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Values for rooted-tree and sink-tree digraph games and sharing a river

Anna B. Khmelnitskaya

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Abstract We introduce values for rooted-tree and sink-tree digraph games axiomatically and provide their explicit formula representation. These values may be considered as natural extensions of the lower equivalent and upper equivalent solutions for line-graph games studied in van den Brink et al. (Econ Theory 33:349–349, 2007). We study the distribution of Harsanyi dividends. We show that the problem of sharing a river with a delta or with multiple sources among different agents located at different levels along the riverbed can be embedded into the framework of a rooted-tree or sink-tree digraph game correspondingly.

Keywords TU game · Cooperation structure · Myerson value · Component efficiency · Deletion link property · Harsanyi dividends · Sharing a river

1 Introduction

In standard cooperative game theory, it is assumed that any coalition of players may form. However, in many practical situations, the collection of feasible coalitions is restricted by some social, economical, hierarchical, communicational, or technical structure. The study of TU games with limited cooperation introduced by means of communication graphs was initiated by Myerson (1977). In this article we restrict our consideration to classes of rooted-tree and sink-tree digraph games in which all players are partially ordered and a possible communication via bilateral agreements between participants is presented by a directed rooted tree or sink tree, respectively. A rooted-tree cooperation structure allows modeling of various splitting processes and

A. B. Khmelnitskaya (✉)
SPb Institute for Economics and Mathematics Russian Academy of Sciences, 1 Tchaikovsky St., 191187 St.Petersburg, Russia
e-mail: a.khmelnitskaya@math.utwente.nl

different hierarchical structures that are a widespread organizational form in a broad range of economic, political and military activities, and evolutionary biology. While a sink-tree cooperation structure provides modeling of different merging processes. Following Myerson (1977), we assume that for a given game with cooperation structure, cooperation is possible only among connected players and focus on component efficient values.

We introduce values for rooted-tree and sink-tree digraph games axiomatically and provide their explicit formula representation. We show that the so-called tree value for a rooted-tree digraph game coincides with a particular marginal vector, first considered in Demange (2004). It turns out that the tree value for rooted-tree digraph games and the sink value for sink-tree digraph games may be considered as natural extensions of the lower equivalent and upper equivalent solutions for line-graph games studied in van den Brink et al. (2007), respectively. We study the distribution of Harsanyi dividends. Furthermore, we show that the problem of sharing a river with a delta or with multiple sources among different agents located at different levels along the riverbed can be embedded into the framework of a rooted-tree or sink-tree digraph game correspondingly.

The structure of the article is as follows. Basic definitions and notation are introduced in Sect. 2. Section 3 provides an axiomatic axiomatization of the tree value for a rooted-tree digraph game via component efficiency and subordinate equivalence. In Sect. 4 we discuss application to the water distribution problem of a river with a delta.

2 Preliminaries

First recall some definitions and notation. A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of the coalition S. The set of all games with a fixed player set N we denote $\mathcal{G}_N$. For simplicity of notation, and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when refering to a game. A *subgame* of a game v is a game $v|_T$ with a player set $T \subseteq N$, $T \neq \emptyset$, and $v|_T(S) = v(S)$ for all $S \subseteq T$. A game v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$. A *value* is an operator $\xi : \mathcal{G}_N \to \mathbb{R}^n$ that assigns to any game $v \in \mathcal{G}_N$ a vector $\xi(v) \in \mathbb{R}^n$; the real number $\xi_i(v)$ represents the *payoff* to the player i in the game v. In what follows we use standard notation $x(S) = \sum_{i\in S} x_i$ and $x_S = \{x_i\}_{i\in S}$, for all $x \in \mathbb{R}^n$, $S \subseteq N$.

It is well known (Shapley 1953) that *unanimity games* $\{u_T\}_{\substack{T\subseteq N\\ T\neq\emptyset}}$, defined as $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise, create a basis for the game space $\mathcal{G}_N$, i.e., every game $v \in \mathcal{G}_N$ can be uniquely presented in the linear form $v = \sum\limits_{\substack{T\subseteq N\\ T\neq\emptyset}} \lambda^v_T\, u_T$, where

$\lambda^v_T = \sum\limits_{S\subseteq T} (-1)^{t-s}\, v(S)$, for all $T \subseteq N$, $T \neq \emptyset$. Moreover,

$$v(S) = \sum_{T \subseteq S} \lambda_T^v, \quad \text{for all } S \subseteq N. \tag{1}$$

Following Harsanyi (1959), the coefficient λ_T^v is referred to as the *dividend* of the coalition T in the game v.

