigma_1, \ldots, \sigma_{p_0}) \in R^{p_0}$ such that

$$\sum_{p\in P} \sigma_p = V(P)$$

and

$$\sum_{p\in C} \sigma_p \ge V(C), \quad \forall C \subseteq P.$$

It implies that $(\mu^*, \sigma) \in \Delta_P \times R^{p_0}$ is in the TU core of Γ. □

To better understand the coalitional population game and two cooperative solutions in our model, we next illustrate an example with finitely many pure strategies.

Example 3.1 Assume that there exist two populations of firms in a market, and $m_1 = m_2 = m > 0$. The inverse demand function is given by $p = a - q$, where q is the total products in the market, and the cost function of every firm is $C(q) = cq$, where

$a, c > 0, a - c > 0$ is sufficiently large. The sets of pure strategies of populations 1 and 2 are $S_1 = S_2 = \{0, 1\}$, where the strategy 0 means that the firm stops production, and the strategy 1 means that the firm produces products in one unit of goods. We construct a coalitional population game:

$$\Gamma = (P, (m_p, S_p, \Delta_p)_{p \in P}, (F_{p,i})_{i \in S_p, p \in P}, (G_C)_{C \subseteq P}),$$

as follows:

(i) $P = \{1, 2\}, S_1 = S_2 = \{0, 1\}, \Delta_{12} = \Delta_1 \times \Delta_2,$

$$\Delta_1 = \{(\mu_{11}, \mu_{12}) \in R_+^2 \mid \mu_{11} + \mu_{12} \leq 2m\},$$
$$\Delta_2 = \{(\mu_{21}, \mu_{22}) \in R_+^2 \mid \mu_{21} + \mu_{22} \leq 2m\},$$

where μ_{11} is the mass of firms selecting the pure strategy 0 in population 1, μ_{12} is the mass of firms selecting the pure strategy 1 in population 1, μ_{21} is the mass of firms selecting the pure strategy 0 in population 2 and μ_{22} is the mass of firms selecting the pure strategy 1 in population 2.

(ii) For any $\mu \in \Delta_{12}$,

$$G_1(\mu) = \{\mu_1' \in \Delta_1 \mid \mu_{11}' + \mu_{12}' \leq \min\{m, \mu_{11} + \mu_{12}\}\},$$
$$G_2(\mu) = \{\mu_2' \in \Delta_2 \mid \mu_{21}' + \mu_{22}' \leq \min\{m, \mu_{21} + \mu_{22}\}\},$$
$$G_{12}(\mu) = \{\mu' \in \Delta_1 \times \Delta_2 \mid \mu_{11}' + \mu_{12}' + \mu_{21}' + \mu_{22}' \leq 2m\}.$$

(iii) For any $\mu \in \Delta_{12}$,

$$F_{1,1}(\mu) = (a - c - 0\mu_{11} - 1\mu_{12} - 0\mu_{21} - 1\mu_{22}) \bullet 0 = 0,$$
$$F_{1,2}(\mu) = (a - c - \mu_{12} - \mu_{22}) \bullet 1 = a - c - \mu_{12} - \mu_{22},$$
$$F_{2,1}(\mu) = 0,$$
$$F_{2,2}(\mu) = a - c - \mu_{12} - \mu_{22}.$$

We say that $\mu^* \in G_{12}(\mu^*)$ is in the NTU core if there exist no $\mu_1' \in G_1(\mu^*), \mu_2' \in G_2(\mu^*), and \mu'' \in G_{12}(\mu^*)$ such that

$$\mu_{11}' F_{1,1}(\mu^*) + \mu_{12}' F_{1,2}(\mu^*) > \mu_{11}^* F_{1,1}(\mu^*) + \mu_{12}^* F_{1,2}(\mu^*),$$
$$\mu_{21}' F_{21}(\mu^*) + \mu_{22}' F_{22}(\mu^*) > \mu_{21}^* F_{2,1}(\mu^*) + \mu_{22}^* F_{2,2}(\mu^*),$$

