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A refinement of the uncovered set in tournaments

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Abstract

We introduce a new solution for tournaments called the unsurpassed set. This solution lies between the uncovered set and the Copeland winner set. We show that this solution is more decisive than the uncovered set in discriminating among alternatives, and avoids a deficiency of the Copeland winner set. Moreover, the unsurpassed set is more sensitive than the uncovered set but less sensitive than the Copeland winner set to the reinforcement of the chosen alternatives. Besides, it turns out that this solution violates the other standard properties including independence of unchosen alternatives, stability, composition consistency and indempotency.

Keywords Uncovered set · Unsurpassed set · Copeland winner set · Monotonicity

1 Introduction

A tournament is presented by an ordered pair (X, P) where X is a set of alternatives and P is a complete and asymmetric binary relation on X. In this paper, it is assumed that X is nonempty and finite and that xPy is read as x dominates y for any x, y in X. A tournament (X, P) is said to be regular if any two alternatives in X dominate an equal number of alternatives. The main subject in the theory of tournaments are solutions which assign a nonempty subset of X to any given (X, P).

Since the Condorcet winner, which is an alternative that dominates every other alternative, may not exist, the top cycle set was proposed by Schwartz (1972) as a generalized notion of the Condorcet winner. It has been shown in Schwartz (1986)

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that the top cycle set is nonempty for every possible tournament. However, this set may include all the alternatives under consideration. In this case, this solution may not discriminate among alternatives at all. Furthermore, it may contain Pareto inefficient alternatives when the dominance relation is derived from pairwise majority comparisons, as pointed out by Deb (1977). In view of these defects, the uncovered set was formulated by Miller (1980) as a refinement of the top cycle set. As shown in Miller (1980), the uncovered set precludes the presence of Pareto inefficient alternatives successfully. It may be worth noting that the uncovered set is also the largest tournament solution that cannot contain Pareto dominated alternatives (See Brandt et al. 2016b). However, the uncovered set may also include every alternative under consideration in spite of the fact that the tournament is irregular. That means, as a tournament solution, the uncovered set may not be discriminating enough.

There are several noted tournament solutions in the literature that may be seen as refinements of the uncovered set. These are the Copeland winner set in Copeland (1951), the Banks set in Banks (1985), the minimal covering set in Dutta (1988), the tournament equilibrium set in Schwartz (1990) and the union of minimal extending sets in Brandt et al. (2017). A pleasant advantage of the Copeland winner set is that it is always a proper subset of X as long as (X, P) is irregular. This property is not satisfied by the uncovered set, and in this sense, the Copeland winner set is a more decisive solution than the uncovered set. However, in some cases, the Copeland winner set may be a dominated subset of the uncovered set in the sense that alternatives in the Copeland winner set are dominated by any other alternative in the uncovered set. That means, the Copeland winner set may only choose the alternative(s) from the uncovered set which are worse than any other unchosen alternatives within the uncovered set. Given this point, it might be argued that the Copeland winner set is an unsatisfactory refinement of the uncovered set. By comparison, it has been shown in Laslier (1997) that none of the Banks set, the minimal covering set and the tournament equilibrium set may be a dominated subset of the uncovered set. Moreover, due to the minimal extending sets being included in the Banks set (Brandt et al. 2017), the union of minimal extending sets is not a dominated subset of the uncovered as well. Regrettably, given an irregular tournament, all of them may choose all the alternatives under consideration. In other words, these refinements fail to remove the weakness of the uncovered set in discriminating among alternatives.

In view of the above facts, it is worth exploring a restrictive theory with more discriminatory power than the uncovered set, but which avoids the weakness of the mentioned refinements of the uncovered set. With this motivation, we put forward a new solution called *the unsurpassed set*. It will be shown that the unsurpassed set is nested between the uncovered set and the Copeland winner set. More importantly, the unsurpassed set avoids the mentioned weaknesses of the uncovered set and that of the Copeland winner set. We also investigate how the unsurpassed set changes when the dominion of a chosen alternative is reinforced and the dominance relation among the unchosen alternatives is unaltered. Moreover, we make a comparison with the solutions of the uncovered set and of the Copeland winner set in this respect. Besides, it will be shown that the unsurpassed set fails to satisfy the choice-theoretic properties: stability, composition consistency and idempotence.

The rest of this paper is organized as follows. In Sect. 2, we recall necessary definitions, notations and conclusions concerning tournaments and their solutions which are partly from Brandt et al. (2016a). Section 3 is devoted to the solutions of the top cycle set, of the uncovered set and of the Copeland winner set. Here, we display the disadvantages of these solutions in a detailed way. We propose the unsurpassed set and investigate which properties are satisfied or violated by this solution in Sect. 4. Finally, we end this paper with some short concluding remarks in Sect. 5 where our results and future research topics are roughly discussed.

2 Preliminaries

A tournament is a pair (X, P) where X is a nonempty finite set of alternatives and where P is a binary relation on X satisfying

- **asymmetry** $\forall x, y \in X$, if xPy then not yPx;
- **completeness** $\forall x, y \in X$, if $x \neq y$, then either xPy or yPx.