The *core* (Gillies 1953) of a game $v \in \mathcal{G}_N$ is defined as

$$C(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S), \quad \text{for all } S \subseteq N\}.$$

A value ξ is *stable* if for any game $v \in \mathcal{G}_N$ with a nonempty core $C(v)$, $\xi(v) \in C(v)$.

A *cooperation structure* on a player set N is specified by a graph L, undirected or directed. An *undirected graph* is given by a collection of unordered pairs of nodes/players $L \subseteq L_N^c = \{\{i, j\} \mid i, j \in N,\ i \neq j\}$, where L_N^c is the complete undirected graph without loops on N and an unordered pair $\{i, j\}$ is a *link* between $i, j \in N$. A *directed graph*, or *digraph*, is a collection of directed links $L \subseteq \bar{L}_N^c = \{(i, j) \mid i, j \in N, i \neq j\}$ while in a directed link $(i, j) \in L$, i is an *origin* and j is a *terminus*, or else i is a *parent* of j and j is a *child* of i. All players having the same parent in L are called *brothers*. In a digraph L, we say that $i \neq j$ is a *predecessor* of j and j is a *successor* of i if there is a sequence of directed links $(i_h, i_{h+1}) \in L$, $h = 1, \ldots, k$, such that $i_1 = i$ and $i_{k+1} = j$. For any node $i \in N$, we denote by $P_L(i)$ the set of all predecessors of i in L, by $O_L(i)$ the set of all parents of i in L, by $T_L(i)$ the set of all children of i in L, by $S_L(i)$ the set of all successors of i in L, and by $B_L(i)$ the set of all brothers of i in L; moreover, $\bar{P}_L(i) = P_L(i) \cup i$, $\bar{S}_L(i) = S_L(i) \cup i$, and $\bar{B}_L(i) = B_L(i) \cup i$. An undirected graph L is *cycle-free*, if it contains no cycles. A sequence of nodes $\{i_1, \ldots, i_{k+1}\} \subseteq N$ presents a *cycle* in an undirected graph L if (i) $k \geq 2$, (ii) $i_h \neq i_l$, for all $h, l = 1, \ldots, k, h \neq l$, (iii) $i_{k+1} = i_1$, and (iv) $\{i_h, i_{h+1}\} \in L$, for all $h = 1, \ldots, k$. An undirected cycle-free connected graph is called a *tree*. A directed graph is cycle-free, if the corresponding undirected graph is cycle-free. A directed graph L is a *rooted tree*, if there is one node in N, called a *root*, having no predecessors in L, and there is a unique sequence of directed links in L from this node to any other node in N. In a rooted-tree graph the root plays a roll of the source of the stream presented via this graph. A directed graph L is a *sink tree*, if the directed graph, composed by the same set of links as L but with the opposite orientation, appears to be a rooted tree; in this case, the root of a tree changes its meaning to the absorbing sink. A *line-graph* is a directed graph that contains links only between subsequent nodes.

A pair $\langle v, L\rangle$ of a game $v \in \mathcal{G}_N$ and a communication graph L on N composes a *game with cooperation structure* or, in other terms, a *game with graph structure*, a *graph game* or a *digraph game*, when we want to emphasize that a graph L is directed. The set of all games endowed with a cooperation structure on a fixed player set N we denote $\mathcal{G}_N^{\mathcal{L}}$. Under a *value for a game with graph structure*, or a *G-value* we understand an operator $\xi : \mathcal{G}_N^{\mathcal{L}} \to \mathbb{R}^n$ that assigns to a graph game $\langle v, L\rangle$, $v \in \mathcal{G}_N$, a vector of payoffs $\xi(v, L) \in \mathbb{R}^n$.

