Theory Dec. (2016) 80:227–243
DOI 10.1007/s11238-015-9503-2

Risk aversion, prudence, and asset allocation: a review and some new developments

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Published online: 21 May 2015

Abstract In this paper, we consider the composition of an optimal portfolio made of two dependent risky assets. The investor is first assumed to be a risk-averse expected utility maximizer, and we recover the existing conditions under which all these investors hold at least some percentage of their portfolio in one of the assets. Then, we assume that the decision maker is not only risk-averse, but also prudent and we obtain new minimum demand conditions as well as intuitively appealing interpretations for them. Finally, we consider the general case of investor's preferences exhibiting risk apportionment of any order and we derive the corresponding minimum demand conditions. As a byproduct, we obtain conditions such that an investor holds either a positive quantity of one of the assets (positive demand condition) or a proportion greater than 50 % (i.e., the "50 % rule").

Keywords Optimal portfolio · Diversification · Risk aversion · Downside risk · Prudence · Risk apportionment

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1 Introduction and motivation

Consider a risk-averse decision maker who has to invest a given initial wealth in two risky assets with possibly correlated performances over a given reference period. The agent is assumed to act in order to maximize the expected utility of terminal wealth. In this paper, we investigate two closely related problems. On the one hand, we further study the conditions on the joint probability distribution of the assets such that all risk-averse decision makers agree to diversify their position, i.e., to invest a positive fraction of their initial wealth in each of the two assets. This first question is referred to as the positive demand problem in the literature. On the other hand, we consider a risk-averse investor holding a given portfolio made of these two risky assets and we derive criteria expressed in terms of the joint probability distribution of the assets and of the underlying portfolio to ensure that the share of an asset is increased at the expense of the other in the portfolio. This second question is referred to as the minimum demand problem in the literature.

Let us now briefly summarize the results available in the literature about these two problems. Negative expectation dependence introduced by Wright (1987) is the key concept to solve the positive demand problem. The expectation dependence concept has been extended to higher orders by Li (2011). Denuit et al. (2015) introduced a dual version of this dependence concept, called excess dependence, and provided an application to the positive demand problem, generalizing the results of Wright (1987) to investors' preferences exhibiting higher-order risk apportionment in the sense of Eeckhoudt and Schlesinger (2006).

The positive demand condition for risk-averse investors obtained by Wright (1987) has been extended by Hadar and Seo (1988) to ensure that the proportion of a given asset in the optimal portfolio exceeds a given threshold. This minimum demand condition is closely related to the concept of marginal conditional stochastic dominance (MCSD) introduced by Yitzhaki and Olkin (1991) and Shalit and Yitzhaki (1994) as a condition under which all risk-averse investors agree to increase the share of one risky asset over the other.

In this paper, we revisit the minimum demand problem by considering risk-averse investors exhibiting prudence. In the expected utility setting, prudence is defined by the non-negativity of the third derivative of the utility function. This risk attitude was initially justified by reference to the decision of building up precautionary savings in order to better face future income risk (Kimball 1990[1]). The role of prudence has also been illustrated in other contexts, including self-protection activities (Chiu 2005), optimal audits (Fagart and Sinclair-Desgagné 2007), or decreasing sensitivity to an increase in correlation when the initial wealth increases (Denuit and Rey 2010). Quite surprisingly, although prudence originates in savings problems, this concept does not seem to have been applied to optimal portfolio selection so far. The present paper investigates the role of this additional assumption of prudence in asset allocation.

[1] Kimball (1990) coined the term prudence in his analysis of savings under future income risk. However, as indicated by Kimball (1990), this question had already been analyzed earlier, e.g., by Drèze and Modigliani (1972), Sandmo (1970), and Leland (1968).

By restricting the class of risk-averse decision makers to the subset of those investors exhibiting prudence, we can weaken the condition imposed by Hadar and Seo (1988) on the joint probability distribution of the assets ensuring that the optimal portfolio comprises at least a given percentage of one of them. In particular, imposing prudence in addition to risk aversion also weakens the positive demand condition in Wright (1987) and the MCSD criterion. This produces new interpretations of the results, extending Denuit et al. (2015) who only considered positive demand problem and provided sufficient conditions (whereas our conditions are both necessary and sufficient).

The results obtained both by Wright (1987) and with the MCSD criterion for risk-averse decision makers can be interpreted in terms of covariances between asset returns and payoffs of digital options written on the performances of the reference portfolio comprising the desired proportion of the assets when the expected returns of the risky assets are equal. When one assumes prudence beyond risk aversion, digital options are replaced by European put options written on the reference portfolio. As such puts can theoretically be replicated by means of digital options protecting against weak performances of the reference portfolio, we thus get a weaker condition.

As is now well known, prudence is one of the risk attitudes beyond risk aversion and this is related to the notion of higher-order risk apportionment, as defined by Eeckhoudt and Schlesinger (2006). The notion of risk apportionment is a preference for a particular class of lotteries combining sure reductions in wealth and zero-mean risks. These higher-order risk attitudes entail a preference for combining relatively good outcomes with bad ones and can be interpreted as a desire to disaggregate the harms of unavoidable risks and losses. Risk apportionments of orders 2 and 3 correspond to risk aversion and prudence, respectively. Increasing the order of risk apportionment further restricts the class of investors and thus gives weaker conditions on the joint probability distribution of the assets to ensure that the optimal portfolio comprises at least a given percentage of one of them.

