Theory Dec. (2016) 80:427–442
DOI 10.1007/s11238-015-9510-3

Egalitarian–utilitarian bounds in Nash's bargaining problem

Shiran Rachmilevitch[1]

Published online: 11 July 2015

Abstract For every 2-person bargaining problem, the Nash bargaining solution selects a point that is "between" the relative (or normalized) utilitarian point and the relative egalitarian (i.e., Kalai–Smorodinsky) point. Also, it is "between" the (non-normalized) utilitarian and egalitarian points. I improve these bounds. I also derive a new characterization of the Nash solution which combines a bounds property together with strong individual rationality and an axiom which is new to Nash's bargaining model, the *sandwich axiom*. The sandwich axiom is a weakening of Nash's IIA.

Keywords Bargaining · Bounds · Egalitarianism · Nash bargaining solution · Sandwich axiom · Utilitarianism

1 Introduction

A (2-person) *bargaining problem* is a compact and convex set $S \subset \mathbb{R}^2_+$ that contains $\mathbf{0} \equiv (0, 0)$ and satisfies $S \cap \mathbb{R}^2_{++} \neq \emptyset$. The set S is interpreted as a menu of available utility pairs, out of which the two players (bargainers) need to chose a single point. If they agree on x then player i ends up with the utility payoff x_i, and failing to reach agreement leads to the implementation of the status quo payoffs—$\mathbf{0}$; since $S \cap \mathbb{R}^2_{++} \neq \emptyset$, there are strict incentives to avoid disagreement. Let $\mathcal{B}$ denote the collection of all bargaining problems. A *bargaining domain* is a non-empty subset $\mathcal{D} \subset \mathcal{B}$, and a *solution* (on $\mathcal{D}$) is any function $\sigma : \mathcal{D} \to \mathbb{R}^2$ that satisfies $\sigma(S) \in S$ for all $S \in \mathcal{D}$.

✉ Shiran Rachmilevitch
shiranrach@gmail.com; shiranrach@econ.haifa.ac.il

1 Department of Economics, University of Haifa, Mount Carmel, 31905 Haifa, Israel

The best-known solution in the literature is the *Nash solution*, N (due to Nash (1950)), which assigns to each problem S the point $x \in S$ that maximizes the payoffs-product, $x_1 \cdot x_2$. Other solutions that will be referred to in the sequel are the *egalitarian solution* and the *utilitarian solution*.

The egalitarian solution, E, which was introduced into the bargaining literature and axiomatized by Kalai (1977a), assigns to each problem S the intersection point of its Pareto frontier and the 45° line. Denote this point by $E(S) \equiv (e(S), e(S))$; that is, $e(S)$ is the maximum payoff that can be provided to both players simultaneously in the problem S. For a *comprehensive* problem S, the point $E(S)$ maximizes $\min\{x_1, x_2\}$ over $x \in S$.[1]

A utilitarian solution, by contrast, maximizes the utility sum over S. In general, this sum may be maximized at multiple points of S, hence there are multiple utilitarian solutions, but they only differ in their tie-breaking among utility-sum maximizers; a generic such solution will be denoted by U.

Utilitarianism and egalitarianism are basic principles of distributive justice (see, among many others, Fleurbaey et al. (2008)). The utilitarian philosophy takes the sum of the individual utilities to be society's utility; by contrast, the egalitarian philosophy, the most prominent representative of which in the twentieth century is John Rawls, puts the emphasis on advancing the welfare of society's worst-off members. Thus, U and E are the formal expressions of these competing principles in the context of the bargaining model.

Both utilitarianism and egalitarianism are based on interpersonal utility comparisons (Elster and Roemer 1991); for instance, if utilities are not comparable, then their summation bears no ethical significance. Unfortunately, however, even if utilities are comparable *in principle*, they may not be comparable *in practice*. To be specific, the bargaining model does not contain any information about the nature of the utility numbers described by any point $x \in S$. Therefore, without out-of-the-model information about the meaning of utilities, the solutions U and E do not necessarily implement their respective underlying philosophies.

A natural reaction to this difficulty is to adopt suitable normalizations. In particular, a *relative utilitarian solution* (Dhilon and Mertens 1999; Pivato 2009; Segal 2000; Sobel 2001) assigns to each S a maximizer of $\frac{x_1}{a_1(S)} + \frac{x_2}{a_2(S)}$ over $x \in S$, where $a_i(S) \equiv \max\{s_i : s \in S\}$ ($a(S)$ is called the *ideal point* of S); the aforementioned sum may be maximized at multiple points—each such point is called a *relative utilitarian point*—hence there are multiple relative utilitarian solutions; a generic such solution will be denoted by RU. Relative egalitarianism is expressed by the *Kalai–Smorodinsky solution*, KS, due to Kalai and Smorodinsky (1975), which assigns to each S the point $\lambda a(S)$, where λ is the maximum possible. The solutions KS and RU are the normalized versions of E and U, respectively. Mathematically, this normalization finds its expression in the fact that, as opposed to E and U, KS and RU are *scale invariant* solutions, meaning that for every S and every pair of positive linear transformations

[1] S is *comprehensive* if $x \in S$ and $\mathbf{0} \leq y \leq x$ implies that $y \in S$. Vector inequalities: uRv iff $u_i R v_i$ for both i, for each $R \in \{>, \geq\}$. Comprehensiveness captures the idea of utility-free disposal.

$l = (l_1, l_2)$ the solution σ satisfies $\sigma(l \circ S) = l \circ \sigma(S)$ (the solution N is also scale invariant).[2]

In broad terms, the subject of this paper is the relationships between the solutions in $\{E, U, KS, RU\}$ on the one hand, and N on the other hand. To the best of my knowledge, the first result in the literature about a non-trivial such relationship is the following, due to Cao (1982). Cao proved that for every comprehensive problem S for which the relative utilitarian point is unique, the point $N(S)$ is "between" $RU(S)$ and $KS(S)$. That is,

$$N_i(S) \geq \min\{RU_i(S), KS_i(S)\} \quad \forall i = 1, 2. \tag{1}$$

Say that a solution, σ, *satisfies Cao's bounds*, if it satisfies the counterpart of (1); that is, σ satisfies Cao's bounds if the following is true for every problem S whose relative utilitarian point is unique:

$$\sigma_i(S) \geq \min\{RU_i(S), KS_i(S)\} \quad \forall i = 1, 2.$$

Adhering to Cao's bounds can be interpreted as respecting a minimal degree of both (relative) utilitarianism and egalitarianism.

