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Implementing egalitarianism in a class of Nash demand games

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Abstract We add a stage to Nash's demand game by allowing the greedier player to revise his demand if the demands are not jointly feasible. If he decides to stick to his initial demand, then the game ends and no one receives anything. If he decides to revise it down to $1 - x$, where x is his initial demand, the revised demand is implemented with certainty. The implementation probability changes linearly between these two extreme cases. We derive a condition on the feasible set under which the two-stage game has a unique subgame perfect equilibrium. In this equilibrium, there is first-stage agreement on the egalitarian demands. We also study two n-player versions of the game. In either version, if the underlying bargaining problem is "divide-the-dollar," then equal division is sustainable in a subgame perfect equilibrium if and only if the number of players is at most four.

Keywords Nash demand game · Divide-the-dollar · Fair division

1 Introduction

Nash (1953) studied the following bargaining game, now known as the *Nash Demand Game* (NDG): two players face a set of feasible utility allocations, S, and they simultaneously make utility-demands: player 1 demands x and player 2 demands y; if the demands are jointly feasible, that is if $(x, y) \in S$, each player receives his demand; otherwise, both receive null payoffs. The special case where S is the unit simplex is

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called the *Divide-the-Dollar* game (DD). Both DD and NDG suffer from the following drawbacks: (a) every efficient demand pair is a Nash equilibrium outcome, and (b) no matter how close (x, y) is to S, if it is outside of S then the entire available social surplus is wasted.

These shortcomings led researchers to modify the game in various ways. This dates back to Nash himself (1953), who proposed the following *smoothing* approach. Denote by $h(x, y)$ the probability that the demand-pair (x, y) is feasible. In the original NDG, h is a step function that only assumes the values zero and one, depending on whether (x, y) is outside or inside S. Nash proposed to replace h by a function whose value is one on S and drops to zero in a continuous fashion outside of S, look at the equilibrium of the resulting (perturbed) game, and take the limit of equilibria as the perturbation function approaches the original step function. He argued that the utility-product maximizer of the original game is the only point that would necessarily emerge from this procedure, if one restricted attention to "regular" perturbations (which he did not define). Abreu and Pearce (2015) recently formalized Nash's suggestion and showed that it indeed holds for a large class of perturbations.[1]

Besides smoothing, a natural way to deal with the drawbacks of NDG/DD is to add a second stage to the game, which is played in case the demands are not feasible. That is, infeasibility does not immediately destroy the social surplus, but leads to a continuation game in which the players have a chance to "correct their stage-one mistake." This route has been particularly prominent in the DD context. Here are a couple of examples from this literature.

Brams and Taylor (1994) introduce a second stage where each player is given two alternative actions: sticking to his original demand or usurping the other player's original demand. Then, the rules of DD are applied to these second stage demands. They show that the resulting game induces *dominance solvability*, and consequently the equal division (in the first stage) is the only outcome that survives. Cetemen and Karagözoğlu (2014) introduce a second stage where the *excess* (i.e., $x + y - 1$, when it is positive) can be eliminated in an ultimatum game. More precisely, their mechanism grants the proposer role in the ultimatum game to the less greedy player in the first stage (or to a randomly chosen player, if the first-stage demands are equal). This player makes a proposal about how the excess should be shared between the two. If this proposal is accepted, then the corresponding excess amounts are deducted from the players' first stage demands, and each player receives the amount remaining from his first-stage demand after this deduction; if the proposal is rejected, then no one receives anything. The authors show that the competition for the proposer role pulls each player's first-stage demand to $\frac{1}{2}$. Thus, the equal division is the only outcome that is sustainable in subgame perfect equilibrium.[2]

[1] An earlier result by Binmore (1987) shows that Nash's suggestion is valid for a certain class of parametrized perturbations. An alternative approach to smoothing is to apply perturbations not to the feasible set, but to the players' strategies. This possibility has been explored by Carlsson (1991), who added a noise component to the players' demands.

[2] Anbarcı (2001), Ashlagi et al. (2012), and Rachmilevitch (2017) are some other papers, which modified the "punishment clause" in DD to tackle the drawbacks mentioned above, and consequently obtained equal division of the surplus in equilibrium. Multi-stage extensions of NDG have been studied by Howard (1992) and by Anbarcı and Boyd III (2011).

In this paper, we follow a similar route in that we modify the game by adding a second stage in case the demands are not jointly feasible. Our framework, however, is different from both the DD framework and the NDG framework. We focus on NDGs which, in a specific sense (to be described shortly), are "sufficiently close" to DD. Thus, in terms of generality, our model is in between DD and NDG.

In our game, the more greedy player among the two (or a randomly chosen player, if the first-stage demands are equal) receives a chance to revise his demand. The revised demand cannot be greater than his original demand, and is also bounded from below by $1 - x$, where x is his original demand. The revised demand is implemented with some probability, λ, and with the complementary probability the game terminates and no one receives anything. The key point of our mechanism is that λ is endogenous and, in particular, is decreasing in the revised demand. A more modest reviser has a greater chance of having his second-stage offer implemented.