The remainder of this paper is organized as follows. Section 2 describes the problem investigated in the present paper and gives the solution for risk-averse decision makers, summarizing in a unified way the results available in the literature. The derivation of these existing results from common grounds allows for an extension in Sect. 3 to risk-averse investors exhibiting prudence and in Sect. 4 to investors whose preferences exhibit higher-order risk attitudes. Specifically, an extension of the results obtained for risk-averse and prudent decision makers to investors exhibiting risk apportionment of orders 1 to 4, 5, ... is proposed in Sect. 4. The final Sect. 5 discusses the results obtained in this paper. Some technical results are gathered in the appendix to this paper, which contains the derivation of a new expansion formula for expected utility.

2 Optimal asset allocation by risk-averse investors

2.1 Two-asset portfolio problem

Consider the following standard 2-asset portfolio problem as described, e.g., in Hadar and Seo (1988). Let X_j, $j = 1, 2$, be the random return per monetary unit invested in risky asset j valued in some interval $[a, b]$ of the real line. Assume that the initial wealth is equal to unity and must be invested in one of these two risky assets by a non-satiated risk-averse decision maker. This agent is assumed to act in order to maximize the expected utility of terminal wealth which is the end-of-period value $\lambda X_1 + (1-\lambda)X_2$ of the portfolio, where λ represents the fraction of the initial wealth invested in asset 1.

Define $\mathcal{U}_{\text{ra}}$ as the class of all utility functions $u : [a, b] \to \mathbb{R}$ with first derivative $u' \geq 0$ and second derivative $u'' \leq 0$, expressing risk aversion. For an investor with utility function $u \in \mathcal{U}_{\text{ra}}$, the optimal λ maximizes the objective function

$$\mathcal{O}(\lambda) = E[u(\lambda X_1 + (1-\lambda)X_2)].$$

The first-order condition is

$$\mathcal{O}'(\lambda) = 0 \Leftrightarrow E[(X_1 - X_2)u'(\lambda X_1 + (1-\lambda)X_2)] = 0. \tag{2.1}$$

Denote as $\lambda^\star$ the solution to Eq. (2.1), assumed to be unique. Notice that the concavity of u ensures that the objective function is also concave. The concavity of the function $\lambda \mapsto \mathcal{O}(\lambda)$ plays a central role in the developments appearing in the remainder of this paper.

2.2 Minimum demand condition for risk-averse investors

The next Proposition 2.1 summarizes the results obtained so far in the literature in the case of risk-averse decision makers. Precisely, it provides different conditions ensuring that the optimal share $\lambda^\star$ invested in the first asset by every risk-averse decision maker is at least equal to some given percentage π. For the sake of completeness and to ease the extension to prudent investors in the next section, we provide a direct proof of these results based on a useful expansion formula given in Appendix. Henceforth, we denote as $I[\cdot]$ the indicator function, i.e., $I[A] = 1$ if event A is realized and $I[A] = 0$ otherwise.

Proposition 2.1 *Consider a fixed percentage* $\pi \in [0, 1]$. *Define the reference portfolio*

$$\overline{X}_\pi = \pi X_1 + (1-\pi)X_2$$

comprising asset 1 in proportion π. *The optimal share* $\lambda^\star$ *invested in the first asset is at least equal to* π *for every* $u \in \mathcal{U}_{ra}$ *if, and only if, one of the following equivalent conditions is fulfilled:*

$$E\big[X_1 I[\overline{X}_\pi \leq z]\big] \geq E\big[X_2 I[\overline{X}_\pi \leq z]\big] \quad \textit{for all } z \in [a, b] \tag{2.2}$$

$$\Leftrightarrow Cov\big[X_1 - X_2, I[\overline{X}_\pi \leq z]\big] \geq E[X_2 - X_1]P[\overline{X}_\pi \leq z] \quad \textit{for all } z \in [a, b]. \tag{2.3}$$

Proof Considering (2.1), the concavity of the objective function $\mathcal{O}$ ensures that $\lambda^\star \geq \pi$ when

$$\mathcal{O}'(\pi) = E[(X_1 - X_2)u'(\overline{X}_\pi)] \geq 0. \tag{2.4}$$

Let us apply formula (5.1) in appendix to $Z_1 = X_1$, $Z_2 = \overline{X}_\pi$ and $g(z_1, z_2) = z_1 u'(z_2)$. This gives

$$E[X_1 u'(\overline{X}_\pi)] = u'(b)E[X_1] - \int_a^b bu''(z_2)P[\overline{X}_\pi \leq z_2]dz_2 + \int_a^b u''(z_2)E\big[(b - X_1)I[\overline{X}_\pi \leq z_2]\big]dz_2. \tag{2.5}$$

Hence,

$$E[(X_1 - X_2)u'(\overline{X}_\pi)] = u'(b)E[X_1 - X_2] - \int_a^b u''(z_2)E\big[(X_1 - X_2)I[\overline{X}_\pi \leq z_2]\big]dz_2.$$

As a consequence, if (2.2) is valid, then condition (2.4) is fulfilled for every $u \in \mathcal{U}_{\text{ra}}$. Conversely, if (2.4) holds for all $u \in \mathcal{U}_{\text{ra}}$, then it holds in particular for the utility function $u(x) = \min\{x, z\}$ such that $u'(x) = I[x \leq z]$, which shows that inequality (2.2) must hold true. To get the equivalence (2.3), it suffices to notice that

$$\begin{aligned} E\big[(X_1 - X_2)I[\overline{X}_\pi \leq z]\big] &= E[X_1 - X_2]E\big[I[\overline{X}_\pi \leq z]\big] + \text{Cov}\big[X_1 - X_2, I[\overline{X}_\pi \leq z]\big] \\ &= E[X_1 - X_2]P[\overline{X}_\pi \leq z] + \text{Cov}\big[X_1 - X_2, I[\overline{X}_\pi \leq z]\big]. \end{aligned}$$

This ends the proof. □

Condition (2.2) involves the average return of assets 1 and 2 computed over scenarios where the portfolio underperforms, i.e., where $\overline{X}_\pi \leq z$. The proportion of wealth invested in asset 1 should be increased above the current fraction π comprised in $\overline{X}_\pi$ if this asset performs on average better over these adverse scenarios.