In a recent paper (Rachmilevitch 2014), I showed that the fact that N satisfies Cao's bounds can be strengthened: it was proved in that paper that the following is true for any comprehensive problem S whose utilitarian point, $U(S)$, is unique:

$$N_i(S) \geq \min\{U_i(S), e(S)\} \quad \forall i = 1, 2. \tag{2}$$

Inequality (2) implies (1) in the following sense: for every problem S such that $U(l \circ S)$ is unique for every rescaling l, (2) implies (1). To see why, consider such an S. Apply to it a transformation l such that $a_1(S') = a_2(S')$, where $S' = l \circ S$. By assumption (2) holds for S', and therefore (1) holds for it as well, because on *normalized* problems—problems the ideal point of which is on the 45°-line—there is no difference between the normalized and non-normalized versions of egalitarianism and utilitarianism. Since (1) holds for S' it also holds for S, because (1) involves only scale-invariant solutions.

Say that a solution, σ, *satisfies the EU bounds*, if it satisfies the counterpart of (2); that is, σ satisfies the EU bounds if the following is true for every problem S whose utilitarian point is unique:

$$\sigma_i(S) \geq \min\{U_i(S), e(S)\} \quad \forall i = 1, 2.$$

Cao's bounds and the EU bounds bring about two questions:

- Can N be axiomatized on the basis of these bounds?
- Can the bounds be improved for N?

In Rachmilevitch (2014), I derived an axiomatization of N on the basis of the EU bounds (on the domain of comprehensive problems, N is the unique scale-invariant solution that respects them), and I showed that for any S for which $RU(S) \neq KS(S)$,

[2] The function l_i is a positive linear transformation if $l_i(t) \equiv \alpha_i t$ for some $\alpha_i > 0$. A pair of such transformations, $l = (l_1, l_2)$, is also called a *rescaling*.

the part of S's Pareto frontier that ranges between $RU(S)$ and $KS(S)$ can be narrowed to a smaller range in which $N(S)$ is guaranteed to lie. In particular, it was shown that, in certain senses, $N(S)$ is closer to $RU(S)$ than it is to $KS(S)$.

In the present paper, I derive analogous results: I characterize N on the basis of Cao's bounds and I show that the EU bounds can be tightened in a way that lends formal meaning to "N is closer to U than it is to E."

The characterization of N involves an axiom which is new to the literature, the *sandwich axiom*. This axiom requires that for every triple of nested problems, $S \subset V \subset T$, if $\sigma(S) = \sigma(T) = x$, then $\sigma(V) = x$. This axiom is a weakening of Nash's (1950) *independence of irrelevant alternatives* (IIA), which requires $\sigma(S) = \sigma(T)$ whenever $\sigma(T) \in S \subset T$.[3] The typical justification for IIA is that if the agreement $\sigma(T)$ is "revealed by the solution" to be superior to any alternative in $(T\backslash\{\sigma(T)\})$, then it is obviously superior to any alternative in $(S\backslash\{\sigma(T)\})$; in particular, whether the alternatives in $(T\backslash S)$ are available for choice is irrelevant as long as $\sigma(T)$ is available. Along similar lines, the sandwich axiom, when applied to a triple $S \subset V \subset T$, can be interpreted as follows: *if* the alternatives in $(T\backslash S)$ were proved to be irrelevant, then those in $(T\backslash V)$ are surely irrelevant. Namely, the sandwich axiom requires that a subset of irrelevant alternatives be a set of irrelevant alternatives.

The Nash solution has a well-known generalization: σ is an *asymmetric Nash solution* if there is a $\beta \in (0, 1)$ such that $\sigma(S)$ maximizes $x_1^{\beta} \cdot x_2^{1-\beta}$ over $x \in S$, for every S. This solution was first axiomatized by Kalai (1977b), on the basis of an axiom list that contains IIA. I show that in this axiomatization, IIA can be weakened to the sandwich axiom.

The rest of the paper is organized as follows. Section 2 formally defines several important bargaining domains; Sect. 3 contains the axiomatization of the Nash solution and other results that are related to the sandwich axiom (among them is the just-mentioned axiomatization of the asymmetric Nash solution); Sect. 4 is dedicated to the improvement of the EU bounds—an improvement which is based on a certain "more-utilitarian-than" ordering of EU bounds respecting bargaining solutions; in Sect. 5, alternative "more-egalitarian/utilitarian-than" orderings of solutions are studied.

2 Bargaining domains

Two important bargaining domains are (1) the grand domain, $\mathcal{B}$, and (2) the domain of comprehensive problems, hereafter denoted by $\mathcal{C}$.

One of the simplest domains is the domain of polytopes in $\mathbb{R}^2_+$ that contain $\mathbf{0}$. Denote this domain by $\mathcal{P}$. This domain can serve to model a variety of finite 2-person strategic-form games (with public randomization).

The *Pareto frontier* of S is $P(S) \equiv \{x \in S : (y \geq x)\&(y \in S) \Rightarrow y = x\}$. In the next Section, I will consider bargaining domains $\mathcal{D}$ that satisfy the following two conditions:

[3] The sandwich axiom resembles (but is not identical to) Weak WARP of choice theory; see Manzini and Mariotti (2007). I thank Kemal Yildiz for pointing this out to me.

- (I) $(\mathcal{P} \cap \mathcal{C}) \subset \mathcal{D}$.
- (II) $\forall S \in \mathcal{D},\ \forall x \in S \cap \mathbb{R}^2_{++}$: $\exists V \in \mathcal{D}$ such that $V \subset S$ and $P(V) = \{x\}$.

(I) and (II) are weak conditions. In particular, they are satisfied by any of the domains mentioned above: $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{P}$.[4] There are, however, economically meaningful bargaining domains that do not satisfy conditions (I) and (II). For example, the domain of comprehensive problems with a Pareto frontier that has strict curvature is such a domain (this domain violates both (I) and (II)).[5]

Another domain that will be referred to in the sequel is the domain of straight lines, $\mathcal{L} \equiv \{\text{conv}\{\mathbf{0}, v\} : v > \mathbf{0}\}$. Though problems in $\mathcal{L}$ may be viewed as degenerate, and they are usually not considered in the literature, they enjoy a straightforward interpretation: these are problems in which the parties' interests are completely aligned. An example of a work on bargaining in which these problems play an important role can be found in Anbarci (2002).

3 The characterization

A solution σ satisfies *strong individual rationality* (due to Roth 1977), if $\sigma(S) > \mathbf{0}$ for all S.

Theorem 1 *Let $\mathcal{D}$ be a bargaining domain that satisfies* (*I*)–(*II*) *and let σ be a solution on $\mathcal{D}$. Then σ satisfies Cao's bounds, strong individual rationality, and the sandwich axiom if and only if it is the Nash solution.*

Theorem 1 will be proved on the basis of the following lemma, for the statement of which the following axiom (due to Anbarci 1998) and definition are needed. A solution on $\mathcal{D}$, σ, satisfies *midpoint outcome on a linear frontier* (MOL) if for every triangular $S \in \mathcal{D}$, $\sigma(S) = \frac{1}{2}a(S)$. This axiom is a weakening of *midpoint domination* (due to Sobel 1981), which requires $\sigma(S) \geq \frac{1}{2}a(S)$ for all $S \in \mathcal{D}$.