For any $x, y \in X$, we read xPy as x dominates[1] y which means that x, if taken into consideration, excludes the acceptance of y. Denote by $\mathcal{T}(X)$ the collection of tournaments on X. Given a subset T of X, define the restriction of P to T by $P_T = P \cap (T \times T)$. Any $(X, P) \in \mathcal{T}(X)$ may be represented by a digraph where X is a vertex set and P is the set of directed edges.

For any $x \in X$, let $B_{(X,P)}(x)$ be the set of alternatives that dominate x; and $D_{(X,P)}(x)$ be the set of alternatives that are dominated by x. An alternative in $B_{(X,P)}(x)$ is called a *dominator* of x and $D_{(X,P)}(x)$ is termed the *dominion* of x. In order to improve readability, we omit the respective subscript whenever X and P are known from the context. Denote by $b(x)$ and $d(x)$ the cardinalities of $B(x)$ and of $D(x)$, respectively.

A tournament (X, P) is called *regular* if $b(x) = b(y)$ and $d(x) = d(y)$ for any $x, y \in X$. (X, P) is called *irregular* if it is not regular. Any $Y \subseteq X$ is called a *dominated subset* of X if xPy for any $x \in X \setminus Y$ and $y \in Y$.

A dominance relation P on X is said to be

- **transitive** $\forall x, y, z \in X$, if xPy and yPz, then xPz;
- **cyclic** $\exists\, x_1, x_2, \ldots, x_m \in X$ such that $x_1Px_2, \ldots, x_{m-1}Px_m, x_mPx_1$;
- **acyclic** if it is not cyclic.

A *solution* for tournaments is a mapping $\mathcal{S}\colon \mathcal{T}(X) \to 2^X \backslash \{\emptyset\}$. $\mathcal{S}(X, P)$ is called the *choice set* of (X, P). For any $B \subseteq X$, we write $\mathcal{S}(B)$ instead of $\mathcal{S}(B, P_B)$ whenever the tournament (B, P_B) is clear from the context. Let $\mathcal{S}, \mathcal{S}'$ be two solutions for tournaments. $\mathcal{S}'$ is called a *refinement* of $\mathcal{S}$ if $\mathcal{S}'(X, P) \subseteq \mathcal{S}(X, P)$ for any $(X, P) \in \mathcal{T}(X)$.

Definition 1 A tournament solution $\mathcal{S}$ is said to be

[1] Note that, xPy is also frequently referred to as x beats y instead of x dominates y in the tournament literature. In essence, a tournament is an abstract game. For this, we employ the notion of the dominance that was originally defined in (Von Neumann and Morgenstern 1944, p. 37).

(i) α**-monotonic**: for any $(X, P) \in \mathcal{T}(X)$, $x \in \mathcal{S}(X, P)$ and $y \in X$, if $P' = P$ except $xP'y$, then $x \in \mathcal{S}(X, P')$;
(ii) β**-monotonic**: for any $(X, P) \in \mathcal{T}(X)$, $x \in \mathcal{S}(X, P)$ and $y \in X$, if $P' = P$ except $xP'y$, then $x \in \mathcal{S}(X, P')$ but $y \notin \mathcal{S}(X, P')$;
(iii) γ**-monotonic**: for any $(X, P) \in \mathcal{T}(X)$, $x \in \mathcal{S}(X, P)$ and $y \in X$, if $P' = P$ except $xP'y$, then $\mathcal{S}(X, P') = \{x\}$.

The property of α-monotonicity prescribes that a chosen alternative should still be chosen if it is reinforced by changing from yPx to xPy and everything else is unchanged. The property of β-monotonicity says that any chosen alternative x is still in the choice set but excludes the acceptance of y when x is reinforced by changing from yPx to xPy and everything else is unchanged. The property of γ-monotonicity states that any chosen alternative x becomes the unique chosen alternative whenever it is reinforced by changing from yPx to xPy and everything else is unchanged. Notice that α-monotonicity and γ-monotonicity are also referred to as monotonicity and strong monotonicity, respectively, in the literature. As for other well-known monotonicity concepts, one may study cover-monotonicity and Maskin monotonicity in Özkal-Sanver and Sanver (2010).

Lemma 1 *Let $\mathcal{S}$ be a solution for tournaments. Then*

(i) *$\mathcal{S}$ is α-monotonic if it is β-monotonic;*
(ii) *$\mathcal{S}$ is β-monotonic if it is γ-monotonic.*

This lemma states that β-monotonicity is a more demanding constrain than α-monotonicity, but a less demanding constrain than γ-monotonicity. This statement follows straightforwardly from the above definition.

A tournament solution $\mathcal{S}$ is *independent of unchosen alternatives* if

$$\mathcal{S}(X, P) = \mathcal{S}(X, P') \text{ for all } (X, P), (X, P') \in \mathcal{T}(X)$$

such that for all $x \in \mathcal{S}(X, P)$ and $y \in X$, xPy if and only if $xP'y$.