For any graph L on N and any coalition $S \subseteq N$, the subgraph of L on S is the graph $L|_S = \{\{i, j\} \in L \mid i, j \in S\}$. A sequence of different nodes $\{i_1, \ldots, i_k\} \subseteq N$ is a *path* in a graph L if $\{i_h, i_{h+1}\} \in L$, for all $h = 1, \ldots, k-1$. For a directed graph L, we may consider paths of two types—a *directed path* that contains only directed links $(i_h, i_{h+1}) \in L$ and an *undirected path* that takes no care of orientation in L and consists of undirected links $\{i_h, i_{h+1}\}$ such that either $(i_h, i_{h+1}) \in L$, or $(i_{h+1}, i_h) \in L$. Two nodes $i, j \in N$ are *connected* in a graph L, directed or undirected, if there exists an undirected path $\{i_1, \ldots, i_k\}$ with $i_1 = i$ and $i_k = j$. Note that in the directed case, a link is directed to indicate which player initiated the communication, but at the same time it also represents a fully developed communication link. Due to that reason, we consider connectedness in a directed graph with respect to undirected paths. A graph is *connected* if any two nodes in it are connected. Given a graph L, a coalition $S \subseteq N$ is said to be *connected* if the subgraph $L|_S$ is connected. For a given graph L and a coalition $S \subseteq N$, denote by $C^L(S)$ the set of all connected subcoalitions of S. Any coalition $S \subseteq N$ splits by any graph L into maximally connected coalitions called *components*. By S/L we denote the set of components of S and let $(S/L)_i$ be the component of S containing player $i \in S$. Notice that S/L is a partition of S. For a graph game $\langle v, L\rangle \in \mathcal{G}_N^{\mathcal{L}}$, a payoff vector $x \in \mathbb{R}^n$ is *component efficient* if it holds that $x(C) = v(C)$, for every component $C \in N/L$.

Following Myerson (1977), we assume that for a given game with cooperation structure $\langle v, L\rangle$, cooperation is possible only among connected players and consider a *restricted game* $v^L \in \mathcal{G}_N$ defined as

$$v^L(S) = \sum_{C \in S/L} v(C), \quad \text{for all } S \subseteq N.$$

The *core* $C(v, L)$ of a graph game $\langle v, L\rangle$ is defined as a set of component efficient payoff vectors that are not dominated by any connected coalition, i.e.,

$$C(v, L) = \{x \in \mathbb{R}^n \mid x(C) = v(C),\ \forall C \in N/L, \text{ and } x(T) \geq v(T),\ \forall T \in C^L(N)\}.$$

It is easy to see that the core of a graph game $\langle v, L\rangle$ coincides with the core of the restricted game v^L, i.e., $C(v, L) = C(v^L)$.

In the article we concentrate on values for two special subclasses of cycle-free digraph games, the so-called rooted-tree and sink-tree digraph games. A *rooted-tree/sink-tree digraph game* $\langle v, L\rangle$ is specified by a digraph L, such that all subgraphs $L|_C$ on components $C \in N/L$, are rooted trees or sink trees correspondingly.

In the sequel for the cardinality of a given set A we use a standard notation $|A|$ along with lower case letters like $n = |N|$, $s = |S|$, and so on.

3 Component efficient values

3.1 Axiomatic characterizations

Our approach to the value is close to that of Myerson (1977) being based on ideas of component efficiency and a certain deletion link property.

A G-value ξ is *component efficient* (CE) if, for any graph game $\langle v, L\rangle$, for all $C \in N/L$,

$$\sum_{i\in C} \xi_i(v, L) = v(C).$$

A G-value ξ is *successor equivalent* (SE) if, for any digraph game $\langle v, L\rangle$, for every link $(i, j) \in L$, for all $k \in \bar{S}_L(j)$,

$$\xi_k(v, L\backslash(i, j)) = \xi_k(v, L).$$

It turns out that two axioms of component efficiency and successor equivalence uniquely define a G-value for a rooted-tree digraph game.

Theorem 1 *On the class of rooted-tree digraph games, there is a unique G-value that satisfies CE and SE, and for any rooted-tree digraph game $\langle v, L\rangle$, it is given by*

$$t_i(v, L) = v\left(\bar{S}_L(i)\right) - \sum_{j\in T_L(i)} v\left(\bar{S}_L(j)\right), \quad \textit{for all } i \in N. \tag{2}$$

From now we refer to the G-value t as to the *tree value*.

The tree value of a rooted-tree digraph game assigns to every player the payoff equal to the worth of the coalition composed of this player and all his successors minus the sum of the worths of all coalitions composed of any child of the considered player and all successors of this child. The tree value was first introduced in Demange (2004) where it was also shown that under the mild condition of superadditivity, it belongs to the core of the restricted game. Besides, afterwards the tree value was used as a basic element in the construction of the average tree solution for cycle-free graph games in Herings et al. (2008).