The paper is organized as follows. We describe our 2-player game in Sect. 2. In Sect. 3, we present our results for this game. The main result is that, under a certain condition on the feasible set, the game has a unique subgame perfect equilibrium; in it, there is first-stage agreement on the egalitarian demands. The DD case does *not* satisfy the aforementioned condition; however, it is a limit case of the family of bargaining problems that do satisfy the condition. In this limit case, an additional equilibrium "pops up," in which each player demands the entire dollar in the first stage of the game, and the randomly chosen player revises his demand to $\frac{1}{2}$ in the second stage. This "extreme demands equilibrium", however, is inferior to the equilibrium with egalitarian demands in several respects (for example, it is Pareto-dominated by the latter). In Sect. 4, we consider two n-player versions of our game. Interestingly, in either version, if the underlying bargaining problem is the (n-dimensional) unit simplex, then equal division is sustainable in a subgame perfect equilibrium if and only if $n \leq 4$. We are not aware of any result in the bargaining literature that depends on whether the number of players is greater than four or not.[3] One of our n-player generalizations applies to non-DD problems, but the other works only in the DD case. In Sect. 5, we conclude.

2 The 2-player game

Two players, player 1 and player 2, face a set of feasible utility allocations, $S \subset \mathbb{R}^2_+$. This set—the *bargaining problem*—is compact, convex, contains the origin, and is comprehensive. Comprehensiveness means that if $(x, y) \in S$ then $(x', y') \in S$, for every $(x', y') \in \mathbb{R}^2_+$ that satisfies $z' \leq z$ for both $z \in \{x, y\}$. Additionally, $\max\{x : \exists y \; s.t \; (x, y) \in S\} = \max\{y : \exists x \; s.t \; (x, y) \in S\} = 1$.[4] This means that the problem is *normalized*, in the sense that the utility of either player is normalized to a $0 - 1$ scale. The strong Pareto boundary of S is denoted ∂S. The *egalitarian payoff* in S—the maximal x such that $(x, x) \in S$—is denoted $e(S)$. Let $\psi^2_S(x)$ be the maximum

[3] There is no shortage of results in the literature that show that it matters whether the number of players is equal to or greater than 2. For example, Brams and Taylor (1994) show that in their version of DD that we described above, the equal division is dominance inducible if and only if $n = 2$.

[4] A player's maximal payoff in a bargaining problem is called his *ideal payoff*.

possible payoff for player 2 in S, given that player 1's payoff is x. Similarly, let $\psi_S^1(y)$ be the maximum possible payoff for player 1 in S, given that player 2's payoff is y.

The players play the following game, $G(S)$. They simultaneously announce demands: player 1 demands a utility payoff $x \in [0, 1]$ and player 2 demands a utility payoff $y \in [0, 1]$. If $(x, y) \in S$, then each player receives his demand. If $(x, y) \notin S$, then play proceeds to a second stage, which is as follows: if $x > y$ then player 1—the more greedy player—gets a chance to revise his demand. Denote his revised demand by z. This quantity is subject to the restriction $z \in [1 - x, x]$. For every possible value of $z \in [1 - x, x]$, there is a unique $\lambda \in [0, 1]$ such that

$$z = (1 - \lambda)x + \lambda(1 - x).$$

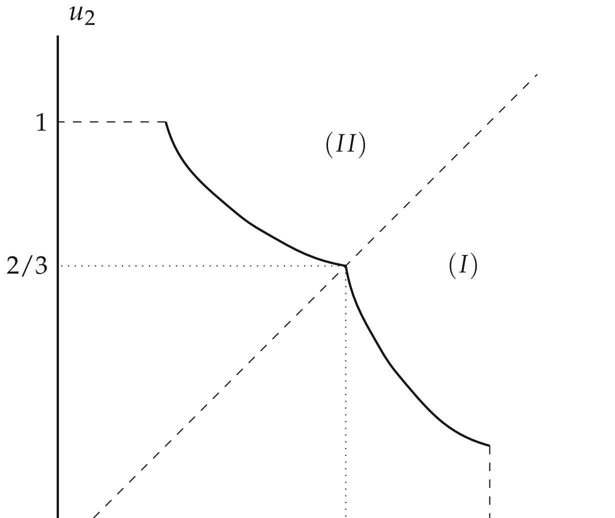
With probability λ the revised demand, z, is implemented: the reviser, player 1, receives z, and player 2 receives $\psi_S^2(z)$; with probability $1-\lambda$ no one receives anything. If $y > x$ then the definition is analogous, and if $y = x$ the reviser is selected at random, with each player being equally likely to be selected.[5]

$G(S)$ is a modified version of the *Divide-the-Dollar* (DD) game if $S = \Delta_2 \equiv \{u \in \mathbb{R}_+^2 : u_1 + u_2 \leq 1\}$. We abuse terminology a bit and call $G(\Delta_2)$ the DD game (or simply DD), even though this term usually denotes the symmetric linear frontier version of Nash's demand game.