Lemma 1 *Let $\mathcal{D}$ be a bargaining domain that satisfies* (*I*)–(*II*) *and let σ be a solution on $\mathcal{D}$. Then, if σ satisfies Cao's bounds, strong individual rationality, and the sandwich axiom, then it satisfies MOL.*

Proof Let $\mathcal{D}$ and σ be as above and let $S \equiv \text{conv}\{\mathbf{0}, (1,0), (0,k)\} \in \mathcal{D}$, where $k > 1$. I will prove that $\sigma(S) = (\frac{1}{2}, \frac{k}{2})$.

Case 1: $\sigma(S) \in P(S)$.

Let $t \equiv \sigma_1(S)$. Since $\sigma(S) \in P(S)$, it is enough to prove that $t = \frac{1}{2}$. Assume by contradiction that $t \neq \frac{1}{2}$.

Case 1.1: $t < \frac{1}{2}$. For a small $\epsilon > 0$, consider $S_\epsilon \equiv \{s \in S : s_1 \leq 1 - \epsilon\}$. By (I), $S_\epsilon \in \mathcal{D}$. In the problem S_ϵ the relative utilitarian point is unique—$RU(S_\epsilon) = (1-\epsilon, -k(1-\epsilon)+k)$. Also, $KS(S_\epsilon) \sim KS(S) = (\frac{1}{2}, \frac{k}{2})$ for all small ϵ's. The

[4] That each of these three domains satisfies (I) is obvious; $\mathcal{P}$ satisfies (II) because it contains straight lines and $\mathcal{C}$ satisfies (II) because it contains rectangles.

[5] However, Theorem 1 extends to this domain by a standard continuity argument.

combination of strong individual rationality and condition (II) implies that there is a $V \in \mathcal{D}$ such that $V \subset S$ and $P(V) = \{\sigma(S)\}$. For a sufficiently small ϵ, $V \subset S_\epsilon$. By Cao's bounds $\sigma(V) = \sigma(S)$, and therefore, by the sandwich axiom, $\sigma(S_\epsilon) = \sigma(S)$. Therefore, Cao's bounds are not satisfied, as $\sigma_1(S_\epsilon) = t < \min\{KS_1(S_\epsilon), 1-\epsilon\} = \min\{KS_1(S_\epsilon), RU_1(S_\epsilon)\}$.

Case 1.2: $t > \frac{1}{2}$. For a small $\epsilon > 0$, let $S'_\epsilon \equiv \{s \in S : s_2 \leq k - \epsilon\}$. By arguments analogous to the ones from Case 1.1, it follows that Cao's bounds are not satisfied on S'_ϵ.

Case 2: $\sigma(S) \notin P(S)$. Let $T \equiv \theta S$, where $\theta \in (0, 1)$ is the unique number such that $\sigma(S) \in P(T)$. Such a $\theta \in (0, 1)$ exists, because, by individual rationality, $\sigma(S) > \mathbf{0}$. By (I), $T \in \mathcal{D}$. It follows from Case 1 that $\sigma(T) = \frac{1}{2}a(T) = (\frac{\theta}{2}, \frac{\theta k}{2})$. Moreover, I argue that $\sigma(T) = \sigma(S)$. To see this, note that by condition (II) there is a $V' \in \mathcal{D}$ such that $V' \subset T$ and $P(V') = \{\sigma(S)\}$. By Cao's bounds, $\sigma(V') = \sigma(S)$. Since V' satisfies $V' \subset T$ it follows from the sandwich axiom that $\sigma(T) = \sigma(S)$.

Next, I argue that $\theta > \frac{1}{2}$. To see this, consider the point on $P(S)$ whose first coordinate equals θ: $v^* \equiv (\theta, -k\theta + k)$. Note that if $\theta \leq \frac{1}{2}$, then the rectangle whose Pareto frontier is $\{v^*\}$, call it R, satisfies $T \subset R \subset S$. Since, as we just noted in the previous paragraph, $\sigma(T) = \sigma(S)$, the sandwich axiom implies that $\sigma(R) = \sigma(S)$ and therefore $\sigma(R) = \sigma(T)$. However, by Cao's bounds it follows that $\sigma(R) = v^*$. This is a contradiction, as $\sigma_1(R) = v^*_1 = \theta$ is inconsistent with $\sigma_1(R) = \sigma_1(T) = \frac{\theta}{2}$. Therefore, $\theta > \frac{1}{2}$.

Let $l \equiv \text{conv}\{v^*, (0, \theta k)\}$. Let Q be the (unique) comprehensive problem such that $P(Q) = l$; namely, $Q = \text{conv}(\{\mathbf{0}, (\theta, 0)\} \cup l)$. By (I), $Q \in \mathcal{D}$. Note that $T \subset Q \subset S$. Therefore, by the sandwich axiom, $\sigma(Q) = \sigma(S) = \sigma(T)$.

Claim 1 $KS_1(Q) > \frac{\theta}{2}$.

Proof of Claim 1 Every point $(x, y) \in l$ satisfies the equation:

$$y = -\frac{k}{\theta}(2\theta - 1)x + \theta k.$$

Therefore, the point $KS(Q) = (x, y)$ satisfies[6]:

$$-\frac{k}{\theta}(2\theta - 1) + \frac{\theta k}{x} = k.$$

Solving this equation gives $x = KS_1(Q) = \frac{\theta^2}{3\theta - 1}$. Therefore, Claim 1 is proved, as $\frac{\theta^2}{3\theta-1} > \frac{\theta}{2}$ follows from $1 > \theta$. □

Claim 2 $RU(Q) = v^*$.

Proof of Claim 2 The point $RU(Q)$ is the solution to the following optimization problem: maximize $\frac{x}{\theta} + \frac{-\frac{k}{\theta}(2\theta-1)x+\theta k}{\theta k}$ over $x \in [0, \theta]$. This is equivalent to maximizing

[6] The LHS is the ratio of payoffs, and the RHS is the ratio of the ideal point's components.

$\frac{x}{\theta} - \frac{x}{\theta^2}(2\theta - 1) = x(\frac{1}{\theta^2} - \frac{1}{\theta})$. Since the term is parentheses is strictly positive the optimum is obtained at $x = \theta$. Therefore, $RU(Q) = v^*$.