Notice that (2.2) ensures that $E[X_1] \geq E[X_2]$ holds, by letting z tend to b. The condition $E[X_1] \geq E[X_2]$ rules out the cases where X_2 dominates X_1 by first-order stochastic dominance. Second-order stochastic dominance is nevertheless possible provided $E[X_1] = E[X_2]$.

Condition (2.2) can be found in Theorem 3 by Hadar and Seo (1988). Instead of the expansion used in the proof provided here, Hadar and Seo (1988) reduce in their Theorem 1 the class $\mathcal{U}_{\text{ra}}$ to the subset of all representative risk averters whose utility functions consist of two linear pieces (i.e., of the form $x \mapsto \min\{x, z\}$ for some fixed z). The alternative proof provided here appears useful when extending the analysis to prudent investors in Sect. 3.

Considering condition (2.2) , it is easy to see that it can be rewritten as

$$\begin{aligned} &E\big[X_1 I[\overline{X}_\pi \leq z]\big] \\ &\quad \geq \pi E\big[X_1 I[\overline{X}_\pi \leq z]\big] + (1-\pi)E\big[X_2 I[\overline{X}_\pi \leq z]\big] \quad \text{for all } z \in [a,b] \\ &\quad \Leftrightarrow E\big[X_1 I[\overline{X}_\pi \leq z]\big] \geq E\big[\overline{X}_\pi I[\overline{X}_\pi \leq z]\big] \quad \text{for all } z \in [a,b]. \end{aligned} \tag{2.6}$$

All risk-averse investors thus agree to increase the fraction of wealth invested in asset 1 above its current level π if this asset performs on average better than the portfolio $\overline{X}_\pi$ in adverse situations (i.e., when $\overline{X}_\pi \leq z$). Increasing the weight of such an asset in the portfolio thus improves its performances in adverse situations and this is considered as optimal by all risk-averse investors. Condition (2.6) is in turn equivalent to

$$\Leftrightarrow Cov\big[X_1 - \overline{X}_\pi, I[\overline{X}_\pi \leq z]\big] \geq E[\overline{X}_\pi - X_1]P[\overline{X}_\pi \leq z] \quad \text{for all } z \in [a,b]. \tag{2.7}$$

Notice that (2.6)–(2.7) no longer refer explictly to asset 2 but rather compare the return X_1 to the current portfolio return $\overline{X}_\pi$.

2.3 The 50 % rule

Let us now consider the so-called 50% rule, i.e., whether more than 50% of the initial wealth is invested in asset 1. This corresponds to Proposition 2.1 with $\pi = 0.5$. In this case, we denote $\overline{X}_{0.5}$ simply as $\overline{X} = \frac{X_1+X_2}{2}$. Portfolio $\overline{X}$ is the equally weighted portfolio comprising an equal share of both assets. Condition (2.2) ensuring that $\lambda^\star \geq \frac{1}{2}$ is equivalent to condition (9) in Clark and Jokung (1999).

2.4 Digital options

If $E[X_1] = E[X_2]$, then only the covariance remains in (2.3) and one obtains

$$\lambda^\star \geq \pi \Leftrightarrow \text{Cov}\big[X_1, I[\overline{X}_\pi \leq z]\big] \geq [\text{Cov}\big[X_2, I[\overline{X}_\pi \leq z]\big] \quad \text{for all } z \in [a,b].$$

This condition can be interpreted as follows. The indicator $I[\overline{X}_\pi \leq z]$ is the payoff of a digital option paying 1 if the performance of the portfolio $\overline{X}_\pi$ does not reach the threshold z. This digital option protects the investor against weak performances of the portfolio $\overline{X}_\pi$. Now, the optimal proportion invested in asset 1 is larger than π if the covariance between X_1 and the payoff of this digital option is always larger than the corresponding covariance with X_2 whatever the performance threshold z. If X_1 and X_2 are identically distributed, or simply have the same variance, then the dominating asset in the portfolio is the one which is less correlated with the payoff of the digital option written on the performance of the reference portfolio $\overline{X}_\pi$.

As $E[X_1] = E[X_2] = E[\overline{X}_\pi]$ also holds, only the covariance remains in (2.7) which reduces to

$$\lambda^\star \geq \pi \Leftrightarrow \mathrm{Cov}\big[X_1, I[\overline{X}_\pi \leq z]\big] \geq \mathrm{Cov}\big[\overline{X}_\pi, I[\overline{X}_\pi \leq z]\big] \quad \text{for all } z \in [a, b].$$

The minimum demand condition is again based on covariances with the payoffs of digital options written on the current portfolio $\overline{X}_\pi$, protecting the investor against weak performances of $\overline{X}_\pi$ (i.e., against $\overline{X}_\pi$ falling below the threshold z). If the covariance between asset 1 and this digital option payoff is larger than the covariance of the portfolio itself with this payoff, then the percentage invested in X_1 should be increased above the current level π.

2.5 MCSD condition

Given a portfolio $\overline{X}_\pi$, Shalit and Yitzhaki (1994) have established that it is optimal for every risk-averse decision maker to increase the weight of asset 1 at the expense of asset 2 if, and only if,

$$E[X_1|\overline{X}_\pi \leq z] \geq E[X_2|\overline{X}_\pi \leq z] \quad \text{for all } z \in [a, b] \tag{2.8}$$

which is obviously equivalent to condition (2.2). This condition is known in the literature as marginal conditional stochastic dominance (MCSD). In words, MCSD favors assets performing on average better in adverse situations (i.e., when the portfolio underperforms $\Leftrightarrow \overline{X}_\pi \leq z$).