We focus on bargaining problems S that satisfy the following:

- (I) $x > \frac{2}{3} \Rightarrow \psi_S^2(x) < \frac{x(2-x)}{4(2x-1)}$,
- (II) $y > \frac{2}{3} \Rightarrow \psi_S^1(y) < \frac{y(2-y)}{4(2y-1)}$.

The graphs corresponding to conditions (I) and (II) are illustrated in the following figure.



[5] Rubinstein et al. (1992) study a sequential bargaining game which is similar to NDG, in which the second mover gets to chose a probability p that governs how play evolves from the third stage of the game onwards. Our probability λ is similar to the aforementioned p, in the sense that it is determined by one of the players.

In DD, $\psi^i_S(z) = 1 - z$ for both $i \in \{1, 2\}$, and it is easy to check that (I) and (II) are satisfied. The family of games that we focus on generalizes DD, in the sense that (I) and (II) mean that the feasible set S is not "too far" from the simplex Δ_2. In particular, if S satisfies (I) and (II) then S' also satisfies them, where S' is any problem that is "sandwiched" between S and Δ_2; that is, S' is any problem that satisfies $\Delta_2 \subset S' \subset S$.

Note that the following are satisfied in $G(S)$:

1. If the reviser insists on his original demand (i.e., $z = x$), then the game ends and no one receives anything.
2. If $G(S)$ is DD and the reviser chooses to "get into the other player's shoes" in the sense of accepting what he originally offered his opponent (i.e., $z = 1 - x$), then the revised demand is implemented with certainty. That is, the reviser ends up with $1 - x$ and the other player ends up with x.
3. The probability of implementing the revised demand changes linearly in the revised demand.

Finally, a word about our normalization assumption is in order. First, without this normalization the rules of the game, as described above, are not well defined. Second, the rationale behind our game is to select the more greedy player in case of first-stage disagreement, where the more greedy player is the one whose demand is maximal. If the utilities are not measured on the same scale, having a larger demand cannot be interpreted as being more greedy. For example, had player 1's utility been measured in dollars and player 2's utility been measured in cents, then the fact that player 2's demand is larger than player 1's does not mean that player 2 is more greedy than player 1.

3 Results for the 2-player game

3.1 non-DD problems

Theorem 1 *Suppose that $S \neq \Delta_2$ and that it satisfies (I) and (II). Then, $G(S)$ has a unique subgame perfect equilibrium. In equilibrium, the first-stage demands are $(e(S), e(S))$.*

To prove the theorem, we make use of the following lemmas. Wherever we write "equilibrium," we mean "subgame perfect equilibrium."

Lemma 1 *Let (x, y) be equilibrium demands that satisfy $x \neq y$. Then, $(x, y) \in \partial S$.*

Proof Let (x, y) be equilibrium demands that satisfy $x \neq y$. Clearly, $(x, y) \in S$ implies $(x, y) \in \partial S$. Thus, it suffices to prove that $(x, y) \in S$. Assume by contradiction that $(x, y) \notin S$.

W.l.o.g, suppose that $x > y$. In this case, player 1 is called to revise his demand. His revised demand takes the form $z = (1-\lambda)x + \lambda(1-x)$, where λ is the probability that the revised demand is implemented. Specifically,

$$\lambda = \frac{x - z}{2x - 1}.$$

Since player 1's expected utility is λz, he maximizes $(\frac{x-z}{2x-1})z$, which is equivalent to maximizing the following function f over $z \in [1-x, x]$:

$$f(z) = zx - z^2.$$

$f'(z) = x - 2z$; therefore, $f'(x) = -x < 0$ and $f'(1-x) = x - 2(1-x) = 3x - 2$. Therefore, if $x > \frac{2}{3}$ the solution to the optimization, call it z^*, is interior—in which case the FOC yields $z^* = \frac{x}{2}$—and otherwise it is $z^* = 1 - x$.

Let $\pi(x)$ denote player 1's payoff given that his initial demand, x, satisfies $x > y$ and $(x, y) \notin S$, and given that the optimal z is selected at the revision stage. The function π is given by:

$$\pi(x) = \begin{cases} \frac{x^2}{4(2x-1)} & \text{if } x > \frac{2}{3} \\ 1 - x & \text{otherwise} \end{cases}$$

Note that π is strictly decreasing. This is clear on $(\frac{1}{2}, \frac{2}{3})$,[6] and on $(\frac{2}{3}, 1]$ it satisfies $\pi(x) = \frac{x^2}{8x-4}$ hence $\pi'(x) = \frac{8x(x-1)}{(8x-4)^2} \leq 0$, and the inequality is strict for $x < 1$. Therefore, player 1 has a profitable deviation: to decrease x. □

Lemma 2 *Let S satisfy (I) and (II), and let (x, y) be demands that satisfy $x \neq y$. Then, (x, y) are not equilibrium demands.*

Proof W.l.o.g, suppose that $x > y$. Assume by contradiction that (x, y) are equilibrium demands. By Lemma 1, $(x, y) \in \partial S$. We argue that player 2 has a profitable deviation.