By the Claims and Cao's bounds, $\sigma_1(Q) > \frac{\theta}{2}$, in contradiction to $\sigma_1(Q) = \sigma_1(T) = \sigma_1(S) = \frac{\theta}{2}$. □

Proof of Theorem 1 Obviously N satisfies strong individual rationality and the sandwich axiom (because it satisfies IIA), and by Cao (1982) it satisfies the bounds. Conversely, let σ be a solution that satisfies the three axioms. Let $S \in \mathcal{D}$. By (II), let $V \in \mathcal{D}$ be such that $V \subset S$ and $P(V) = \{N(S)\}$, and let $T \equiv \text{conv}\{\mathbf{0}, (2N_1(S), 0), (0, 2N_2(S))\}$. By (I), $T \in \mathcal{D}$. By Lemma 1, $\sigma(T) = N(S)$. By Cao's bounds, $\sigma(V) = N(S)$. Since $V \subset S \subset T$, the sandwich axiom implies $\sigma(S) = N(S)$. □

The axioms in Theorem 1 are independent. The solution KS satisfies Cao's bounds and strong individual rationality, but not the sandwich axiom. The solution E satisfies the sandwich axiom and strong individual rationality on $\mathcal{C}$, but not Cao's bounds. To see that strong individual rationality is indispensable, let $\Delta \equiv \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1\}$ be the unit simplex, let $\mathcal{D}^*$ be an arbitrary domain that contains Δ and satisfies (I)-(II), and consider the following solution on $\mathcal{D}^*$, σ^*: σ^* coincides with N on $\mathcal{D}^* \backslash \{\Delta\}$ and $\sigma^*(\Delta) = (1, 0)$. It is not hard to check that σ^* satisfies the sandwich axiom and Cao's bounds, and $\sigma^* \neq N$.

The proof of Theorem 1 is based on an idea which is due to Moulin (1983), and that has also been utilized by Anbarci (1998). Moulin (1983) characterized the Nash solution on the basis of IIA and midpoint domination, and Anbarci (1998) showed that midpoint domination can be weakened to MOL, and the characterization would still go through. It is worth noting that in both of these results IIA cannot be weakened to the sandwich axiom. To see this, note that the *midpoint solution*, $m(S) \equiv \frac{1}{2}a(S)$, satisfies the sandwich axiom and midpoint domination.

It is easy to see that the sandwich axiom is implied by IIA, and that the converse implication does not hold. One can construct various solutions that satisfy the sandwich axiom but not IIA. Below is such solution; for its description, the following definitions are needed. Given S, let $A(S)$ denote the area of S. Let $p(S) \equiv \frac{A(S)}{1+A(S)}$, and consider the solution that assigns to each problem S the point $(\frac{A(S)}{1+A(S)}e, \frac{1}{1+A(S)}e)$, where e is the maximum possible. It is easy to check that this solution, call it σ^A, satisfies the sandwich axiom but violates IIA, for example, on the domain $\mathcal{C} \cup \mathcal{L}$.

On the abovementioned domain, the solution σ^A is neither *weakly Paretian* nor strongly individually rational, since it assigns the origin to every straight line (σ is *weakly Paretian* if $\sigma(S) \in WP(S) \equiv \{x \in S : y > x \Rightarrow y \notin S\}$ for all S). This is no coincidence: on any domain that contains $\mathcal{L}$, if a weakly Paretian and strongly individually rational solution satisfies the sandwich axiom then it also satisfies IIA.

Proposition 1 *Let $\mathcal{D}$ be a bargaining domain that contains $\mathcal{L}$ and let σ be a weakly Paretian and strongly individually rational solution on $\mathcal{D}$. Then σ satisfies the sandwich axiom if and only if it satisfies IIA.*

Proof It suffices to prove that under the abovementioned assumptions, the sandwich axiom implies IIA. Let $\mathcal{D}$ and σ be as above and let $S \subset T$ be two problems in $\mathcal{D}$

such that $x \equiv \sigma(T) \in S$. Let $V \equiv \text{conv}\{\mathbf{0}, x\}$. By strong individual rationality $x > \mathbf{0}$, therefore $V \in \mathcal{L} \subset \mathcal{D}$. By weak Pareto, $\sigma(V) = x$. Since $V \subset S$, the sandwich axiom implies $\sigma(S) = x$. Therefore, IIA is satisfied. □

Proposition 1 implies that if one restricts attention to weakly Paretian and strongly individually rational solutions, then the sandwich axiom and IIA differ only on domains that do not contain $\mathcal{L}$. Such domains are standard in bargaining theory (e.g., $\mathcal{C}$).

Neither axiom can be dispensed with in Proposition 1. For example, the midpoint solution satisfies the sandwich axiom and strong individual rationality, but not IIA. The solution that assigns to S the point $(a_1(S), 0)$ if $a_1(S) \leq a_2(S)$ and assigns to it $(0, a_2(S))$ otherwise satisfies the sandwich axiom and weak Pareto, but not IIA.

Our last result in this Section concerns the asymmetric Nash solution, which maximizes $x_1^{\beta} \cdot x_2^{1-\beta}$ over $x \in S$, for some $\beta \in (0, 1)$. Kalai (1977b) characterized this solution on the basis of weak Pareto, scale invariance, strong individual rationality, and IIA. Proposition 1 implies the following:

Corollary 1 *Let σ be a solution on $\mathcal{B}$. Then σ satisfies weak Pareto, scale invariance, strong individual rationality, and the sandwich axiom if and only if it is an asymmetric Nash solution.*

Proof Clearly an asymmetric Nash solution satisfies the axioms. Conversely, let σ be a solution that satisfies them. Then, by Proposition 1, it satisfies IIA. By Kalai's (1977b) theorem, σ is an asymmetric Nash solution. □

4 An improvement of the EU bounds

In what follows, I restrict my attention to problems for which the utilitarian point is unique.

Define the *compromise on utilitarianism* at point x given the problem S as the ratio $C_U(x|S) \equiv \frac{U_1(S)+U_2(S)}{x_1+x_2}$ and the *compromise on egalitarianism* as $C_E(x|S) \equiv \frac{e(S)}{\min\{x_1,x_2\}}$.[7] If one views both egalitarianism and utilitarianism as appealing principles of distributive justice, then one would like both of these compromises to be low. Of course, typically there is a trade-off between the two: setting the compromise on utilitarianism at its minimum results in a non-trivial compromise on egalitarianism, and vice versa. In light of this trade-off, we see that it is useful to have a systematic way to *aggregate* the compromises: if we could aggregate them into a single measure, then (under some restrictions on the aggregation method) minimizing this measure would express the desire of having both compromises low. Consider, then, the following procedure.

[7] The compromise on utilitarianism can be viewed as related to the *price of fairness* associated with the solution σ given the problem S (due to Bertsimas et al. 2011), which is defined as the relative reduction in the utility sum, when compared to the optimal sum: namely, it is given by $\frac{[U_1(S)+U_2(S)]-[\sigma_1(S)+\sigma_2(S)]}{U_1(S)+U_2(S)}$. Both of these efficiency measures share a similar flavor with the *price of anarchy* (Koutsoupias and Papadimitriou (1999)), which assigns to each game the ratio of the maximal utility sum over the utility sum in the worst Nash equilibrium. Other related concepts include the *price of stability* (e.g., Anshelevich et al. 2008) and the *price of selfishness* (Blocq and Orda 2012).