and

$$\mu_{11}'' F_{1,1}(\mu^*) + \mu_{12}'' F_{1,2}(\mu^*) > \mu_{11}^* F_{1,1}(\mu^*) + \mu_{12}^* F_{1,2}(\mu^*),$$
$$\mu_{21}'' F_{21}(\mu^*) + \mu_{22}'' F_{22}(\mu^*) > \mu_{21}^* F_{2,1}(\mu^*) + \mu_{22}^* F_{2,2}(\mu^*),$$

hold simultaneously, that is, there exist no $\mu'_1 \in G_1(\mu^*)$, $\mu'_2 \in G_2(\mu^*)$ and $\mu'' \in G_{12}(\mu^*)$ such that

$$\mu'_{12} > \mu^*_{12}, \mu'_{22} > \mu^*_{22},$$
$$(\mu''_{12}, \mu''_{22}) - (\mu^*_{12}, \mu^*_{22}) \in int R^2_+.$$

We obtain that the NTU core is $\{\mu \in \Delta_1 \times \Delta_2 \mid \mu_{11} = \mu_{21} = 0, \mu_{12} + \mu_{22} = 2m\}$.

Furthermore, by solving the problem finding $\mu^* \in G_{12}(\mu^*)$ such that

$$\begin{aligned}&\mu^*_{11}F_{1,1}(\mu^*) + \mu^*_{12}F_{1,2}(\mu^*) + \mu^*_{21}F_{2,1}(\mu^*) + \mu^*_{22}F_{2,2}(\mu^*)\\ &\quad = \max_{\mu \in G_{12}(\mu^*)} \{\mu_{11}F_{1,1}(\mu^*) + \mu_{12}F_{1,2}(\mu^*) + \mu_{21}F_{2,1}(\mu^*) + \mu_{22}F_{2,2}(\mu^*)\},\end{aligned}$$

we have $\mu^*_{12} + \mu^*_{22} = 2m$ and

$$\begin{aligned}&\mu^*_{11}F_{1,1}(\mu^*) + \mu^*_{12}F_{1,2}(\mu^*) + \mu^*_{21}F_{2,1}(\mu^*) + \mu^*_{22}F_{2,2}(\mu^*)\\ &\quad = 2m(a - c - 2m).\end{aligned}$$

By (2) of Definition 2.3, we obtain that the TU core is

$$\begin{aligned}&\{(\mu^*, \sigma) \in \Delta_{12} \times R^2_+ \mid \mu^*_{11} = \mu^*_{21} = 0, \mu^*_{12} + \mu^*_{22} = 2m,\\ &\quad \sigma_1 = \mu^*_{12}(a - c - 2m), \sigma_2 = \mu^*_{22}(a - c - 2m)\}\end{aligned}$$

Remark 3.2 In Example 3.1, we obtain different results relative to the classical duopoly model. We can explain the fact by the following reasons. (i) The hypotheses of our Example 3.1 are different from the duopoly model. (ii) Following the idea of population biology, we assume that all micro-individuals (firms) form a continuum of mass m, and make the choice between two pure strategies. The final social state is implied by the choice of every agent in two populations. Thus, although every micro-individual is rational, every population may represent the irrational macro-behaviors. (iii) Our notations of cooperative solutions are different than those of Sandholm (2010), Zhao (1999) and classical duopoly models.

4 Some extensions: strong equilibria

Inspired by Nessah and Tian (2014), we introduce a refinement solution concept of the NTU core, that is, strong equilibrium.

Definition 4.1 A social state $\mu^* \in \Delta_P$ is said to be a strong equilibrium of a coalitional population game Γ if for any $C \subseteq P$, $\mu^*_C \in G_C(\mu^*)$ and

$$\sum_{p \in C} \int_{x \in S_p} F_p(x, \mu^*) d\mu^*_p = \max_{\nu_C \in G_C(\mu^*)} \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu^*) d(\nu_C)_p.$$

Now, denote by $S(\Gamma)$, $C_{NTU}(\Gamma)$ the set of strong equilibria and the NTU core of Γ, respectively.

Theorem 4.1 $S(\Gamma) \subseteq C_{NTU}(\Gamma)$.