Case 1: $x > \frac{2}{3}$. Consider a deviation by player 2 to y', where $y < y' < x$. Thus, $(x, y') \notin S$. In this case player 1 is called to revise his demand, and (as we saw in the proof of Lemma 1) he chooses $z^* = \frac{x}{2}$. Thus, with probability $\frac{\frac{x}{2}}{2x-1}$ player 2 will obtain $\psi_S^2(\frac{x}{2})$; with the remaining probability, he will receive zero. Thus, by deviating to y', player 2 can guarantee the expected utility $\frac{\frac{x}{2}}{2x-1}\psi_S^2(\frac{x}{2})$, which is weakly greater than $\frac{x(2-x)}{4(2x-1)}$.[7] Since $y = \psi_S^2(x) < \frac{x(2-x)}{4(2x-1)}$, this deviation is profitable.

Case 2: $x \leq \frac{2}{3}$. Consider the deviation of player 2 to y', where $y < y' < x$. Then, player 1 revises his demand to $z^* = 1 - x$, which is implemented with certainty. The resulting payoff for player 2 is $\psi_S^2(1-x)$; since $\psi_S^2(1-x) \geq x > y$, the deviation is profitable. □

Lemma 3 *Let $x \in (e(S), 1)$. Then, (x, x) is not an equilibrium demand vector.*

Proof If $x \in (\frac{2}{3}, 1)$, then the player who is selected at random to revise his demand revises his demand to $\frac{x}{2}$, and this revised demand is implemented with probability $\lambda = \frac{x}{2(2x-1)}$. A player's expected utility in this case is, therefore, $\frac{1}{2}\lambda\frac{x}{2} + \frac{1}{2}\lambda\psi_S^i(\frac{x}{2})$.

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6 π's domain is $(\frac{1}{2}, 1]$: note that the combination of $x \leq \frac{1}{2}$ and $y < x$ implies that $(x, y) \in \Delta_2 \subset S$.

7 Note that $\psi_S^2(\frac{x}{2}) \geq 1 - \frac{x}{2}$.

By deviating to $x - \epsilon$ for a small $\epsilon > 0$, a player obtains the payoff $\lambda\psi_S^i(\frac{x}{2})$. The deviation is profitable, since $\lambda\psi_S^i(\frac{x}{2}) > \frac{1}{2}\lambda\frac{x}{2} + \frac{1}{2}\lambda\psi_S^i(\frac{x}{2})$.[8]

If $x \leq \frac{2}{3}$, then the player who is selected at random to revise his demand revises his demand to $1 - x$, and this revised demand is implemented with certainty. Thus, any profile of the form (x, x) where $x \in (e(S), \frac{2}{3}]$ gives each player the expected utility $\frac{1}{2}(1 - x) + \frac{1}{2}\psi_S^i(1 - x)$. If, say, player 1 deviates to $x - \epsilon$, for a small $\epsilon > 0$, then player 2 will revise his demand to $1-x$, which will be implemented with certainty, and player 1's payoff would be $\psi_S^1(1-x)$. The profitability of this deviation is equivalent to $\psi_S^1(1 - x) > \frac{1}{2}(1 - x) + \frac{1}{2}\psi_S^1(1 - x)$, or $\psi_S^1(1 - x) > (1 - x)$; this is true, since $x > e(S) \geq \frac{1}{2}$. □

The following lemma speaks of the function π that we derived in the proof of Lemma 1.

Lemma 4 $\pi(x) \leq \frac{1}{2}$ *for all* $x \in (\frac{1}{2}, 1]$.

Proof For $x \in (\frac{1}{2}, \frac{2}{3}]$ this is clear, since in this case $\pi(x) = 1 - x < \frac{1}{2}$. Consider then $x > \frac{2}{3}$. Here, the claim is $\frac{x^2}{4(2x-1)} \leq \frac{1}{2}$, or $2x^2 \leq 8x - 4$. This inequality holds at $x = \frac{2}{3}$ and the derivative of its LHS is $4x$, which is smaller than that of the RHS, 8. Hence, the result follows. □

Lemma 5 $(e(S), e(S))$ *is an equilibrium demand vector.*

Proof Clearly, checking only deviations upward suffices. If a player increases his demand to $x > e(S)$, his payoff will be $\pi(x)$, and by Lemma 4 we know that $\pi(x) \leq \frac{1}{2} \leq e(S)$. □

Lemma 6 *If* $S \neq \Delta_2$*, then* $(1, 1)$ *is not an equilibrium demand vector.*

Proof Consider (1,1). The randomly selected player will revise his demand to $\frac{1}{2}$, which will be implemented with probability $\frac{1}{2}$. Therefore, each player's expected utility in this case is $\frac{1}{8} + \frac{1}{4}\psi_S^i(\frac{1}{2})$. However, if a player deviates to $y < 1$ such that the resulting demands are still not feasible, his payoff from this deviation is $\frac{1}{2}\psi_S^i(\frac{1}{2})$. Thus, it is enough to prove that $\frac{1}{2}\psi_S^i(\frac{1}{2}) > \frac{1}{8} + \frac{1}{4}\psi_S^i(\frac{1}{2})$, which is equivalent to $\psi_S^i(\frac{1}{2}) > \frac{1}{2}$. This is equivalent to $S \neq \Delta_2$. □

Equipped with the lemmas, we can turn to prove Theorem 1.