Remark 1 Since the rooted-tree structure of L, for any $i \in N$, $\bar{S}_L(i)$ is connected and sets $\bar{S}_L(j)$, $j \in T_L(i)$, provide a partition of $S_L(i)$. Wherefrom, as it was already mentioned in Herings et al. (2008), the tree value t of any rooted-tree digraph game $\langle v, L\rangle$ can be equivalently presented in terms of restricted games as

$$t_i(v, L) = v^L(\bar{S}_L(i)) - v^L(S_L(i)), \quad \text{for all } i \in N, \tag{3}$$

i.e., in terms of restricted games, the payoff to each player is equal to this player's contribution to all his successors when he joins them.

Proof I. First show that CE and SE on a subclass of rooted-tree digraph games uniquely define a G-value that satisfies these axioms. Consider a G-value ξ meeting CE and SE, and let $\langle v, L\rangle \in \mathcal{G}_N^{\mathcal{L}}$ be a rooted-tree game. Let $l = |L|$ be the number of links in L and $c = |N/L|$ the number of components in N defined by L. CE implies that

$$\sum_{k\in C}\xi_k(v,L)=v(C),\quad \text{for all } C\in N/L. \tag{4}$$

It is not difficult to see that all c equations of the type (4) are linearly independent.

Further, for a link $(i,j)\in L$, let C^j be the component in $N/\{L\backslash(i,j)\}$ containing player j. Because of CE, it holds that

$$\sum_{k\in C^j}\xi_k(v,L\backslash\{i,j\})=v(C^j). \tag{5}$$

By the rooted-tree structure of L, $C^j=\bar{S}_L(j)$. Then, SE implies that

$$\sum_{k\in C^j}\xi_k(v,L)=\sum_{k\in\bar{S}_L(j)}\xi_k(v,L)\overset{SE}{=}\sum_{k\in\bar{S}_L(j)}\xi_k(v,L\backslash\{i,j\})\overset{(5)}{=}v(C^j). \tag{6}$$

Again due to the rooted-tree structure of L, every $i\in N$ has at most one parent and for all $j,k\in T_L(i)$ such that $j\neq k$, $\bar{S}_L(j)\cap\bar{S}_L(k)=\emptyset$. Wherefrom it follows that all l equations of type (6) are linearly independent. Since the rooted-tree digraph L is cycle-free, $l+c=n$. Moreover, since equations of type (4) involve entire components $C\in N/L$ and equations of type (6) involve proper subcoalitions of these components, all n equations of types (4) and (6) together are linearly independent and, therefore, uniquely determine $\xi(v,L)$.

II. Verify now that the tree value t, defined by (2), satisfies CE and SE. As it was already observed in Herings et al. (2008), CE follows easily from definition (2). Indeed, let $C\in N/L$, then $L|_C$ is a rooted tree, and let $i\in C$ be its root. Obviously, $C=\bar{S}_L(i)$. Moreover, because of the rooted-tree structure of L and by equality (2), the total payoff to any player together with all his successors defined by the tree value is equal to the worth of the coalition composed by this player and all his successors. In particular, the last statement holds for the root player i. Wherefrom,

$$\sum_{j\in C}t_j(v,L)=\sum_{j\in\bar{S}_L(i)}t_j(v,L)=v(\bar{S}_L(i))=v(C),$$

which proves CE.

Next observe that, due to the rooted-tree structure of L, for any link $(i,j)\in L$, for all $k\in\bar{S}_L(j)$, the sets $\bar{S}_{L\backslash(i,j)}(k)$ and $\bar{S}_L(k)$ coincide. Whence and by definition (2), it follows immediately that t meets SE. $\square$

Observe that the claim of Theorem 1 holds true only on the subclass of rooted-tree digraph games, since among cycle-free digraphs, only a rooted-tree graph structure guarantees that every node (player) has at most one parent. Otherwise, there is at least one node $i^*\in N$ having at least two different parents, say j and k, $j\neq k$. In such a case, both links (j,i^*) and (k,i^*) generate the same equation of type (6), i.e., the system of n equations (4) and (6) looses its independence, and therefore, it cannot determine uniquely a G-value.

Consider another deletion link axiom:

A G-value ξ is *predecessor equivalent* (PE) if, for any digraph game $\langle v, L\rangle$, for every link $(i, j) \in L$, for all $k \in \bar{P}_L(i)$,

$$\xi_k(v, L\backslash(i, j)) = \xi_k(v, L).$$

It turns out that the two axioms of component efficiency and predecessor equivalence uniquely define a G-value for a sink-tree digraph game.