Independence of unchosen alternative says that the choice set should be unaffected by changes in the dominance relation between unchosen alternatives.

A tournament solution $\mathcal{S}$ is *stable* if for all tournaments (X, P) and for all nonempty subsets $A, B, C \subseteq X$ with $A \subseteq B \cap C$,

$$A = \mathcal{S}(B) = \mathcal{S}(C) \text{ if and only if } A = \mathcal{S}(B \cup C).$$

This property says that a set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets.[2] Stability was factorized into two conditions $\widehat{\alpha}$ and $\widehat{\gamma}$ in Brandt et al. (2017) by considering each implication in the above equivalence separately.

[2] In comparison to monotonicity, stability is a more demanding property such that most of the tournament solutions may not satisfy this property. We refer to Brandt et al. (2016a) for a more thorough discussion about which tournament solution satisfies the property of stability.

A tournament solution $\mathcal{S}$ satisfies $\widehat{\alpha}$ if for all tournaments (X, P) and for all nonempty $B, C \subseteq X$,

$$\mathcal{S}(B) \subseteq C \subseteq B \text{ implies } \mathcal{S}(C) = \mathcal{S}(B).$$

Condition $\widehat{\alpha}$ requires that if the range of available alternatives is narrowed from B to C but still contains all previously chosen alternatives, the choice from C should be the same as that from B.

A tournament solution $\mathcal{S}$ satisfies $\widehat{\gamma}$ if for all tournaments (X, P) and for all $B, C \subseteq X$,

$$\mathcal{S}(C) = \mathcal{S}(B) \text{ implies } S(B \cup C) = \mathcal{S}(B).$$

Condition $\widehat{\gamma}$ states that a choice set for two different sets of available alternatives should still be the choice set when the range of alternatives is expanded by merging the two sets.

Note that $\widehat{\alpha}$ corresponds to the implication from right to left whereas $\widehat{\gamma}$ is the implication from left to right.[3]

A *component* of (X, P) is a nonempty subset of alternatives $B \subseteq X$ such that for all $x \in X \setminus B$, either xPy for any $y \in B$ or yPx for any $y \in B$. A *decomposition* of (X, P) is a partition of X into components.

For a given tournament $\widetilde{T} = (\widetilde{X}, \widetilde{P})$ where $\widetilde{X} = \{1, 2, \ldots, k\}$, a new tournament $T = (X, P)$ can be constructed by replacing each alternative in $\widetilde{X}$ with a component. Consider a set of tournaments $T_1 = (X_1, P_1), T_2 = (X_2, P_2), \ldots, T_k = (X_k, P_k)$ where $X_1, \ldots, X_k$ are pairwise disjoint sets of alternatives. The *product* of $T_1, T_2, \ldots, T_k$ with respect to $\widetilde{T}$ is the tournament $T = (X, P)$ such that $X = \bigcup_{i=1}^{i=k} X_i$ and for any $s \in X_i, t \in X_j$, sPt if and only if $i = j$ and sP_it, or $i \neq j$ and $i\widetilde{P}j$. $\widetilde{T}$ is called the *summary* of T with regard to the decomposition $\{X_1, X_2, \ldots, X_k\}$.

A tournament solution $\mathcal{S}$ is *composition consistent* if for all tournaments T_1, T_2, ..., T_k and $\widetilde{T}$ such that $T = \prod(\widetilde{T}, T_1, T_2, \ldots, T_k)$,

$$\mathcal{S}(T) = \bigcup_{i \in \mathcal{S}(\widetilde{T})} \mathcal{S}(T_i).$$

This property expresses that a tournament solution is composition consistent if it selects the "best" alternatives from the "best" components.

A tournament solution $\mathcal{S}$ is *idempotent* if $\mathcal{S}(\mathcal{S}(T)) = \mathcal{S}(T)$ for all $T = (X, P) \in \mathcal{T}(X)$. This property requires that the choice set be invariant under repeated application of the solution concept.

[3] The property of stability was further factorized into four conditions: $\widehat{\alpha}_{\subseteq}$, $\widehat{\alpha}_{\supseteq}$, $\widehat{\gamma}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$ in Brandt et al. (2017). We refer to Brandt et al. (2017) for a more thorough discussion of these conditions. Note that $\widehat{\alpha}$ is also known as *Chernoff's postulate* 5* in Chernoff (1954), the *strong superset property* in Bordes (1979) and the *attention filter axiom* in (Masatlioglu et al. 2012).

3 Top cycle set, uncovered set and Copeland winner set

An alternative $x \in X$ is said to be *maximal* in (X, P) if yPx for no $y \in X$. As the dominance relation may contain cycles, a given tournament may not admit a maximal alternative.[4] Due to the possible non-existence of maximal alternatives, the top cycle set[5] was proposed as a generalization of the concept of maximal alternative.

The *top cycle set* of (X, P), denoted by $\mathcal{TC}(X, P)$, is defined as follows:

$$\mathcal{TC}(X, P) = \{x \in X \mid xP^{\tau}y \text{ for any } y \in X\}.$$

where $xP^{\tau}y$ if there exists a sequence $x = x_0, x_1, \ldots, x_m = y \in X$ such that $x_i P x_{i+1}$ for all $i = 0, \ldots, (m-1)$.