Proof of Theorem 1 By Lemma 5, $(e(S), e(S))$ are equilibrium demands. Assume by contradiction that there exists another equilibrium. Let (x, y) be the equilibrium demands. By Lemma 2, $x = y$. Clearly, $x \geq e(S)$. By Lemma 3, $x \in \{e(S), 1\}$ and by Lemma 6 $x < 1$. This contradicts the assumption that the equilibrium in question is not the one with the egalitarian demands. □

[8] The above inequality is equivalent to $\psi_S^i(\frac{x}{2}) > \frac{x}{2}$; the latter holds, since $\psi_S^i(\frac{x}{2}) \geq 1 - \frac{x}{2}$ and $x \in (0, 1)$.

For problems S that do not satisfy (I) and (II), it may be possible to construct equilibria of $G(S)$ different from the one described in Theorem 1. To see an example, take $m \in [\frac{1}{2}, 1]$ and consider the bargaining problem S which is the convex hull of $\{(0,0), (0,1), (m,1), (1,0)\}$. The demands $(m, 1)$ constitute an equilibrium of $G(S)$. To see this, note first that player 2 receives his ideal payoff so he obviously does not have a profitable deviation. As for player 1, if he deviates to some $x \in (m, 1)$ his payoff would be $\frac{1}{2}\psi_S^1(\frac{1}{2}) \leq \frac{1}{2} \leq m$; if he deviates to $x = 1$ and is called to be the reviser, then his payoff would be $\pi(1) = \frac{1}{4}$.

3.2 Divide-the-dollar

For non-DD problems, part of the incentive to be the *less* greedy player stems from the fact that the residual share after the revision, $\psi_S^i(z)$, is strictly greater than $1 - z$. In DD, this is not the case. That the incentive to be the less greedy player is less powerful in DD translates to the emergence of an additional equilibrium, one in which both players are maximally (and equally) greedy.

Proposition 1 $G(\Delta_2)$ *has precisely two subgame perfect equilibria. In one equilibrium, the (first-stage) demands are* $(\frac{1}{2}, \frac{1}{2})$ *and in the other the (first-stage) demands are* $(1, 1)$. *In the latter, the randomly chosen player revises his original demand (i.e.,* 1*) to* $\frac{1}{2}$ *in the second stage.*

Proof It is easy to verify that Lemmas 1–5 apply to $S = \Delta_2$. It only remains to show that $(1, 1)$ is an equilibrium demand vector. Note that under these demands one player will be selected at random (with probability half), and he will revise his demand to $\frac{1}{2}$, in which case the payoff vector $(\frac{1}{2}, \frac{1}{2})$ is implemented with probability half. In short, in this case each player's expected utility is $\frac{1}{4}$. To see that there is no profitable deviation, consider, w.l.o.g, player 2. Suppose that he deviates to some $y < 1$. If $(1, y) \notin S$, then player 1 will be the reviser and player 2's expected utility from the revision will be without a change, $\frac{1}{4}$. If $(1, y) \in S$, then $y = 0$, which is clearly not profitable. □

The equilibrium whose demands are $(\frac{1}{2}, \frac{1}{2})$ Pareto-dominates the one whose demands are $(1, 1)$. There is an additional sense in which the former is superior to the latter. Suppose that all that a player knows is that his opponent will demand a quantity which is consistent with *some* equilibrium, though he is not sure according to which equilibrium the opponent is going to play. This is described in the following table:

Player 1 \ Player 2	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}$
1	$\frac{1}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{4}$

In this strategic-form game, there are precisely two Nash equilibria: $(\frac{1}{2}, \frac{1}{2})$ and $(1, 1)$. The former is in weakly dominant strategies, the later is in dominated strategies. Moreover, of these two equilibria, the former is *proper* (Myerson 1978), the latter is not.

Another support for the $(\frac{1}{2}, \frac{1}{2})$ equilibrium is based on the following continuity argument. Take any $S \neq \Delta_2$, which satisfies (I) and (II), and consider $\Delta_2^{\epsilon} \equiv (1-\epsilon)\Delta_2 + \epsilon S$. Note that $G(\Delta_2^{\epsilon})$ has a unique equilibrium for all $\epsilon > 0$; requiring the equilibrium correspondence to be continuous implies that the equilibrium of the limit game, $G(\Delta_2)$, is the $(\frac{1}{2}, \frac{1}{2})$ equilibrium. This argument for equilibrium selection can be viewed as combining the two approaches that have been studied in the literature: it combines limit taking—analogous to Nash's smoothing—and addition of a second stage to the game.