Considering the alternative statement (2.6), it is easy to see that the MCSD condition can be equivalently expressed as

$$E[X_1|\overline{X}_\pi \leq z] \geq E[\overline{X}_\pi|\overline{X}_\pi \leq z] \quad \text{for all } z \in [a, b] \tag{2.9}$$

The share invested in asset 1 should be increased when this asset performs on average better than the current portfolio in adverse situations, i.e., when $\overline{X}_\pi \leq z$ for some threshold z.

2.6 Positive demand

The particular case $\pi = 0$ has been considered by Wright (1987) who established that all risk-averse investors hold a positive amount of each asset in their expected utility maximizing portfolio when (2.2)–(2.3) hold with $\pi = 0$, so that the reference portfolio $\overline{X}_\pi$ reduces to $\overline{X}_0 = X_2$.

The next example considers the case of independent returns X_1 and X_2.

Example 2.2 If X_1 and X_2 are mutually independent and $E[X_1] \geq E[X_2]$, then there is always a positive demand for X_1 as

$$E[X_1|X_2 \leq z] = E[X_1] \geq E[X_2] \geq E[X_2|X_2 \leq z] \quad \text{for all } z \in [a, b]$$

so that condition (2.2) with $\pi = 0$ (hence $\overline{X}_\pi = X_2$) is fulfilled. Every risk-averse investor diversifies his position in such a case.

Stated in the alternative way (2.6)–(2.7), it is clear that positive demand and minimum demand conditions are intimately related. A minimum demand at level π for asset X_1 in presence of asset X_2 is equivalent to a positive demand for asset X_1 in presence of the portfolio $\overline{X}_\pi$ viewed as the second asset. We are allowed to let the portfolio play the role of the second asset as we made no restriction about the dependence structure between the two assets when deriving conditions (2.2)–(2.3).

2.7 Negative expectation dependence

In his conclusion, Wright (1987) suggested to define X_1 as more negatively expectation dependent than X_2 on $\overline{X}_\pi$ when the inequality

$$E[X_1|\overline{X}_\pi \leq t] - E[X_1] \geq E[X_2|\overline{X}_\pi \leq t] - E[X_2] \tag{2.10}$$

holds for all t. If $E[X_1] = E[X_2]$ and (2.2) holds, then (2.10) is necessary fulfilled. Condition (2.10) allows one to derive inequalities involving covariances, as shown next. Consider a decreasing transformation h. From the proof of Theorem 3.1 in Wright (1987), we can write

$$\begin{aligned}
&\mathrm{Cov}[X_1, h(\overline{X}_\pi)] \\
&\quad = \int_a^b \Big(E[X_1] - E[X_1|\overline{X}_\pi \leq x_2]\Big) P[\overline{X}_\pi \leq x_2] h'(x_2) \mathrm{d}x_2 \\
&\quad \geq \int_a^b \Big(E[X_2] - E[X_2|\overline{X}_\pi \leq x_2]\Big) P[\overline{X}_\pi \leq x_2] h'(x_2) \mathrm{d}x_2 \text{ under (2.10)} \\
&\quad = \mathrm{Cov}[X_2, h(\overline{X}_\pi)].
\end{aligned} \tag{2.11}$$

Hence, (2.10) ensures that the inequality $Cov[X_1, h(\overline{X}_\pi)] \geq Cov[X_2, h(\overline{X}_\pi)]$ is valid for every decreasing transformation h. The reverse inequality holds for an increasing h. Condition (2.10) corresponds to Definition 2.9 in Dionne et al. (2012), restricted to the pairs of random variables $(X_1, \overline{X}_\pi)$ and $(X_2, \overline{X}_\pi)$.

3 Optimal asset allocation by risk-averse and prudent investors

So far, we have reviewed the existing literature on optimal portfolio composition by a risk-averse decision maker. We now investigate how the restriction to risk-averse and prudent decision makers affects the portfolio composition. We start our study with the following illustration.

3.1 Introductory example

To motivate the analysis, we start with a simple numerical example. Consider for instance assets with respective returns given by

$$X_1 = \begin{cases} 1.1 \text{ with probability } \frac{3}{4} \\ 1.3 \text{ with probability } \frac{1}{4} \end{cases} \text{ and } X_2 = \begin{cases} 1 \text{ with probability } \frac{1}{4} \\ 1.2 \text{ with probability } \frac{3}{4} \end{cases}.$$

Both returns have the same mean and variance. Assume besides that the returns X_1 and X_2 are correlated, with joint distribution

$$\begin{aligned} P[X_1 = 1.1, X_2 = 1] &= \frac{3}{16} + \rho \\ P[X_1 = 1.1, X_2 = 1.2] &= \frac{9}{16} - \rho \\ P[X_1 = 1.3, X_2 = 1] &= \frac{1}{16} - \rho \\ P[X_1 = 1.3, X_2 = 1.2] &= \frac{3}{16} + \rho, \end{aligned} \tag{3.1}$$

for some correlation parameter $\rho \in \left[-\frac{3}{16}, \frac{1}{16}\right]$. The strength of the dependence between X_1 and X_2 is controlled by the parameter ρ. A positive (resp. negative) ρ entails positive (resp. negative) dependence between X_1 and X_2, i.e., a large return for asset 1 tends to be accompanied by a large (resp. small) return for asset 2. The special case $\rho = 0$ corresponds to independence.