The top cycle set is not only nonempty but also unique for every tournament. Moreover, any alternative in the top cycle set dominates any alternative outside this set, and no proper subset of the top cycle set has this property, as showed in Schwartz (1986). However, this solution may not discriminate the alternatives under consideration even if the tournament is irregular, and it also may contain Pareto inefficient alternatives when the dominance relation is derived from pairwise majority comparisons, as clarified in Moulin (1986).

The next lemma will be useful in the sequel.

Lemma 2 *For any $(X, P) \in \mathcal{T}(X)$ and $Y \subseteq X$, Y is not a dominated subset of X if $Y \cap \mathcal{TC}(X, P) \neq \emptyset$.*

Proof Let $Y \cap \mathcal{TC}(X, P) \neq \emptyset$. Suppose Y is a dominated subset of X. Then xPy for any $x \in X \backslash Y$ and $y \in Y$. Take $z \in Y \cap \mathcal{TC}(X, P)$. Then $D(z) \subseteq Y$. Moreover, not $zP^{\tau}x$ for any $x \in X \setminus Y$, which contradicts $z \in \mathcal{TC}(X, P)$. □

The uncovered set was proposed by Miller (1980) in order to overcome the disadvantages of the top cycle set.[6]

The *uncovered set* of (X, P) is defined as:

$$\mathcal{UC}(X, P) = \{x \in X \mid yP^{c}x \text{ for no } y \in X\}$$

where *y covers x*, denoted by $yP^{c}x$, if yPx and $D(x) \subseteq D(y)$.

The uncovered set selects the maximal alternatives with respect to the covering relation P^c. Since P^c is a transitive sub-relation of P, the uncovered set is nonempty for every possible tournament. Moreover, it has been shown in Miller (1980) that the uncovered set refines the top cycle set and that every alternative in the uncovered set is Pareto efficient when the dominance relation is derived from pairwise majority comparisons.

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4 The maximal alternative is also well-known as the Condorcet winner in the context of tournaments.

5 Here, we follow the definition of the top cycle set in Moulin (1986) which is the same as the notion of the admissible set in Kalai and Schmeidler (1977). See a different but equivalent definition of the top cycle set in Schwartz (1990).

6 An equivalent definition was independently formulated by Fishburn (1977). Here, we follow the definition from Miller (1980).

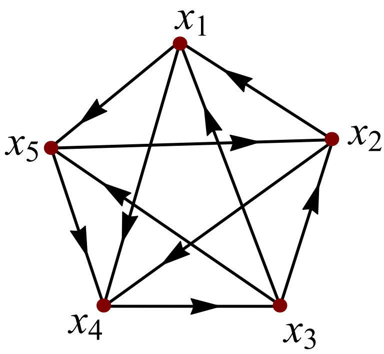


Fig. 1 (X, P)

However, the uncovered set is still not discriminating enough in the sense that it may include every alternative under consideration in the case of some irregular tournaments. To show this point, consider the following example:

Example 1 Consider the tournament in Fig. 1. Because of $d(x_3) \neq d(x_4)$, (X, P) is irregular. But $\mathcal{UC}(X, P) = X$.

Remark 1 It could be verified that for the tournament in Fig. 1, the solutions of the Banks set in Banks (1985), of the minimal covering set in Dutta (1988), of the tournament equilibrium set in Schwartz (1990) and of the minimal extending set[7] in Brandt (2011) do not discriminate among the alternatives in X at all in spite of the fact that the tournament is irregular.

The *Copeland winner set* of (X, P) is defined as follows:

$$\mathcal{CW}(X, P) = \{x \in X |\ yP^{\text{co}}x \text{ for no } y \in X\},$$

where y *co-dominates* x, denoted by $yP^{\text{co}}x$, if $d(y) - b(y) > d(x) - b(x)$.

Due to the asymmetry and completeness of P, we get $xP^{\text{co}}y$ if and only if $d(x) > d(y)$. Since P^{co} is transitive, the nonempty Copeland winner set always exists for every possible tournament. We note that P^{co} is not necessarily a sub-relation of P. That is, it is not necessarily true that $P^{co} \subseteq P$.

It can be easily verified that the Copeland winner set is a proper subset of the alternative set as long as a tournament is irregular. It has been shown in Moulin (1986) that the Copeland winner set is a refinement of the uncovered set and that a deficiency of this solution may arise when X includes more than twelve alternatives. In this case, the Copeland winner set might be a dominated subset of the uncovered set. To confirm this point, we reproduce the example from (Moulin 1986, p. 280) in the following.

Denote by $\mathcal{TC}(\mathcal{UC}(X, P))$ the top cycle set of the restriction of tournament (X, P) to its own uncovered set $\mathcal{UC}(X, P)$.

7 A remarkable feature of the minimal extending set is that it may rule out alternatives even though a tournament is regular, as indicated in (Brandt 2011, p. 1497).