4 *n*-player generalizations

A natural question is how to extend our model to $n > 2$ players. Consider the following generalization, which is played with respect to some n-player bargaining problem, S.[9] First, each player i announces an entire vector of shares, $x^i = (x_1^i, \cdots, x_n^i)$ $(\sum_j x_j^i = 1)$. If $\sum x_i^i \leq 1$, then each player receives his own demand, x_i^i. In case of demand incompatibility, the most (or a most) greedy player is selected to revise his demand. As opposed to the first-stage strategy, in the second stage the reviser only announces a number (own demand), not an entire vector. Assume that k is selected, and denote his revised demand by z^k. This revised demand z^k must satisfy $z^k \in [\frac{1-x_k^k}{n-1}, x_k^k]$. Let λ solve the following equation:

$$z^k = (1-\lambda)x_k^k + \lambda\left(\frac{1-x_k^k}{n-1}\right).$$

With probability λ player k receives z^k and any other player receives $\frac{1-z^k}{n-1}$; with the complementary probability, no one receives anything. We denote this game by $G_n^{Av}(S)$. The superscript stands for "average," reflecting the restriction on the lower bound on the revised demand, namely that it is the average offer to the other players. Note that $G(S) = G_2^{Av}(S)$.

A comprehensive analysis of $G_n^{Av}(S)$ is beyond the scope of this paper. We confine our attention to the possibility of sustaining the vector of egalitarian demands as a (first-stage) equilibrium outcome in this game.

Theorem 2 *The game $G_n^{Av}(S)$ has a subgame perfect equilibrium in which each player's first-stage own demand is $x_i^i = e(S)$ if and only if $e(S) \geq \frac{1}{4}$.*

Proof W.l.o.g, consider a deviation upwards of player 1 from a profile such that $x_i^i = e(S)$ for all i. Similarly to the derivation from the proof of Theorem 1, one can show that the probability λ is given by $\lambda = \frac{(x-z)(n-1)}{nx-1}$, where z is the revised demand. Upon deviating, player 1 maximizes $f(z) = zx - z^2$, where x is his own demand in the first stage of the deviation. The solution to this maximization can be interior or at the left

[9] The definition of a bargaining problem in the n-player case is a straightforward analog of the 2-player definition. Hence, for brevity, we do not repeat the details.

corner. The former case obtains if $x > \frac{2}{n+1}$ and the latter obtains if $x \leq \frac{2}{n+1}$. The optimal revised demand in these two cases is $z^* = \frac{x}{2}$ and $z^* = \frac{1-x}{n-1}$, respectively.

Consider first the possibility of an interior solution, namely when the demand at the first stage of the deviation, x, satisfies $x > \frac{2}{n+1}$. In this case, player 1's payoff is given by $\pi(x) = \frac{(n-1)x^2}{4(nx-1)}$. The FOC $\pi'(x) = 0$ is obtained at $x = \frac{2}{n}$, but this point is a local minimum; thus, the maximum is obtained at one of the corners: $x = \max\{e(S), \frac{2}{n+1}\}$ or $x = 1$.

Consider the left corner, $\max\{e(S), \frac{2}{n+1}\}$.

Case 1: $\max\{e(S), \frac{2}{n+1}\} = e(S)$. In this case, the supremum payoff that the deviation can yield is $\pi(e(S))$. We argue that this number is smaller than $e(S)$, namely $\frac{(n-1)e(S)^2}{4(ne(S)-1)} \leq e(S)$. This inequality simplifies to $e(S) \geq \frac{4}{3n+1}$. To see that the last inequality holds, it is enough to show that $\frac{2}{n+1} \geq \frac{4}{3n+1}$, since $e(S) \geq \frac{2}{n+1}$ by assumption. Since $\frac{2}{n+1} \geq \frac{4}{3n+1}$ is equivalent to $n \geq 1$, the deviation is not profitable.

Case 2: $\max\{e(S), \frac{2}{n+1}\} = \frac{2}{n+1}$. Substituting $x = \frac{2}{n+1}$ in the payoff function gives $\pi(\frac{2}{n+1}) = \frac{1}{n+1} < \frac{1}{n} \leq e(S)$, so the deviation is not profitable.

Therefore, if there exists a profitable deviation such that the first stage of the deviation satisfies $x > \frac{2}{n+1}$, then $x = 1$. Consider then $x = 1$. Substituting $x = 1$ in the payoff function gives the payoff $\frac{1}{4}$, which means that this deviation is profitable if and only if $e(S) < \frac{1}{4}$.