Let us consider the case $\rho = \frac{1}{32}$ so that both returns are positively related. We take $\pi = 0.4$, meaning that the reference portfolio $\overline{X}_{0.4}$ comprises 40 % of unit wealth invested in asset 1 and we wonder whether this proportion should be increased. Figure 1 displays the curves $z \mapsto E\big[X_1 I[\overline{X}_{0.4} \leq z]\big]$ and $z \mapsto E\big[X_2 I[\overline{X}_{0.4} \leq z]\big]$. These are step functions exhibiting jumps at the four possible values of $\overline{X}_{0.4}$, that is 1.04, 1.12, 1.16, and 1.24. The two curves are at zero before 1.04 and at $E[X_1] = E[X_2] = 1.15$ after 1.24. We clearly see on Fig. 1 that the two curves intersect so that condition (2.2) is violated and risk-averse investors do not unanimously agree to invest more that 40 % of their initial wealth in asset 1. We could nevertheless wonder whether a subset of these decision makers would agree to do so. This is why we derive in the next section the minimum demand conditions for a relevant subset of investors. Then, we come back to this example to show that all these decision makers agree to increase the share of X_1 in their portfolio.

3.2 Minimum demand condition for risk-averse, prudent investors

The introductory example described above motivates the restriction of the set of investors. As prudence is usually justified by reference to the decision of building up precautionary savings in order to better face future income risk, this behavioral

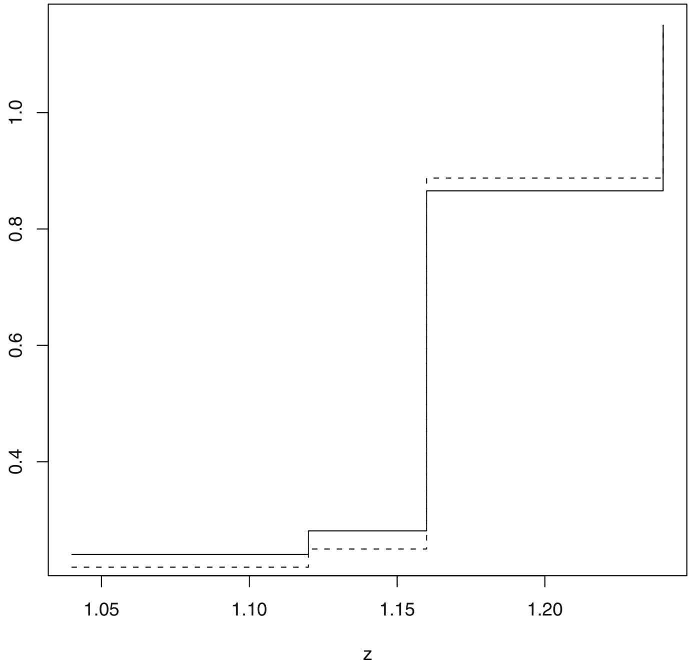

Fig. 1 Graphs of $z \mapsto E\big[X_1 I[\overline{X}_\pi \le z]\big]$ (*continuous line*) and $z \mapsto E\big[X_2 I[\overline{X}_\pi \le z]\big]$ (*broken line*) for $\pi = 0.4$ when the joint distribution of asset returns X_1 and X_2 is given by (3.1) with $\rho = \frac{1}{32}$

trait is quite natural in investment problems. This is why we now consider prudent investors and define the subset $\mathcal{U}_{\text{ra-p}}$ of $\mathcal{U}_{\text{ra}}$ consisting in all $u \in \mathcal{U}_{\text{ra}}$ with third derivative u''' such that $u''' \ge 0$.

Thanks to the expansion formula proposed in Appendix, we have been able to review in a unified way the results obtained from the end of the 1980s for the optimal allocation of a risky portfolio when decision makers are risk-averse. In the present section, we examine the implication of the additional and now well-accepted assumption that decision makers are prudent for optimal portfolio selection. The next result extends Proposition 2.1 to the case of prudent risk-averse decision makers. Precisely, it provides different conditions such that the optimal share $\lambda^\star$ invested in the first asset by every prudent risk-averse decision maker is at least equal to some given percentage. We denote as $x_+ = \max\{x, 0\}$ the positive part of x, i.e., $x_+ = x$ if $x > 0$ and $x_+ = 0$ otherwise.

Proposition 3.1 *Consider a fixed percentage* $\pi \in [0, 1]$. *The optimal share* $\lambda^\star$ *invested in the first asset is at least equal to* π *for every* $u \in \mathcal{U}_{ra\text{-}p}$ *if, and only if,* $E[X_1] \ge E[X_2]$ *and one of the following equivalent conditions is fulfilled:*

$$E\big[X_1(z - \overline{X}_\pi)_+\big] \ge E\big[X_2(z - \overline{X}_\pi)_+\big] \quad \textit{for all } z \in [a, b] \tag{3.2}$$

$$\Leftrightarrow Cov\big[X_1 - X_2, (z - \overline{X}_\pi)_+\big] \ge E[X_2 - X_1]E[(z - \overline{X}_\pi)_+] \quad \textit{for all } z \in [a, b]. \tag{3.3}$$

Proof Condition (2.4) still applies but with u' being now decreasing and convex. Let us use integration by parts in the two integrals appearing in formula (2.5) to get

$$\begin{aligned} &E[X_1 u'(\overline{X}_\pi)] \\ &= u'(b)E[X_1] - bu''(b)E[b - \overline{X}_\pi] + \int_a^b bu'''(z_2)E[(z_2 - \overline{X}_\pi)_+]\mathrm{d}z_2 \\ &\quad + u''(b)E[(b - X_1)(b - \overline{X}_\pi)] - \int_a^b u'''(z_2)E[(b - X_1)(z_2 - \overline{X}_\pi)_+]\mathrm{d}z_2. \end{aligned}$$