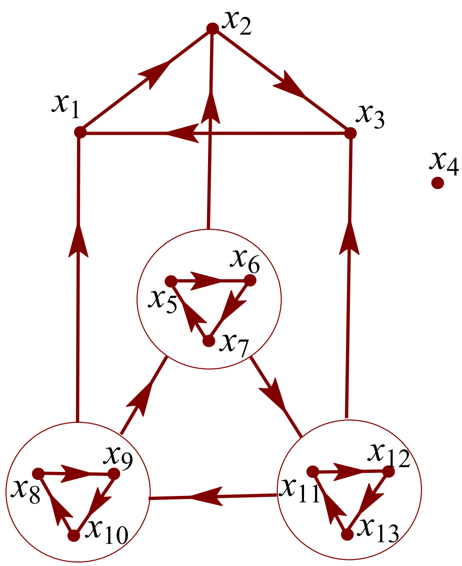


Fig. 2 (X, P): non-depicted arrows go downward

Example 2 Consider the tournament in Fig. 2. It is not difficult to verify that $\mathcal{UC}(X, P) = \{x_1, x_2, x_3, x_4\}$, $\mathcal{CW}(X, P) = \{x_4\}$ and $\mathcal{TC}(\mathcal{UC}(X, P)) = \{x_1, x_2, x_3\}$. Clearly, $x_1 P x_4$, $x_2 P x_4$ and $x_3 P x_4$. Thus, $\mathcal{CW}(X, P)$ is a dominated subset of $\mathcal{UC}(X, P)$.

4 Unsurpassed set and its properties

In this section, we attempt to formulate a new refinement of the uncovered set which may accommodate the mentioned shortcomings of the uncovered set and of the Copeland winner set in the last section. We also analyze this refinement with respect to two different types of properties: dominance-based properties and choice-theoretic properties.

Definition 2 The unsurpassed set of (X, P) is defined as follows:

$$\mathcal{US}(X, P) = \{x \in X \mid y P^s x \text{ for no } y \in X\},$$

where y surpasses x, denoted by $y P^s x$, if $y P x$ and $d(y) > d(x)$.

The idea behind the surpassing relation is that y surpasses x if not only y dominates x but also the size of y's dominion is larger than that of x. The unsurpassed set is then the set of maximal alternatives with respect to P^s.

Example 3 For the tournament in Fig. 1, $\mathcal{US}(X, P) = \{x_3\}$, whereas for the tournament in Fig. 2, $\mathcal{US}(X, P) = \{x_1, x_2, x_3, x_4\}$.

Theorem 1 *Let* $(X, P) \in \mathcal{T}(X)$. *Then*

(i) *P^s is an acyclic sub-relation of P;*
(ii) $P^c \subseteq P^s \subseteq P^{co}$.

Proof (i) It is clear that $P^s \subseteq P$. Since by (ii) below, $P^s \subseteq P^{co}$ and P^{co} is transitive, P^s is acyclic.

(ii) Let $x, y \in X$. If $xP^c y$, then xPy and $D(y) \subseteq D(x)$, which implies $d(x) > d(y)$ and $xP^s y$. If $xP^s y$, then $d(x) > d(y)$. As $b(x) = |X| - d(x) - 1$, $d(x) - b(x) > d(y) - b(y)$. Thus $xP^{co}y$.

□

Theorem 1-(i) ensures that the unsurpassed set is nonempty for every possible tournament. Theorem 1-(ii) expresses the fact that the surpassing relation is weaker than the covering relation but stricter than the co-dominance relation. Thus, the following conclusion can be straightforwardly obtained:

Corollary 1 *Let $(X, P) \in \mathcal{T}(X)$. Then*

(i) $\mathcal{US}(X, P) \neq \emptyset$;
(ii) $\mathcal{CW}(X, P) \subseteq \mathcal{US}(X, P) \subseteq \mathcal{UC}(X, P)$.

The unsurpassed set may be properly included in the uncovered set and the Copeland winner set may be strictly contained in the unsurpassed set. This point can be confirmed by reconsidering the tournaments in Figs. 1 and 2, respectively. Thus, the unsurpassed set can be seen as a refinement of the uncovered set but an extension of the Copeland winner set.

Remark 2 It is worth emphasizing that the unsurpassed set neither contains nor refines the solutions of the Banks set in Banks (1985), of the minimal covering set in Dutta (1988), of the tournament equilibrium set in Schwartz (1990) and of the union of minimal extending sets in Brandt et al. (2017). This statement can be easily verified by examining again the tournaments in Figs. 1 and 2.

4.1 Distinctive properties

This subsection is devoted to verifying whether the unsurpassed set may avoid the mentioned shortcomings of the uncovered set and its refinements.

The following theorem says that the unsurpassed set includes every alternative under consideration whenever the tournament is regular and excludes some alternative(s) as long as the tournament is irregular.

Theorem 2 *Let $(X, P) \in \mathcal{T}(X)$. $\mathcal{US}(X, P) = X$ if and only if (X, P) is regular.*

Proof If (X, P) is regular, $d(x) = d(y)$ for any $x, y \in X$. Then, no $x \in X$ is surpassed by any $y \in X$. Thus, $\mathcal{US}(X, P) = X$.