Proof Suppose that $\mu^* \notin C_{NTU}(\Gamma)$. Then (1) $\mu^* \notin G_P(\mu^*)$ or (2) there exist $C \subseteq P$ and $\nu_C \in G_C(\mu^*)$ such that

$$\int_{x \in S_p} F_p(x, \mu^*) d(\nu_C)_p > \int_{x \in S_p} F_p(x, \mu^*) d\mu^*_p, \quad \forall p \in C.$$

It is obvious that $\mu^* \notin S(\Gamma)$ if the case (1) occurs.

If the case (2) occurs, then

$$\begin{aligned} \max_{\nu'_C \in G_C(\mu^*)} \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu^*) d(\nu'_C)_p &\geq \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu^*) d(\nu_C)_p \\ &> \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu^*) d\mu^*_p. \end{aligned}$$

It shows that $\mu^* \notin S(\Gamma)$. Therefore, $S(\Gamma) \subseteq C_{NTU}(\Gamma)$. □

To prove the existence of strong equilibria, we shall assume, in addition to (A-1):

(A-4) For any $C \subseteq P$, G_C is continuous with nonempty convex compact values.
(A-5) Γ has the coalitional consistency property, that is, for any $\mu \in \Delta_P$, there exists $\mu' \in \Delta_P$ such that for any $C \subseteq P$, $\mu'_C \in G_C(\mu)$ and

$$\sum_{p \in C} \int_{x \in S_p} F_p(x, \mu) d(\mu'_C)_p = \max_{\nu_C \in G_C(\mu)} \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu) d(\nu_C)_p.$$

Theorem 4.2 *(Corollary 17.59 in Aliprantis and Border 2006) Let X be a nonempty convex compact subset of a locally convex Hausdorff topological vector space E and $F : X \rightrightarrows E$ be a correspondence such that (1) F is upper semicontinuous with nonempty convex compact values; (2) for any $x \in X$, there exist $\lambda > 0$ and $y \in F(x)$ such that $x + \lambda y \in X$. Then, there exists $x^* \in X$ such that $0 \in F(x^*)$.*

Theorem 4.3 *Under the assumptions (A-1), (A-4) and (A-5), the set of strong equilibria is nonempty.*

Proof (1) For any $C \subseteq P$, define a correspondence $T_C : \Delta_P \rightrightarrows \Delta_C$ by

$$\begin{aligned} T_C(\mu) = \Big\{ \mu'_C \in G_C(\mu) | \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu) d(\mu'_C)_p \\ = \max_{\nu_C \in G_C(\mu)} \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu) d(\nu_C)_p \Big\}. \end{aligned}$$

By (A-1), (A-4) and Lemma 3.1, it is easy to verify that T_C is upper semicontinuous with nonempty convex compact values.

(2) Let

$$\widehat{\Delta} = \prod_{C \subseteq P} \Delta_C,\ E_C = \prod_{i \in C} E_i,\ \widehat{E} = \prod_{C \subseteq P} E_C$$

and the mapping $g : \Delta_P \longrightarrow \widehat{\Delta}$ be defined by

$$g(\mu) = (\mu_C)_{C \subseteq P}.$$

Obviously, g is linear and $g(\Delta_P)$ is a nonempty convex compact subset of $\widehat{\Delta}$. Define a mapping $\widehat{T} : g(\Delta_P) \rightrightarrows \widehat{E}$ by

$$\widehat{T}((\mu_C)_{C \subseteq P}) = \left\{(\mu'_C)_{C \subseteq P} \in g(\Delta_P) \mid \mu'_C \in T_C(\mu),\ \forall C \subseteq P\right\} - (\mu_C)_{C \subseteq P}.$$

By part (1) and (A-5), we obtain that $\widehat{T}$ is upper semicontinuous with nonempty convex compact values. Furthermore, by (A-5), for any $\mu \in \Delta_P$, there exists $\mu' \in \Delta_P$ such that for any $C \subseteq P$, $\mu'_C \in G_C(\mu)$ and

$$\sum_{p \in C} \int_{x \in S_p} F_p(x, \mu) d(\mu'_C)_p = \max_{\nu_C \in G_C(\mu)} \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu) d(\nu_C)_p.$$