Let F be an increasing and symmetric two-argument function. Given S and a point on its frontier, x, the corresponding compromises are $C_U(x|S)$ and $C_E(x|S)$, which the function F aggregates into a single number, namely $F(C_U(x|S), C_E(x|S))$. Different F's aggregate the compromises in different ways, and if F is such that $F(C_U(x|S), C_E(x|S))$ is minimized at a unique x for every S, then a bargaining solution results:

$$\sigma^F(S) \equiv \text{argmin}_{x \in S} F(C_U(x|S), C_E(x|S)).$$

Let $\mathcal{F}$ be the set of these F's such that σ^F is a well-defined solution and let:

$$\Sigma \equiv \{\sigma^F : F \in \mathcal{F}\}.$$

Every solution in Σ satisfies the EU bounds, but not every EU-bounds-respecting solution is in Σ; for example, the Nash solution is not.

Theorem 2 *There does not exist an $F \in \mathcal{F}$ such that $\sigma^F = N$.*

Proof Assume by contradiction that $F \in \mathcal{F}$ is such that $\sigma^F = N$. Let $S \equiv \text{conv}\{\mathbf{0}, (0, \frac{k}{2}), (\frac{1}{2}, \frac{k}{2}), (1, 0)\}$, where $k > 1$. For this S, we have that $U(S) = N(S) = (\frac{1}{2}, \frac{k}{2})$ and $E(S) = (\frac{k}{1+k}, \frac{k}{1+k})$. Therefore, $C_U(N(S)|S) = 1$ and $C_E(N(S)|S) = \frac{\frac{k}{1+k}}{\frac{1}{2}} = \frac{2k}{1+k}$. At the egalitarian point, the compromises are $C_U(E(S)|S) = \frac{\frac{k+1}{2}}{\frac{2k}{k+1}} = \frac{(k+1)^2}{4k}$ and $C_E(E(S)|S) = 1$. Since F is symmetric, the fact that $N(S)$ uniquely minimizes the F-value of the compromises implies that:

$$F\left(1, \frac{2k}{1+k}\right) < F\left(1, \frac{(k+1)^2}{4k}\right).$$

Therefore, since F is increasing:

$$\frac{2k}{1+k} < \frac{(k+1)^2}{4k}.$$

This implies that $8k^2 < (k+1)^3$, or $f(k) \equiv 2k^{2/3} < k + 1 \equiv g(k)$. However, note that $f(1) = g(1)$ and $f'(k)|_{k=1} = \frac{4}{3} > 1 = g'$, hence, for $k \sim 1$, a contradiction obtains. □

In what follows I will give special attention to the following sub-family of Σ:

$$\Sigma^{\text{diff}} \equiv \{\sigma^F \in \Sigma : F \text{ is differentiable}\}.$$

Below I use the family Σ^{diff} to describe a formal sense according to which N is more oriented towards utilitarianism than towards egalitarianism.

If ϕ and ψ are two EU-bounds-respecting solutions, then the following holds for each S:

$$|\phi_1(S) - U_1(S)| \leq |\psi_1(S) - U_1(S)| \Leftrightarrow |\phi_2(S) - U_2(S)| \leq |\psi_2(S) - U_2(S)|.$$

Namely, either both individual payoffs under ϕ are closer to the utilitarian payoffs than the payoffs under ψ, or the other way around. Consequently, it is natural to label ϕ as *more utilitarian than* ψ *given* S (or ψ as *less utilitarian than* ϕ *given* S) if $|\phi_i(S) - U_i(S)| \leq |\psi_i(S) - U_i(S)|$ for both $i = 1, 2$. Note that if ϕ and ψ are two EU-bounds-respecting solutions, then ϕ is more utilitarian than ψ given S if and only if $\phi_1(S) + \phi_2(S) \geq \psi_1(S) + \psi_2(S)$. If both ϕ and ψ respect the EU bounds and ϕ is more utilitarian than ψ given any S, then ϕ is simply *more utilitarian than* ψ.

The following results can be interpreted as improvements of the EU bounds, and as a formal manifestation of the idea that "the Nash solution is more utilitarian than egalitarian."

Theorem 3 *There does not exist a solution in* Σ^{diff} *that is more utilitarian than the Nash solution.*

Proof Let $\sigma^F \in \Sigma^{\text{diff}}$ and consider the triangle, T, whose boundary is given by the function $y = f(x) = -kx + k$, where $k > 1$. Suppose further that $k \sim 1$. For this problem σ^F select a point $(t, -kt + k)$ that minimizes $F(\frac{k}{t(1-k)+k}, \frac{k}{(k+1)\min\{t, -kt+k\}})$. It is well known (and easy to check) that N satisfies MOL, hence $N(T) = (\frac{1}{2}, \frac{k}{2})$. To show that $\sigma^F(S)$ lies in between $N(S)$ and $E(S)$, it is enough to show that at $t = \frac{1}{2}$ the derivative (wrt t) of F is strictly negative. This derivative is $[F_1 \cdot \frac{-k(1-k)}{[t(1-k)+k]^2} - F_2 \cdot \frac{k(k+1)}{[t(k+1)]^2}]$, which is indeed negative at $t = \frac{1}{2}$ provided that k is close to one. □

Theorem 4 *There exists a solution in* Σ^{diff} *that is less utilitarian than the Nash solution.*

To prove Theorem 4, it is enough to point to one example of a solution with the desired properties. Consider then the solution that corresponds to $F^*(a, b) \equiv ab$. Call it the *minimal product solution*, and denote it for short by MP.

The following lemmas establish that (i) MP is a well-defined solution, and (ii) it is less utilitarian than N.