Assume that (X, P) is irregular. Take the partition $(X_1, X\backslash X_1)$ of X that satisfies: (i) $d(x) = d(y)$ for any $x, y \in X_1$; (ii) $d(x) < d(y)$ for any $x \in X_1$ and $y \in X\backslash X_1$. As (X, P) is irregular, both X_1 and $X\backslash X_1$ are nonempty. If there exist $x \in X_1$ and $y \in X\backslash X_1$ such that $yP^s x$, then $\mathcal{US}(X, P)$ is a proper subset of X. Otherwise, xPy for any $x \in X_1$ and $y \in X\backslash X_1$, which implies that $d(x) > d(y)$ for every $x \in X_1$ and $y \in X\backslash X_1$. This is a contradiction and, hence, $\mathcal{US}(X, P)$ is strictly included in X. □

In contrast, the Copeland winner set is provided with the same property while this is not true for the uncovered set, the Banks set in Banks (1985), the minimal covering set in Dutta (1988), the tournament equilibrium set in Schwartz (1990), and the minimal extending sets in Brandt et al. (2017).

Theorem 3 *For any* $(X, P) \in \mathcal{T}(X)$, $\mathcal{US}(X, P) \cap \mathcal{TC}(\mathcal{UC}(X, P)) \neq \emptyset$.

Proof Suppose $\mathcal{US}(X, P) \cap \mathcal{TC}(\mathcal{UC}(X, P)) = \emptyset$. Take $x \in \mathcal{TC}(\mathcal{UC}(X, P))$ such that $d(x) \geq d(y)$ for any $y \in \mathcal{TC}(\mathcal{UC}(X, P))$. Since $x \notin \mathcal{US}(X, P)$, there exists a $z \in X$ with $zP^s x$, which implies zPx and $d(z) > d(x)$. Since $x_i P x_j$ for any $x_i \in \mathcal{TC}(\mathcal{UC}(X, P))$ and $x_j \in \mathcal{UC}(X, P) \setminus \mathcal{TC}(\mathcal{UC}(X, P))$, $z \notin \mathcal{UC}(X, P)$ and $z \notin \mathcal{US}(X, P)$. Then there exists a $w \in \mathcal{UC}(X, P)$ such that $wP^c z$ implying wPz and $d(w) > d(z) > d(x)$. As $wP^c z$ and zPx, wPx implying $w \in \mathcal{TC}(\mathcal{UC}(X, P))$, which is a contradiction, since $d(x) \geq d(y)$ for any $y \in \mathcal{TC}(\mathcal{UC}(X, P))$. □

The above theorem says that the unsurpassed set always picks out some alternative(s) from the top cycle set of the uncovered set. Because of Lemma 2, the following conclusion is straightforwardly obtained:

Corollary 2 *The unsurpassed set is never a dominated subset of the uncovered set.*

This result shows that the unsurpassed set makes up for the aforementioned flaw of the Copeland winner set as a refinement of the uncovered set.

4.2 Dominance-based properties

In this subsection, we investigate the sensitivity of the unsurpassed set to the changes in the dominance relation. As the notion of the unsurpassed set has a close relation with that of the uncovered set and of the Copeland winner set, we compare its performance with them in this respect.

Intuitively, a chosen alternative should still be chosen whenever it is reinforced (i.e., expanding its dominion). Moreover, it may be desirable that the choice set be independent from the modifications of the dominance relation among alternatives outside this set. The following theorem shows that the unsurpassed set satisfies the former property but violates the latter one.

Theorem 4 *The unsurpassed set satisfies β-monotonicity (and, consequently, α-monotonicity), but does not necessarily satisfy independence of unchosen alternatives.*

Proof Let $x \in \mathcal{US}(X, P)$ and $y \in X$ with yPx. Since $x \in \mathcal{US}(X, P)$, then $d(y) \leq d(x)$. For any $(X, P') \in \Omega(X)$ with $P' = P$ except $xP'y$. Then

$$|\{z \in X | yP'z\}| = d(y) - 1 < |\{z \in X | xP'z\}| = d(x) + 1.$$

Thus, $xP'^s y$ and $y \notin \mathcal{US}(X, P')$. Therefore, the unsurpassed set is β-monotonic. By Lemma 1, we obtain that the unsurpassed set satisfies α-monotonicity.

To confirm the second statement, reconsider the tournament in Fig. 1. $\mathcal{US}(X, P) = \{x_3\}$. Take (X, P') where $P' = P$ except $x_1 P' x_2$. Then, we have $\mathcal{US}(X, P') = \{x_1, x_3\} \neq \mathcal{US}(X, P)$. □

In general, the unsurpassed set does not satisfy γ-monotonicity. To see this point, reconsider the tournament in Fig. 2. Take (X, P') where $P' = P$ except $x_4 P' x_3$. Then, we have $\mathcal{US}(X, P') = \{x_1, x_2, x_4\}$.