It implies that $\mu'_C \in T_C(\mu)$ for any $C \subseteq P$ and

$$(\mu'_C)_{C \subseteq P} - (\mu_C)_{C \subseteq P} \in \widehat{T}((\mu_C)_{C \subseteq P}).$$

By the convexity of $g(\Delta_P)$, we have that for any $\lambda \in [0, 1]$,

$$\begin{aligned}(\mu_C)_{C \subseteq P} &+ \lambda[(\mu'_C)_{C \subseteq P} - (\mu_C)_{C \subseteq P}] \\ &= (1 - \lambda)(\mu_C)_{C \subseteq P} + \lambda(\mu'_C)_{C \subseteq P} \in g(\Delta_P).\end{aligned}$$

(3) By parts (1) and (2), $\widehat{T}$ satisfies all conditions of Theorem 4.2. Then, there exists $\mu^* \in \Delta_P$ such that $0 \in \widehat{T}(\mu^*)$, that is, for any $C \subseteq P$, $\mu^*_C \in G_C(\mu^*)$ and

$$\sum_{p \in C} \int_{x \in S_p} F_p(x, \mu^*) d\mu^*_p = \max_{\nu_C \in G_C(\mu^*)} \sum_{p \in C} \int_{x \in S_p} F_p(x, \mu^*) d(\nu_C)_p.$$

This completes the proof.

□

Remark 4.1 Since the strong equilibrium is a stronger solution concept than the NTU core for coalitional population games, we adjust some existence sufficient conditions relative to Theorem 3.1.

Remark 4.2 To prove the existence of strong Nash equilibria for normal-form games, Nessah and Tian (2014) introduced the notion of coalitional consistency property for normal-form games. Thus, following the idea of Nessah and Tian (2014), we also introduce the notion of coalitional consistency property for coalitional population games.

Example 4.1 In Example 3.1, by Definition 4.1, we can obtain the strong equilibrium $\mu^* \in \Delta_P$ of Γ such that $\mu^* \in G_{12}(\mu^*)$, $\mu_1^* \in G_1(\mu^*)$, $\mu_2^* \in G_2(\mu^*)$,

$$\begin{aligned}
&\mu_{11}^* F_{1,1}(\mu^*) + \mu_{12}^* F_{1,2}(\mu^*) + \mu_{21}^* F_{2,1}(\mu^*) + \mu_{22}^* F_{2,2}(\mu^*) \\
&\quad = \max_{\mu \in G_{12}(\mu^*)} (\mu_{11} F_{1,1}(\mu^*) + \mu_{12} F_{1,2}(\mu^*) + \mu_{21} F_{2,1}(\mu^*) + \mu_{22} F_{2,2}(\mu^*)), \\
&\mu_{11}^* F_{1,1}(\mu^*) + \mu_{12}^* F_{1,2}(\mu^*) = \max_{\mu_1 \in G_1(\mu^*)} (\mu_{11} F_{1,1}(\mu^*) + \mu_{12} F_{1,2}(\mu^*)), \\
&\mu_{21}^* F_{2,1}(\mu^*) + \mu_{22}^* F_{2,2}(\mu^*) = \max_{\mu_2 \in G_2(\mu^*)} (\mu_{21} F_{2,1}(\mu^*) + \mu_{22} F_{2,2}(\mu^*)),
\end{aligned}$$

implying that $(\mu_{11}^*, \mu_{12}^*, \mu_{21}^*, \mu_{22}^*) = (0, m, 0, m)$. It is a element of the NTU core.

5 Conclusions

In this paper, inspired by Scarf (1971), Zhao (1999), Sandholm (2010), and Yang and Zhang (2018), we introduce the notion of NTU cores and TU cores for coalitional population games, and prove their existence theorems. Inspired by Nessah and Tian (2014), we introduce the notion of strong equilibria for coalitional population games. By adjusting some conditions, we also show the existence of strong equilibria. Our model extends the work of Yang and Zhang (2018) to the case with infinitely many pure strategies.

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