Lemma 2 $F^* \in \mathcal{F}$. *Namely, the minimal product solution,* MP*, is a well-defined bargaining solution.*

Proof Let S be a problem. Suppose first that $U(S)$ is to the north-west of $E(S)$. In this case, MP minimizes $\frac{U_1(S)+U_2(S)}{x+y} \cdot \frac{e(S)}{x}$ over $x \in [U_1(S), e(S)]$, where $y \equiv \max\{y' : (x, y') \in S\}$. Namely, it maximizes $(x + y)x$ over this domain. To see that the solution to this maximization is unique, look at the function $(x, y) \mapsto x^2 + xy$. The MRS of the level curves of this function is $2 + \frac{y}{x}$, which is continuously decreasing as one moves from the north-west to the south-east, implying that the level curves are convex, hence the intersection of the highest level curve and $WP(S)$ is a singleton. The case where $U(S)$ is to the south-east of $E(S)$ is analogous. □

Lemma 3 *The Nash solution is more utilitarian than the minimal product solution.*

Proof It is enough to prove the above claim for *smooth* problems: S for which there is a differentiable and strictly decreasing function f, such that for all (a, b) in the Pareto frontier of S, $b = f(a)$. Consider such an S, and wlog suppose that $U(S)$ is to the north-west of $E(S)$. Note that MP minimizes $\frac{U_1(S)+U_2(S)}{a+f(a)} \cdot \frac{e(S)}{a}$ over $a \in [U_1(S), e(S)]$. Namely, it maximizes $[a + f(a)]a \equiv H(a)$ over this domain. We have that $H'(a) = 2a + f(a) + af'(a)$. Since $f(N_1(S)) + N_1(S)f'(N_1(S)) = 0$, $H'(N_1(S)) = 2N_1(S) > 0$; therefore, $MP(S)$ is to the right of $N(S)$. □

The proofs of Theorems 3–4 rely on differentiability arguments. An interesting and easily interpretable solution that belongs to Σ but not to Σ^{diff} is the one corresponding to $F(a, b) = \max\{a, b\}$. It is easy to check that this solution assigns to every S the unique $x \in WP(S)$ that satisfies the EU bounds and the following constraint:

$$C_U(x|S) = C_E(x|S).$$

I therefore call this solution the *equal compromise solution*. Denote it by EC. This solution, as opposed to any solution in Σ^{diff}, satisfies midpoint outcome on a linear frontier, MOL.

Proposition 2 *The equal compromise solution EC, which belongs to $\Sigma \backslash \Sigma^{diff}$, satisfies MOL. By contrast, there does not exist any solution in Σ^{diff} that satisfies MOL.*

Proof First, I will prove that EC satisfies MOL. Wlog, consider the triangle $T = \text{conv}\{\mathbf{0}, (1, 0), (0, k)\}$ for some $k > 1$. An $x \in WP(T)$ takes the form $x = (t, -kt+k)$ and $C_U(x|T) = C_E(x|T)$ is:

$$\frac{k}{t(1-k)+k} = \frac{k}{(k+1)\cdot\min\{t, -kt+k\}}.$$

This equation has two solutions, $t = \frac{1}{2}$ and $t = \frac{k^2}{1+k^2}$. It is immediate to check that $C_U((\frac{1}{2}, \frac{k}{2})|T) < C_U((\frac{k^2}{1+k^2}, \frac{k^2}{1+k^2}(1-k))|T)$, therefore $EC(T) = \frac{1}{2}a(T)$.

Now, consider an arbitrary $\sigma^F \in \Sigma^{\text{diff}}$. Assume by contradiction that it satisfies MOL and consider the triangle T whose boundary is given by the function $y = f(x) = -kx + k$, where $k > 1$. The solution σ^F selects the point $(t, -kt + k)$ that minimizes $F(\frac{k}{t(1-k)+k}, \frac{k}{(k+1)\min\{t,-kt+k\}})$. Since for such a problem $t = \frac{1}{2}$ is strictly in between the egalitarian and the utilitarian point, the following FOC needs to hold at $t = \frac{1}{2}$:

$$F_1 \cdot \frac{-k(1-k)}{[t(1-k)+k]^2} = F_2 \cdot \frac{k(k+1)}{[t(k+1)]^2}.$$

Since $t = \frac{1}{2}$ is a point at which the compromises on utilitarianism and egalitarianism are equal and since F is symmetric, it follows that $F_1 = F_2$ at that point. This implies the contradiction $-1 = 1$. □

The fact that EC satisfies MOL together with the (easily checked) fact that $EC(S)$ is strictly in between $E(S)$ and $U(S)$ whenever $E(S) \neq U(S)$ implies that EC is neither more nor less utilitarian than the Nash solution. To see this, consider again the triangle $T = \text{conv}\{\mathbf{0}, (1, 0), (0, k)\}$ where $k > 1$. Let $T' \equiv \{s \in T : s_2 \leq \frac{k}{2}\}$; for this problem $N(T') = U(T') = (\frac{1}{2}, \frac{k}{2})$, and $EC(T')$ is to the south-east of this point, hence EC is less utilitarian than N given T'. On the other hand, for $T'' \equiv \{s \in T : s_1 \leq \frac{1}{2}\}$, we have that the Nash and utilitarian points are unchanged relatively to the original T—$N(T'') = (\frac{1}{2}, \frac{k}{2})$ and $U(T'') = (0, k)$—and the point $EC(T'')$ is strictly in between them, so EC is more utilitarian than N given T''. Note that on T'', the Nash solution coincides with the *lexicographic extension of* E—the highest feasible point that dominates E according to the standard partial order on $\mathbb{R}^2$. As the following result shows, this is a manifestation of a more general principle: if EC is more utilitarian than N given some problem S, then there is a bound on "how far" $N(S)$ can be from $E(S)$.

Proposition 3 *Let S be such that EC is more utilitarian than N given S. Then the following holds for each $i \in \{1, 2\}$:*

$$N_i(S) \geq \left[\frac{\min\{a_1(S), a_2(S)\}}{\max\{a_1(S), a_2(S)\}} \cdot \left(\sqrt{2} - \frac{1}{2}\right) + \frac{1}{2} - \frac{\min\{a_1(S), a_2(S)\}}{a_1(S) + a_2(S)}\right] \cdot e(S).$$

Proposition 3 implies that if S is such that $a_1(S) \sim a_2(S)$, then the equal compromise solution can be more utilitarian than the Nash solution given S only if both of the Nash payoffs realize a fraction no smaller than 0.91 of the egalitarian payoff.

Proof of Proposition 3 Let S be such that EC is more utilitarian than N given S. Wlog, suppose that $U(S)$ is to the north-west of $E(S)$ and that $U_1(S) < EC_1(S) < N_1(S) \leq e(S)$. This means that the compromise on utilitarianism at $n \equiv N(S)$ is greater than the compromise on egalitarianism at that point. Therefore,

$$\frac{u^*}{n_1 + n_2} \geq \frac{e(S)}{n_1}, \tag{3}$$

where $u^* \equiv U_1(S) + U_2(S)$. By Theorem 2 from Bertsimas et al. (2011), we know that $\frac{u^* - (n_1 + n_2)}{u^*} \leq 1 - X$, where $X \equiv [\frac{\min\{a_1(S), a_2(S)\}}{\max\{a_1(S), a_2(S)\}} \cdot (\sqrt{2} - \frac{1}{2}) + \frac{1}{2} - \frac{\min\{a_1(S), a_2(S)\}}{a_1(S) + a_2(S)}]$. Hence $1 - \frac{n_1 + n_1}{u^*} \leq 1 - X$. Therefore, $\frac{n_1 + n_2}{u^*} \geq X$, or $X^{-1} \geq \frac{u^*}{n_1 + n_2}$. Combining this inequality with (3) yields $X^{-1} \geq \frac{e(S)}{n_1}$, and so $n_1 = \min\{N_1(S), N_2(S)\} \geq X \cdot e(S)$, as desired. □

The solution EC illustrates the "integrability problem" associated with the family Σ: it may very well be that two functions in $\mathcal{F}$ induce different orders on the plane, but, nevertheless, give rise to the same solution. For example, by definition $EC = \sigma^F$ for $F(a, b) = \max\{a, b\}$, but it is also true that $EC = \sigma^G$, for $G(a, b) = |a - b|$. I conjecture that such an integrability problem does not arise if one restricts attention to the family Σ^{diff}; that is, my conjecture is that if $F, G \in \mathcal{F}$ are both differentiable, then $\sigma^F = \sigma^G$.