As a comparison, the uncovered set is α-monotonic, which follows from the fact that any uncovered alternative is still uncovered by enlarging its dominion set without expanding its dominator set whereas the Copeland winner set is γ-monotonic, which was clarified in Henriet (1985). Note that the uncovered set is neither β-monotonic nor γ-monotonic. To show this statement, reconsider the tournament in Fig. 2. Take (X, P') where $P' = P$ except $x_4 P' x_3$. Then $\mathcal{UC}(X, P') = \mathcal{UC}(X, P) = \{x_1, x_2, x_3, x_4\}$. In spite of the reinforcement of x_4 in (X, P'), x_4 fails to exclude the acceptance of x_3.

The above result states that the unsurpassed set may change with a modification of the dominance relation between alternatives outside it. Similarly, neither the uncovered set nor the Copeland winner set satisfies independence of unchosen alternatives. This can be shown by reexamining the tournaments in Figs 1 and 2, respectively.

Example 4 Reconsider the tournament in Fig. 2. Take (X, P') where $P' = P$ except $x_{10} P' x_{11}$. We have

$$\mathcal{UC}(X, P') = \{x_1, x_2, x_3, x_4, x_{10}\} \neq \mathcal{UC}(X, P) = \{x_1, x_2, x_3, x_4\}.$$

Reconsider the tournament in Fig. 1. Take (X, P') where $P' = P$ except $x_5 P' x_1$. We have $\mathcal{CW}(X, P') = \{x_3, x_5\} \neq \mathcal{CW}(X, P) = \{x_3\}$.

4.3 Choice-theoretic properties

This subsection is dedicated to investigating whether the unsurpassed set satisfies the properties that concern the consistency of choices from different sub-tournaments of the same tournament to each other.

Theorem 5 *The unsurpassed set satisfies neither $\widehat{\alpha}$ nor $\widehat{\gamma}$.*

The above theorem shows that if the range of available alternatives is reduced by removing alternatives outside the unsurpassed set, the unsurpassed set for the reduced alternative set may change, and that the unsurpassed set for two different alternative sets may not necessarily be the unsurpassed set for their union. To verify this statement, see the following counterexample:

Example 5 Consider the tournament in Fig. 3. Take $B = X$ and $C = \{x_1, x_2, x_4, x_5\}$. We have $\mathcal{US}(B) = \{x_1, x_5\}$, but $\mathcal{US}(C) = \{x_1, x_2, x_5\}$, which violates $\widehat{\alpha}$.[8]

Consider the tournament in Fig. 4. Take $B = X \setminus \{x_4, x_5, x_6\}$ and $C = X \setminus \{x_1, x_2, x_8\}$. We have $\mathcal{US}(B) = \mathcal{US}(C) = \{x_3, x_7\}$, but $\mathcal{US}(B \cup C) = \{x_7\}$, which violates $\widehat{\gamma}$.

[8] This counterexample also shows that the unsurpassed set does not satisfy the Aizerman property. A tournament solution $\mathcal{S}$ satisfies the *Aizerman property* if for all tournaments (X, P) and for all nonempty $B, C \subseteq X$, $\mathcal{S}(B) \subseteq C \subseteq B$ implies $\mathcal{S}(C) \subseteq \mathcal{S}(B)$. Note that the Aizerman property is exactly the $\widehat{\alpha}_{\subseteq}$ in Brandt et al. (2017).

Fig. 3 (X, P)

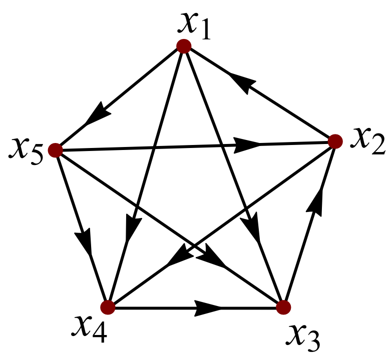


Fig. 4 (X, P): omitted arrows go rightwards

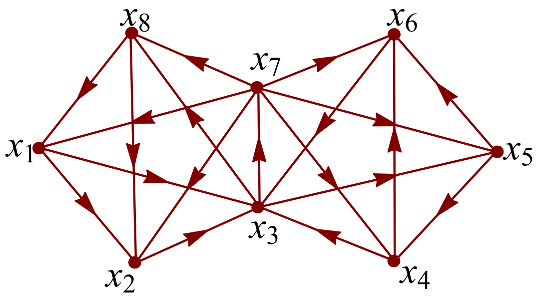
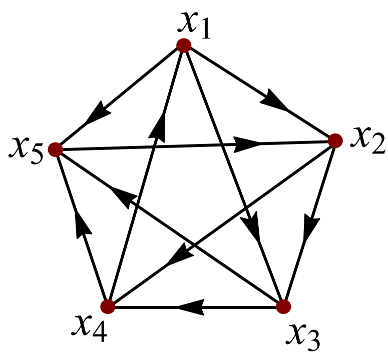


Fig. 5 (X, P)