This gives

$$\begin{aligned} E[(X_1 - X_2)u'(\overline{X}_\pi)] &= u'(b)E[X_1 - X_2] + u''(b)E\big[(X_2 - X_1)(b - \overline{X}_\pi)\big] \\ &\quad + \int_a^b u'''(z_2)E[(X_1 - X_2)(z_2 - \overline{X}_\pi)_+]dz_2. \end{aligned} \tag{3.4}$$

Hence, the proportion invested in asset 1 is at least π for every $u \in \mathcal{U}_{\text{ra-p}}$ if $E[X_1] \geq E[X_2]$ and (3.2) is fulfilled. To get the converse implication, notice that condition (2.4) with $u(x) = x$ ensures that $E[X_1] \geq E[X_2]$. Inserting the utility function $u(x) = -(z-x)_+^2$ in (2.4) shows that condition (3.2) must also hold. Finally, condition (3.3) easily follows as

$$\begin{aligned} E[(X_1 - X_2)(z - \overline{X}_\pi)_+] &= E[X_1 - X_2]E[(z - \overline{X}_\pi)_+] \\ &\quad + Cov[X_1 - X_2, (z - \overline{X}_\pi)_+] \end{aligned}$$

and this ends the proof. □

Notice that compared to Proposition 2.1, we now need an additional condition imposed on the first moments of X_1 and X_2, i.e., we have to impose that $E[X_1] \geq E[X_2]$ in addition to (3.2)–(3.3).

Let us come back to the introductory example of this section. Figure 2 displays the curves $z \mapsto E\big[X_1(z - \overline{X}_{0.4})_+\big]$ and $z \mapsto E\big[X_2(z - \overline{X}_{0.4})_+\big]$. We see that condition (3.2) is fulfilled in this case. Whereas risk-averse investors did not all agree to invest more than 40% of their initial wealth in asset 1, all the risk-averse and prudent ones among them agree about this decision.

As it was the case for risk-averse investors, we can rewrite condition (3.2) as

$$\begin{aligned} &E\big[X_1(z - \overline{X}_\pi)_+\big] \\ &\quad \geq \pi E\big[X_1(z - \overline{X}_\pi)_+\big] + (1-\pi)E\big[X_2(z - \overline{X}_\pi)_+\big] \quad \text{for all } z \in [a,b] \\ &\quad \Leftrightarrow E\big[X_1(z - \overline{X}_\pi)_+\big] \geq E\big[\overline{X}_\pi(z - \overline{X}_\pi)_+\big] \quad \text{for all } z \in [a,b]. \end{aligned} \tag{3.5}$$

Compared to (3.2), return X_2 does not explicitly appear in condition (3.5) which involves X_1 and the reference portfolio $\overline{X}_\pi$. Then, (3.3) can be rewritten as

$$Cov\big[X_1 - \overline{X}_\pi, (z - \overline{X}_\pi)_+\big] \geq E[\overline{X}_\pi - X_1]E[(z - \overline{X}_\pi)_+] \quad \text{for all } z \in [a,b]. \tag{3.6}$$

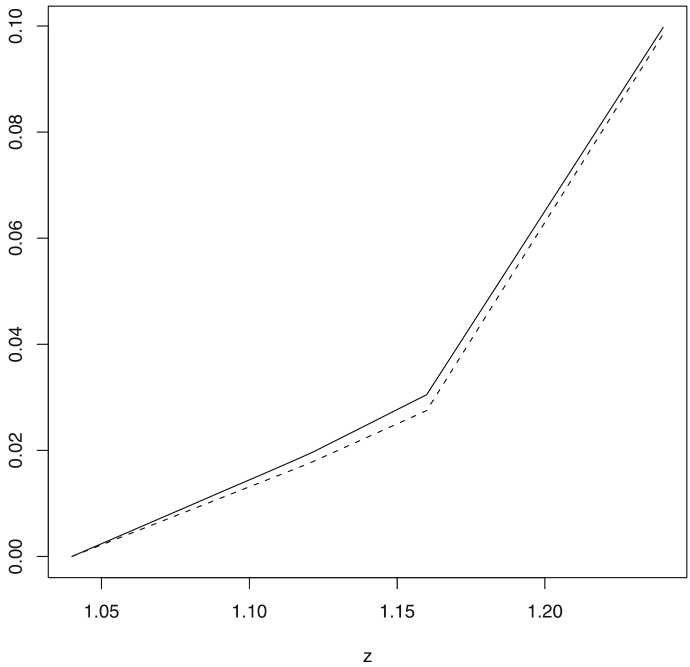

Fig. 2 Graphs of $z \mapsto E\big[X_1(z-\overline{X}_\pi)_+\big]$ (*continuous line*) and $z \mapsto E\big[X_2(z-\overline{X}_\pi)_+\big]$ (*broken line*) for $\pi = 0.4$ when the joint distribution of asset returns X_1 and X_2 is given by (3.1) with $\rho = \frac{1}{32}$

3.3 The 50 % rule

Setting $\pi = 0.5$ in (3.2)-(3.3) allows us to obtain the conditions under which all risk-averse, prudent decision makers invest at least 50 % of their initial wealth in asset 1.

Now, consider two asset returns with the same means and variances, i.e., such that $E[X_1] = E[X_2]$ and $E[X_1^2] = E[X_2^2]$. Then, (3.4) with $\pi = \frac{1}{2}$ gives

$$E[(X_1 - X_2)u'(\overline{X}_\pi)] = \int_a^b u'''(z_2)E\left[(X_1 - X_2)\left(z_2 - \frac{X_1 + X_2}{2}\right)_+\right]\mathrm{d}z_2.$$

If (3.2) holds true with $\pi = 0.5$, then we see that

(i) all prudent decision makers ($u''' \geq 0$) invest at least π in asset 1 as $\mathcal{O}'(\pi) \geq 0$.
(ii) all imprudent decision makers ($u''' \leq 0$) invest at most π in asset 1 as $\mathcal{O}'(\pi) \leq 0$.