Finally, consider the corner solution $z^* = \frac{1-x}{n-1}$ (which corresponds to the case $x \leq \frac{2}{n+1}$). Player 1's payoff is $z = \frac{1-x}{n-1}$, which is strictly decreasing in x; the optimum is obtained at $x = e(S)$, which gives the utility $\frac{1-e(S)}{n-1}$. Since $\frac{1-e(S)}{n-1} \leq e(S)$, such a deviation is not profitable. □

Let $\Delta_n \equiv \{u \in \mathbb{R}^n_+ : \sum_{j=1}^n u_j \leq 1\}$. The following is an immediate consequence of Theorem 2.

Corollary 1 *The game $G_n^{Av}(\Delta_n)$ has a subgame perfect equilibrium in which each player's first-stage own demand is $x_i^i = \frac{1}{n}$ if and only if $n \in \{2, 3, 4\}$.*

The game G_n^{Av} is not the only natural n-player generalization of our model. Consider the following generalization, which is identical to G_n^{Av}, except that the lower bound on the revised demand is $\min_j x_j^k$, not $\frac{1-x_k^k}{n-1}$. That is, the lower bound on the revised demand is the minimum amount which is offered to any of the other players, not the average of what is offered to them. In this case, the probability λ is defined by the following equation:

$$z^k = (1-\lambda)x_k^k + \lambda \min_j x_j^k.$$

The rules of the game are as in G_n^{Av}: demand incompatibility leads to a second stage, in which, with probability λ, the reviser, call him player k, receives z^k and any other player receives $\frac{1-z^k}{n-1}$, and with the complementary probability no one receives anything. We denote this game by G_n^{Min}. The superscript refers to the fact that the lower bound on the revised demand is the minimum offer to the other players.

Our result for G_n^{Min} is less general than our aforementioned result for G_n^{Av}: it only covers the case where the underlying bargaining problem is Δ_n.[10] Note that $G_2^{Min}(\Delta_2) = G_2^{Av}(\Delta_2) = G(\Delta_2)$.

Theorem 3 *The game $G_n^{Min}(\Delta_n)$ has a subgame perfect equilibrium in which each player's first-stage own demand is $x_i^i = \frac{1}{n}$ if and only if $n \in \{2, 3, 4\}$.*

Theorem 3 is the parallel of Corollary 1, when the game is G_n^{Min} rather than G_n^{Av}. The intuition behind these results is as follows. Consider the equal split vector, $(\frac{1}{n}, \cdots, \frac{1}{n})$. When every player announces this vector, each player's payoff is $\frac{1}{n}$, which is decreasing in n. However, upon deviating upwards and obtaining the position of the reviser, a player's payoff is independent of n.[11] Hence, such a deviation is necessarily profitable if n is sufficiently large. The remarkable feature of these results is that the cutoff on the number of players is $n = 4$. As far as we know, there is no other result in the bargaining literature that depends on such a cutoff.[12]

Proof of Theorem 3 By Theorem 1, we know that the equal split is sustainable in equilibrium for $n = 2$. Let then $n > 2$. Consider a profile of first-stage vectors such that $x_i^i = \frac{1}{n}$ for all i. We consider the following cases.

Case 1: $n > 4$. If player 1, say, deviates to $(1, 0, \cdots, 0)$, he becomes the most greedy player is therefore called to revise his demand. Let $z \in [0, 1]$ denote his revised demand. The expected utility corresponding to z is $(1 - z)z$, hence player 1's optimal z is $z^* = \frac{1}{2}$, and the utility it brings about is $\frac{1}{4}$. Clearly, the deviation is profitable (had player 1 not deviated, his utility would have been $\frac{1}{n} < \frac{1}{4}$).

Case 2: $n = 4$. We show that, w.l.o.g, player 1 does not have a profitable deviation. Assume by contradiction that he has one. Clearly, the deviation is such that it makes him the most greedy player, and besides his own demand it only matters what is the minimum that he offers the other players. Let α denote his own demand and β denote this minimum. Then $\alpha > \frac{1}{4} > \beta$. Player 1's revised demand, z, satisfies $z = (1-\lambda)\alpha + \lambda\beta$, so $\lambda = \frac{\alpha - z}{\alpha - \beta}$ and the expected utility is $\lambda z = (\frac{\alpha - z}{\alpha - \beta})z$. The solution to the maximization over z must be interior, since $z = \beta$ gives the utility β, which makes the deviation not profitable. Therefore, the deviation is such that $z^* = \frac{\alpha}{2}$ and the expected utility from the deviation is $\frac{\alpha^2}{4(\alpha - \beta)}$. The profitability of the deviation implies:

$$\frac{\alpha^2}{4(\alpha - \beta)} > \frac{1}{4}.$$

10 The reason for this can be seen in footnote 13.

11 More precisely: *if* there is a profitable deviation, then the deviation to the vector where the player asks the maximal payoff for himself and offers zero to any other player is profitable deviation; conditional on *this* deviation, the deviator's payoff is independent of n.