5 Alternative orderings of solutions

Below I present alternative methods for ordering solutions in terms of their utilitarianism and egalitarianism. I restrict my attention to smooth problems with a distinct utilitarian and egalitarian points: those $S \in \mathcal{B}$ such that:

- $WP(S) = \{(x, f(x)) : x \in [0, M]\}$ where f is a strictly concave and differentiable function, and
- $U(S) \neq E(S)$.

Also, I restrict my attention to Pareto optimal solutions.

Fix a problem S as above and consider the following function, $\Psi_\alpha : [0, M] \to \mathbb{R}$:

$$\Psi_\alpha(x) \equiv x + f(x) - \alpha|\max\{x, f(x)\} - \min\{x, f(x)\}|.$$

Let σ be a solution such that $\sigma(S) \neq E(S)$. Then there is a unique $\alpha = \alpha_\sigma(S) \in \mathbb{R}$ such that $\sigma(S)$ maximizes Ψ_α.[8] The higher is α the greater is the importance which is assigned by the solution to promoting egalitarianism—namely, the equality of payoffs. If σ is a utilitarian solution it attaches no importance whatsoever to this equality, and $\alpha = 0$. For a general, not-necessarily-utilitarian σ, the number α need not be zero, and, informally speaking, the further it is from zero, the further is σ from utilitarianism (given S).

If $U(S)$ is to the north-west of $E(S)$, then $\alpha = \alpha_\sigma(S)$ satisfies the following FOC:

$$-f'(\sigma_1(S)) = \frac{1+\alpha}{1-\alpha}. \tag{4}$$

It is easy to see from (4) that $\alpha = 0$ corresponds to $\sigma(S) = U(S)$. Moreover, if $\sigma(S) \neq U(S)$ and σ satisfies the EU bounds, then $\alpha > 0$ and a higher α means closer proximity to $E(S)$.[9] The case where $U(S)$ is to the south-east of $E(S)$ is analogous; thus, for brevity, and wlog, from here on I will consider only smooth S's such that $U(S)$ is to the north-west of $E(S)$.

The above analysis gives rise to the following definition, which applies to any pair of EU-bounds-respecting solutions, ϕ and ψ: ϕ is *more egalitarian than* ψ if for any S such that $\alpha_\phi(S)$ and $\alpha_\psi(S)$ are well defined, it is true that $\alpha_\phi(S) \geq \alpha_\psi(S)$.

An alternative approach to measure where a bargaining solution "stands" on an egalitarianism-to-utilitarianism spectrum, is to consider, given a fixed problem, the following function, $\Phi_\beta : [0, M] \to \mathbb{R}$:

$$\Phi_\beta(x) \equiv x + f(x) + \beta \min\{x, f(x)\}.$$

[8] If $\sigma(S) = E(S)$ then such an α may not exist. Here is an example of a smooth problem S for which (i) $U(S) \neq E(S)$, and (ii) there does not exist an α such that $E(S)$ maximizes Ψ_α. Consider the problem whose Pareto frontier is $\{(x, f(x)) : x \in [0, 1]\}$, where $f(x) = 2\sqrt{1-x^2}$. Assume by contradiction that such an α exists. Since $E(S) = (\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, it follows that $\Psi'_\alpha(x) \geq 0$ for $x \leq \frac{2}{\sqrt{5}}$ and $\Psi'_\alpha(x) \leq 0$ for $x \geq \frac{2}{\sqrt{5}}$. The first inequality implies $\alpha \geq \frac{3}{5}$ and the second implies $\alpha \leq -\frac{5}{3}$—a contradiction.

[9] $\alpha < 0$ is possible if σ does not respect the EU bounds.

The objective Φ expresses the Rawlsian goal of maximizing the utility of the worst-off individual (Rawls 1971), which, given that the bargaining problem is comprehensiveness, is equivalent to the goal which Ψ expresses.

Let $\beta = \beta_\sigma(S)$ be such that $\sigma(S)$ is the maximizer of Φ_β. Since the FOC that corresponds to $\alpha = \alpha_\sigma(S)$ is $-f'(x) = \frac{1+\alpha}{1-\alpha}$ and the FOC that corresponds to $\beta = \beta_\sigma(S)$ is $-f'(x) = 1 + \beta$, we obtain the following 1-to-1 correspondence between α and β:

$$\beta = \frac{2\alpha}{1-\alpha}. \tag{5}$$

Thus, ϕ is more egalitarian than ψ if and only if $\beta_\phi(S) \geq \beta_\psi(S)$.

A special solution whose α and β are pinned down immediately, and without reference to any FOC's, is the *average solution*, A, which assigns to each smooth S the maximizer of $(x_1 + x_2) + \min\{x_1, x_2\}$ over $x \in S$.[10] It is easy to see that A respects the EU bounds. By definition, $\beta_A \equiv 1$, and therefore, by (5), $\alpha_A \equiv \frac{1}{3}$. Equipped with this observation, we can turn to the following result.

Proposition 4 *The minimal product solution, MP, is more egalitarian than the average solution, A.*

Proof Since $\alpha_A = \frac{1}{3}$, what needs to be proved is:

$$\alpha_{MP}(S) \geq \frac{1}{3}.$$

Consider then a smooth S such that $U(S)$ is to the north-west of $E(S)$. As we saw in the proof of Lemma 3, MP maximizes $x(x + f(x))$, which gives rise to the FOC $2x + f(x) + xf'(x) = 0$. Substituting in the FOC for $\alpha = \alpha_{MP}(S)$ (namely $-f'(x) = \frac{1+\alpha}{1-\alpha}$) and rearranging we get $x(\frac{1-3\alpha}{1-\alpha}) = -f(x)$, and so $\alpha > \frac{1}{3}$. □

In Rachmilevitch (2014), I showed that N not only respects the EU bounds, but, moreover, $N(S)$ is between $U(S)$ and $A(S)$ (for every S on which U and A are single-valued). Combining this result with the analysis developed here implies that, on the part on S's frontier which stretches from $E(S)$ to $U(S)$, the point $A(S)$ is between $N(S)$ and $MP(S)$. Thus, the average point $A(S)$ lies between the point that maximizes the payoffs-product and the one that minimizes the compromise-product.