Notice that decision makers with quadratic utilities ($u''' = 0$) are indifferent between the various portfolio compositions as $\mathcal{O}'(\pi) = 0$ for every proportion π in this case.

3.4 European put options

If $E[X_1] = E[X_2]$, then Proposition 3.1 shows that $\lambda^\star \geq \pi$ for any $u \in \mathcal{U}_{\text{ra-p}}$ when

$$\text{Cov}\big[X_1, (z - \overline{X}_\pi)_+\big] \geq \text{Cov}\big[X_2, (z - \overline{X}_\pi)_+\big] \quad \text{for all } z \in [a, b].$$

In particular, we get for $z = b$

$$\text{Cov}\big[X_1, \overline{X}_\pi\big] \leq \text{Cov}\big[X_2, \overline{X}_\pi\big].$$

It is thus necessary (but not sufficient) for investing at least π in asset 1 that the covariance of X_1 with the reference portfolio $\overline{X}_\pi$ is smaller than that of X_2.

Notice that $(z - \overline{X}_\pi)_+$ is the payoff of a put option written on the performance of the reference portfolio $\overline{X}_\pi$, with exercise price z. If $E[X_1] = E[X_2]$, then the optimal proportion invested in asset 1 is larger than π if the covariance between X_1 and the payoff of this put option is always larger than the corresponding covariance with X_2. If X_1 and X_2 are identically distributed, or simply have the same variance, then the dominating asset in the portfolio is the one which is more correlated with the put option payoff on the performance of the portfolio $\overline{X}_\pi$.

As $E[X_1] = E[X_2] = E[\overline{X}_\pi]$ holds, condition (3.6) ensures that $\lambda^\star \geq \pi$ for any $u \in \mathcal{U}_{\text{ra-p}}$ is guaranteed when

$$Cov\big[X_1, (z - \overline{X}_\pi)_+\big] \geq Cov\big[\overline{X}_\pi, (z - \overline{X}_\pi)_+\big] \quad \text{for all } z \in [a, b].$$

This condition is stated in terms of the covariance between X_1 and the put option written on the portfolio $\overline{X}_\pi$ compared to the covariance of $\overline{X}_\pi$ with the same put option.

3.5 Positive demand

The particular case $\pi = 0$ has been considered by Denuit et al. (2015) who extended the analysis conducted in Wright (1987) to higher-order risk attitudes. Considering the results obtained by Denuit et al. (2015) in their Sect. 6.1 (with $n = 2$, letting ϵ tend to 0 to recover the class of risk-averse and prudent decision makers), it is worth to mention that Proposition 3.1 with $\pi = 0$ provides a characterization for the positive demand case, not only a sufficient condition.

3.6 Negative second-order expectation dependence

We can relate (2.10) to conditions (3.2)–(3.3) in Proposition 3.1, noting that

$$\begin{aligned}\int_a^z \big(E[X_1] - E[X_1|\overline{X}_\pi \leq x_2]\big) P[\overline{X}_\pi \leq x_2] dx_2 &= E[X_1]E[(z - \overline{X}_\pi)_+] \\ &\quad - E[X_1(z - \overline{X}_\pi)_+] \\ &= -Cov[X_1, (z - \overline{X}_\pi)_+].\end{aligned}$$

This also allows us to derive inequalities involving covariances, as integration by parts in formula (2.11) gives

$$
\begin{aligned}
Cov[X_1, h(\overline{X}_\pi)] &= \int_a^b h'(z)\big(E[X_1] - E[X_1|\overline{X}_\pi \le z]\big)P[\overline{X}_\pi \le z]dz \\
&= h'(b)Cov[X_1, \overline{X}_\pi] + \int_a^b h''(z)Cov[X_1, (z - \overline{X}_\pi)_+]dz.
\end{aligned}
$$

Therefore, provided $E[X_1] = E[X_2]$, conditions (3.2)-(3.3) ensure that $Cov[X_1, h(\overline{X}_\pi)] \ge Cov[X_2, h(\overline{X}_\pi)]$ for any decreasing and convex transformation h.

4 Optimal asset allocation by investors with higher-order risk apportionment

The results obtained for risk-averse and prudent investors in Sects. 2–3 can be extended to investors whose preferences exhibit risk apportionment up to any order in the sense of Eeckhoudt and Schlesinger (2006). Henceforth, we write $u^{(n)}$ for the nth derivative of u, $n = 1, 2, 3, 4, \ldots$; the notations u', u'', and u''' and $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$, respectively, are used interchangeably. Recall that the preferences expressed by a differentiable utility function u satisfy risk apportionment of order n if u fulfills the condition $(-1)^{n+1}u^{(n)} \ge 0$. Prudence, temperance, and edginess, respectively, correspond to risk apportionment of order 3, 4, and 5.