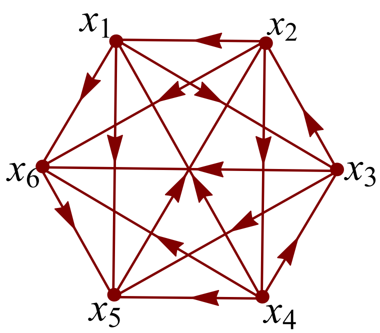


The same is true for the uncovered set and the Copeland winner set. That is, both satisfy neither $\widehat{\alpha}$ nor $\widehat{\gamma}$. To confirm this statement, see the following example:

Example 6 (i) Consider the tournament in Fig. 5. If $B = X$ and $C = \{x_1, x_2, x_3, x_4\}$, then $\mathcal{UC}(B) = \{x_1, x_2, x_3, x_4\}$, but $\mathcal{UC}(C) = \{x_1, x_2, x_4\}$, which violates $\widehat{\alpha}$. (ii) Consider the tournament in Fig. 3, but changing the preference between x_1 and x_4 to $x_4 P x_1$. If $B = \{x_1, x_2, x_3, x_5\}$ and $C = \{x_1, x_2, x_4, x_5\}$, then $\mathcal{UC}(B) = \mathcal{UC}(C) = \{x_1, x_2, x_5\}$, but $\mathcal{UC}(B \cup C) = \{x_1, x_2, x_4, x_5\}$, which violates $\widehat{\gamma}$.

Consider the tournament in Fig. 6. (i) If $B = \{x_2, x_3, x_4, x_5\}$ and $C = \{x_2, x_3, x_4\}$, then $\mathcal{CW}(B) = \{x_3, x_4\}$, but $\mathcal{CW}(C) = \{x_2, x_3, x_4\}$, which violates $\widehat{\alpha}$. (ii) If

Fig. 6 (X, P)



$B = \{x_1, x_2, x_3, x_4\}$ and $C = \{x_2, x_4, x_5, x_6\}$, then $\mathcal{CW}(B) = \mathcal{CW}(C) = \{x_2, x_4\}$, but $\mathcal{CW}(B \cup C) = \{x_4\}$, which violates $\widehat{\gamma}$.

The following example shows that the unsurpassed set is neither composition consistent nor idempotent:

Example 7 Reconsider the tournament in Fig. 1. $\{\{x_1, x_2, x_5\}, \{x_3\}, \{x_4\}\}$ is a decomposition of (X, P). Composition consistency requires $x_1, x_2, x_4, x_5 \in \mathcal{US}(X, P)$, which is not the case.

Reconsider the tournament in Fig. 2. It can be verified that

$$\mathcal{US}(\mathcal{US}(X, P)) = \{x_1, x_2, x_3\} \text{ whereas } \mathcal{US}(X, P) = \{x_1, x_2, x_3, x_4\}.$$

Thus, the unsurpassed set is not idempotent.

In contrast, it has been shown in Laslier (1997) that the uncovered set is composition consistent but not idempotent, whereas the Copeland winner set is nether composition consistent nor idempotent.

5 Concluding remark

In this paper, we proposed the notion of surpassing relation which is weaker than the covering relation but stricter than the co-dominance relation. By using this notion, we formulated the unsurpassed set which is a refinement of the uncovered set but an extension of the Copeland winner set. The theory of the unsurpassed set always yields a proper subset of the alternative set for all irregular tournaments while neither the uncovered set, nor the Banks set in Banks (1985), nor the minimal covering set in Dutta (1988), nor the tournament equilibrium set in Schwartz (1990), nor the minimal extending sets in Brandt et al. (2017) have this property. In this respect, the unsurpassed set is distinctive from the solution of the uncovered set and the mentioned refinements. The unsurpassed set and the Copeland winner set both refine the uncovered set. More important, both theories have an aspect of decisiveness in discriminating among alternatives. That is, both solution theories are able to rule out some alternative(s) under consideration as long as the tournament is irregular. However, the Copeland winner set may exclusively choose alternatives from the uncovered set which are worse than any other unchosen alternatives in the uncovered set, whereas the unsurpassed set in contrast will not be provided with this feature. Based on this point, we argue that the unsurpassed set refines the uncovered set in a more convincing way than the Copeland winner set.

One of the main advantages of the unsurpassed set over the uncovered set and its mentioned refinements is the former's decisive power. But more precise and analytical results about the decisive power of this tournament solution still need to be obtained. Hence, it would be an interesting topic to see how discriminatory the unsurpassed set really is. As we have showed, the unsurpassed set satisfies a moderate monotonicity which is more demanding than α-monotonicity possessed by the uncovered set but less demanding than γ-monotonicity fulfilled by the Copeland winner set. However, it is still open whether the unsurpassed set is the maximal (with respect to set inclusion) tournament solution satisfying the property of β-monotonicity. Moreover, it has been proved that the unsurpassed set violates independence of unchosen alternatives, stability, composition consistency and idempotence. However, whether these negative conclusions impair the usefulness of the unsurpassed set is still unclear. For this, an axiomatic characterization of the unsurpassed set is waiting for to be discovered.

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