Corollary 2 *For every S such that $\{\alpha_{MP}(S), \alpha_A(S), \alpha_N(S)\}$ are well defined, it is true that $\alpha_{MP}(S) \geq \alpha_A(S) \geq \alpha_N(S)$.*

Proof In view of Proposition 4, only the inequality $\alpha_A(S) \geq \alpha_N(S)$ needs to be proved. This inequality follows from the combination of: (i) the above cited result from Rachmilevitch (2014), and (ii) Eq. (4). □

Similarly to the derivation of parameter that measures the importance of the egalitarian/Rawlsian goal, and where a zero value of the parameter means utilitarianism, one

[10] Multiplying the abovementioned objective by $\frac{1}{2}$ results in a 50–50 average of the utilitarian and egalitarian objectives, hence the name of this solution.

can consider a dual approach, where a zero value of the parameter is associated with egalitarianism. Specifically, consider, for a fixed S, the function $\Lambda_\gamma : [0, M] \to \mathbb{R}$:

$$\Lambda_\gamma(x) \equiv \min\{x, f(x)\} + \gamma(x + f(x)).$$

Let $\gamma = \gamma_\sigma(S)$ be the number such that $\sigma(S)$ maximizes Λ_γ. If ϕ and ψ are EU-bounds-respecting solutions, ϕ is *more utilitarian than* ψ if for any S such that $\gamma_\phi(S)$ and $\gamma_\psi(S)$ are well defined, it is true that $\gamma_\phi(S) \geq \gamma_\psi(S)$.[11]

Proposition 5 *Let S be a problem and let σ be an EU-bounds-respecting solution. If $\beta_\sigma(S)$ and $\gamma_\sigma(S)$ are well defined, then:*

$$\gamma_\sigma(S) = \frac{1}{\beta_\sigma(S)}.$$

Proof Let S and σ be as above, and suppose that $U(S)$ is to the north-west of $E(S)$. The FOC associated with Λ_γ is $-f'(x) = 1 + \frac{1}{\gamma}$. Combining this with Φ_β's FOC, $1 + \beta + f'(x) = 0$, we obtain the result. □

Proposition 5 implies, in particular, that $\gamma_A \equiv 1$. The 1-to-1 relationship between β and γ motivates the following definition, this paper's last: an EU-bounds-respecting solution, σ, is *more utilitarian than egalitarian*, if $\gamma_\sigma(S) \geq \beta_\sigma(S)$ (whenever these numbers are well defined).

Theorem 5 *The Nash solution is more utilitarian than egalitarian.*

Proof In Rachmilevitch (2014), it is proved that N satisfies the EU bounds. Then, by Proposition 5, it is enough to prove that $\beta_N \leq 1$. By (5), this is equivalent to $\alpha_N \leq \frac{1}{3}$, which was proved in Corollary 2. □

Acknowledgments I thank the participants of the microeconomics seminar at Bilkent University, the participants of the T.S. Kim Memorial Seminar at Seoul National University, and the participants of ORSIS 2015 for their helpful comments. Reports from three anonymous referees are also gratefully acknowledged.

References

Anbarci, N. (1998). Simple characterizations of the Nash and Kalai/Smorodinsky solutions. *Theory and Decision*, *45*, 255–261.

Anbarci, N. (2002). A simple two-axiom characterization of the Nash solution. *Operations Research Spectrum*, *24*, 215–218.

Anshelevich, E., Dasgupta, A., Kleinberg, J., Tardos, E., Wexler, T., & Roughgarden, T. (2008). The price of stability for network design with fair cost allocation. *SIAM Journal on Computing*, *38*, 1602–1623.

Bertsimas, D., Farias, V., & Trichakis, N. (2011). The price of fairness. *Operations Research*, *59*, 17–31.

Blocq, G., & Orda, A. (2012). How good is bargained routing? In *Proceedings of IEEE INFOCOM'2012*. Orlando, FL, U.S.A., March.

Cao, X.-R. (1982). Performance functions and bargaining solutions. *Proceedings of the 21st IEEE Conference on Decision and Control* (pp. 164–171). Orlando, Florida, Dec.

[11] Recall that in Sect. 4 "more utilitarian" meant something else. I trust that no confusion should arise because of this little abuse of terminology.

Dhilon, A., & Mertens, J.-F. (1999). Relative utilitarianism. *Econometrica*, *67*, 471–498.
Elster, J., & Roemer, J. E. (Eds.). (1991). *Interpersonal comparisons of well-being*. Cambridge: Cambridge University Press.
Fleurbaey, M., Salles, M., & Weymark, J. A. (Eds.). (2008). *Justice, political liberalism, and utilitarianism: Themes from Harsanyi and Rawls*. Cambridge: Cambridge University Press.
Kalai, E. (1977a). Proportional solutions to bargaining situations: Interpersonal utility comparisons. *Econometrica*, *45*, 1623–1630.
Kalai, E. (1977b). Nonsymmetric Nash solutions and replications of 2-person bargaining. *International Journal of Game Theory*, *6*, 129–133.
Kalai, E., & Smorodinsky, M. (1975). Other solution to Nash's bargaining problem. *Econometrica*, *43*, 513–518.
Koutsoupias, E., & Papadimitriou, C. (1999). Worst-case equilibria. *Lecture Notes in Computer Science*, *1563*, 404–413.
Manzini, P., & Mariotti, M. (2007). Sequentially rationalizable choice. *American Economic Review*, *97*, 1824–1839.
Moulin, H. (1983). Le choix social utilitariste, Ecole Politechnique DP.
Nash, J. F. (1950). The bargaining problem. *Econometrica*, *18*, 155–162.
Pivato, M. (2009). Twofold optimality of the relative utilitarian bargaining solution. *Social Choice and Welfare*, *32*, 79–92.
Rachmilevitch, S. (2014). The Nash solution is more utilitarian than egalitarian. *Theory and Decision* doi:10.1007/s11238-014-9477-5.
Rawls, J. (1971). *A Theory of Justice*. Belknap: Belknap Press.
Roth, A. E. (1977). Individual rationality and Nashs solution to the bargaining problem. *Mathematics of Operations Research*, *2*, 64–65.
Segal, U. (2000). Let's agree that all dictatorships are equally bad. *Journal of Political Economy*, *108*, 569–589.
Sobel, J. (1981). Distortion of utilities and the bargaining problem. *Econometrica*, *49*, 597–619.
Sobel, J. (2001). Manipulation of preferences and relative utilitarianism. *Games and Economic Behavior*, *37*, 196–215.