Assume now that the risk-averse decision maker exhibits prudence and temperance, i.e., $(-1)^{n+1}u^{(n)} \ge 0$ holds for $n = 1, 2, 3, 4$. Integrating (3.4) by parts gives

$$
\begin{aligned}
E[(X_1 - X_2)u'(\overline{X}_\pi)] &= u'(b)E[X_1 - X_2] + u''(b)E\big[(X_2 - X_1)(b - \overline{X}_\pi)\big] \\
&\quad + u'''(b)E\left[(X_1 - X_2)\frac{(b - \overline{X}_\pi)^2}{2}\right] \\
&\quad + \int_a^b u^{(4)}(z_2)E\left[(X_2 - X_1)\frac{(z_2 - \overline{X}_\pi)^2}{2}\right]\mathrm{d}z_2. \qquad (4.1)
\end{aligned}
$$

Provided $E[X_1] \ge E[X_2]$, the analog of condition (3.2) for the subset of temperant investors becomes

$$
E\big[X_1(b - \overline{X}_\pi)\big] \ge E\big[X_2(b - \overline{X}_\pi)\big]
$$

and

$$
E\big[X_1(z - \overline{X}_\pi)_+^2\big] \ge E\big[X_2(z - \overline{X}_\pi)_+^2\big] \quad \text{for all } z \in [a, b].
$$

Proceeding in the same way, we see that every investor whose preferences exhibit risk apportionment of orders 1 to n includes a proportion at least equal to π of asset 1 in his optimal portfolio when $E[X_1] \ge E[X_2]$ if

$$E\big[X_1(b-\overline{X}_\pi)^k\big] \geq E\big[X_2(b-\overline{X}_\pi)^k\big] \quad \text{for } k=1,\ldots,n-3$$

and

$$E\big[X_1(z-\overline{X}_\pi)_+^{n-2}\big] \geq E\big[X_2(z-\overline{X}_\pi)_+^{n-2}\big] \quad \text{for all } z \in [a,b].$$

The minimum demand conditions are thus structured similarly, whatever the order n of risk apportionment, except that the shortfall $(z-\overline{X}_\pi)_+$ in the performances of the reference portfolio $\overline{X}_\pi$ used for prudent investors is replaced by its increasing powers. Compared to Sect. 6.1 in Denuit et al. (2015), we have thus identified here the positive demand condition, not only a sufficient one, and we have extended the analysis to the minimum demand case.

5 Discussion

The notion of prudence is now well accepted in the economics literature, almost at parity with that of risk aversion. Besides its initial implications for the analysis of savings decision, it has been useful also to analyze other problems such as self-protection or optimal audits. Surprisingly however its implications for minimum demand in portfolio composition have not been analyzed so far and we have tried here to compensate for this deficiency.

The existing literature looks at the role of only risk aversion in the optimal composition of a portfolio of two possibly correlated risky assets. Thanks to an expansion formula presented in appendix, we have been able to summarize and extend the existing literature in a unified way. Then in Sect. 3, we have made the additional assumption that the decision maker is risk-averse and prudent. This additional requirement of prudence has lead to new results about diversification or about the 50% rule. Besides, when the two risky assets have the same mean, these conditions can be interpreted in terms of covariances with the payoffs of European put options written on the reference portfolio, replacing the digital options protecting against weak performances of this portfolio for risk-averse investors. An extension to higher-order risk apportionments has been proposed in Sect. 4.

Acknowledgments The financial support of PARC "Stochastic Modelling of Dependence" 2012–2017 awarded by the Communauté française de Belgique is gratefully acknowledged by Michel Denuit.

Appendix: A useful expansion formula

Consider two random variables Z_1 and Z_2 valued in $[a,b]$ and a real valued function g with domain $[a,b]\times[a,b]$. Let $g^{(i,j)}$ denote the (i,j)th partial derivative of g, i.e., $g^{(i,j)}(z_1,z_2)=\frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j}g(z_1,z_2)$. Integration by parts shows that

$$E[g(Z_1,Z_2)] = E[g(Z_1,b)] - \int_a^b g^{(0,1)}(b,z_2)P[Z_2 \leq z_2]\mathrm{d}z_2 \\ + \int_a^b \int_a^b \Pr[Z_1 \leq z_1, Z_2 \leq z_2]g^{(1,1)}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2.$$

Integrating by parts the last double integral gives

$$\begin{aligned}\int_a^b \int_a^b \Pr[Z_1 \le z_1, Z_2 \le z_2] g^{(1,1)}(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 \\ = \int_a^b g^{(1,1)}(b, z_2) \left(\int_a^b \Pr[Z_1 \le \xi_1, Z_2 \le z_2] \mathrm{d}\xi_1 \right) \mathrm{d}z_2 \\ - \int_a^b \int_a^b \left(\int_a^{z_1} \Pr[X_1 \le \xi_1, X_2 \le z_2] \mathrm{d}\xi_1 \right) g^{(2,1)}(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2.\end{aligned}$$

Now, as

$$\begin{aligned}\int_a^{z_1} \Pr[Z_1 \le \xi_1, Z_2 \le z_2] \mathrm{d}\xi_1 &= \int_a^{x_1} E\Big[I[Z_1 \le \xi_1] I[Z_2 \le z_2] \Big] \mathrm{d}\xi_1 \\ &= E\left[\int_a^{z_1} I[Z_1 \le \xi_1] \mathrm{d}\xi_1 I[Z_2 \le z_2] \right] \\ &= E\Big[(z_1 - Z_1)_+ I[Z_2 \le z_2] \Big]\end{aligned}$$

the expectation $E[g(Z_1, Z_2)]$ can be expanded as follows:

$$\begin{aligned}E[g(Z_1, Z_2)] = E[g(Z_1, b)] &- \int_a^b g^{(0,1)}(b, z_2) P[Z_2 \le z_2] \mathrm{d}z_2 \\ &+ \int_a^b g^{(1,1)}(b, z_2) E\Big[(b - Z_1) I[Z_2 \le z_2] \Big] \mathrm{d}z_2 \\ &- \int_a^b \int_a^b g^{(2,1)}(z_1, z_2) E\Big[(z_1 - Z_1)_+ I[Z_2 \le z_2] \Big] \mathrm{d}z_1 \mathrm{d}z_2.\end{aligned} \tag{5.1}$$

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