12 It is easy to check that, as opposed to equal division, the first-stage demands where each player demands the entire dollar for himself are consistent with a subgame perfect equilibrium of either $G_n^{Min}(\Delta_n)$ or $G_n^{Av}(\Delta_n)$, for any n.

This inequality implies that either $\alpha < \frac{1-\sqrt{1-4\beta}}{2}$ or $\alpha > \frac{1+\sqrt{1-4\beta}}{2}$. We will treat these two cases separately, showing that neither of them is possible.

Consider first $\alpha < \frac{1-\sqrt{1-4\beta}}{2}$. Note that $z^* = \frac{\alpha}{2} \geq \beta$, hence $\alpha \geq 2\beta$. Therefore, we obtain $1 - \sqrt{1-4\beta} > 4\beta$. Rearranging this gives $\sqrt{1-4\beta} > 1$—a contradiction.

Now consider $\alpha > \frac{1+\sqrt{1-4\beta}}{2}$. Suppose that the amount β is offered to player 2. Then, the sum of offers to players 3 and 4 is at least 2β (because β is the minimal offer). On the other hand, the sum of offers to players 3 and 4 is $1-\alpha-\beta < 1-(\frac{1+\sqrt{1-4\beta}}{2})-\beta$.[13] Therefore:

$$1 - \left(\frac{1+\sqrt{1-4\beta}}{2}\right) - \beta > 2\beta,$$

which implies $\frac{1-\sqrt{1-4\beta}}{2} > 3\beta$, or equivalently $1 - 6\beta > \sqrt{1-4\beta}$. Since the LHS must be positive, $\beta < \frac{1}{6}$. On the other hand, rearranging this inequality gives $\beta > \frac{2}{9}$—a contradiction.

Therefore, equal division is sustainable in equilibrium in the 4-player case.

Case 3: $n = 3$. Similarly to the 4-player case, consider a deviation of player 1 where he asks $\alpha > \frac{1}{3}$ for himself, and offers β to player 2, where $\beta < 1 - \alpha - \beta$ (i.e., β is the minimum he offers to any other player). If there exists a profitable deviation, then there is a deviation as above that satisfies:

$$\frac{\alpha^2}{4(\alpha-\beta)} > \frac{1}{3}.$$

This inequality implies that either $\alpha < \frac{2-2\sqrt{1-3\beta}}{3}$ or $\alpha > \frac{2+2\sqrt{1-3\beta}}{3}$.

Consider first $\alpha < \frac{2-2\sqrt{1-3\beta}}{3}$. Since $z^* = \frac{\alpha}{2} \geq \beta$, hence $\alpha \geq 2\beta$. Therefore we obtain $2 - 2\sqrt{1-3\beta} > 6\beta$. Rearranging this yields $\sqrt{1-3\beta} > 1$—a contradiction.

Now consider $\alpha > \frac{2+2\sqrt{1-3\beta}}{3}$. Player 3 is offered $1-\alpha-\beta < 1-(\frac{2+2\sqrt{1-3\beta}}{3})-\beta$, which is more than that offered to player 2. Hence

$$1 - \left(\frac{2+2\sqrt{1-3\beta}}{3}\right) - \beta > \beta,$$

or

$$1 - 6\beta > 2\sqrt{1-3\beta}.$$

Since the LHS is positive, $\beta < \frac{1}{6}$. On the other hand, rearranging this inequality gives $\beta > \sqrt{\frac{1}{12}}$—a contradiction.

Therefore, equal division is sustainable in equilibrium in the 3-player case. □

13 This argument relies on the specific DD structure.

5 Conclusion

In line with other papers in the DD/NDG literature, we have presented a two-stage mechanism, in which a second stage is played in case of demand incompatibility at the first-stage NDG. Under a certain condition on the feasible set, our game has a unique subgame perfect equlibrium; in this equilibrium, there is immediate agreement on the egalitarian payoffs. On the other hand, in the DD version of our game, an additional equilibrium exists: one in which each player demands the entire dollar in the first stage and the randomly chosen player revises his demand to $\frac{1}{2}$.

We have also studied two n-player generalizations of our game. In either generalization, when the underlying bargaining problem is the unit simplex, sustaining the egalitarian outcome in a subgame perfect equilibrium can be achieved if and only if the number of players is at most four. The general theme behind these results is as follows. Consider two-stage games with the following properties: (a) the first stage can end with "success" or "failure," (b) the second stage is played only after "failure," (c) if the second stage is reached, the game's outcome is determined by a subset of players, (d) the players are symmetric, and (e) a player can trigger "failure" and make sure that he gets to be one of the influencing players at the second stage. In such games, the group of players who determine the outcome in the second stage has to be sufficiently large relative to the grand set of players; alternatively, the influence of each individual player who participates at the second stage must be bounded. Otherwise, an equilibrium with first-round "success" would be impossible: any individual player will have an incentive to upset it and trigger the second stage, in which he will be over-proportionally influential. Alternatively, the payoffs from "success" need to be sufficiently large in order to make such deviations non-profitable.

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