Theorem 2 *On the class of sink-tree digraph games, there is a unique G-value that satisfies CE and PE, and for any sink-tree digraph game $\langle v, L\rangle$, it is given by*

$$s_i(v, L) = v\left(\bar{P}_L(i)\right) - \sum_{j \in O_L(i)} v(\bar{P}_L(j)), \quad \textit{for all } i \in N. \tag{7}$$

From now we refer to the G-value s as to the *sink value*.

The proof of Theorem 2 is a pure mimicking of the proof of Theorem 1, and we skip it. Moreover, applying similar arguments as in the case of Theorem 1, we may declare that the claim of Theorem 2 holds true only on the subclass of sink-tree digraph games.

Remark 2 The sink value s of any sink-tree digraph game $\langle v, L\rangle$ can be equivalently presented in terms of restricted games as

$$s_i(v, L) = v^L(\bar{P}_L(i)) - v^L(P_L(i)), \quad \text{for all } i \in N, \tag{8}$$

i.e., in terms of restricted games, the payoff to each player is equal to this player's contribution to all his predecessors when he joins them.

From Charnes and Littlechild (1975) it follows that the core of the restricted game of any superadditive rooted-tree or sink-tree graph game is not empty. The core stability property of the tree value was already mentioned above with reference to Demange (2004). Applying the similar type arguments, we can show that the sink value on a class of sink-tree superadditive games also belongs to the core of the restricted game.

Notice that a line-graph is a particular case of both, a rooted tree and a sink tree. It is not difficult to see that the tree/sink value of an arbitrary line-graph game coincides, respectively, with the lower/upper equivalent solution proposed in van den Brink et al. (2007). Thus, the tree/sink value can be considered as a natural extension of the lower/upper equivalent solution defined on the class of line-graph games to rooted/sink-tree digraph games. Moreover, the axiom of successor/predecessor equivalence applied to a line-graph game coincides with the lower/upper equivalence axiom that, as it is shown in van den Brink et al. (2007), together with component efficiency characterizes the lower/upper equivalent solution on the class of line-graph games.

Recall that CE together with axiom of fairness[1] (F) uniquely define the Myerson value (Myerson 1977) of any graph game $\langle v, L\rangle \in \mathcal{G}_N^{\mathcal{L}}$ with arbitrary undirected graph L. In particular, CE and F define the Myerson value of any given rooted-tree or sink-tree graph game. However, the Myerson value, being equal to the Shapley value of the restricted game, does not respect the graph orientation and therefore, does not reflect extra information provided by the given orientation in the digraph structure. Furthermore, CE together with component fairness[2] (CF) uniquely define the average tree (AT) solution (Herings et al. 2008) of any undirected cycle-free graph game. In particular, CE and CF define the AT solution of any rooted-tree or sink-tree graph game, and we may apply its formula representation to a rooted-tree or sink-tree game as well. However, the AT solution is defined as an average of tree values assuming that any node in a given cycle-free graph can be chosen as a root of a directed rooted tree. Having a priori prescribed orientation of links in a given rooted-tree (or sink-tree) graph structure, the procedure of constructing the AT solution becomes meaningless.

3.2 Distribution of Harsanyi dividends

We consider now the tree and the sink values with respect to the distribution of Harsanyi dividends. By (1), the worth of any coalition is equal to the sum of Harsanyi dividends of the coalition itself and all its proper subcoalitions. Whence the Harsanyi dividend of a coalition has a natural interpretation as the extra revenue from cooperation among the players of the coalition that they did not already realize cooperating in proper subcoalitions. How the value under scrutiny distributes the dividend of a coalition among the players provides important information concerning the interest of different players to create the coalition. This information is especially important in games with limited cooperation structure when it might happen that one player (or some group of players) is responsible for the creation of a coalition. If in such a case the player(s) responsible for the creation of a coalition obtain(s) no quota from the dividend of this coalition, he (they) may simply block the creation of this coalition at all. This happens, for example, with some values for games with line-graph cooperation structure, see discussion concerning the topic in van den Brink et al. (2007).

From the equivalent definition of the tree value in terms of restricted games (3) and the representation of the worth of a coalition via Harsanyi dividends (1), it follows

[1] A G-value ξ is *fair* (F) if, for any graph game $\langle v, L\rangle$, for every link $\{i, j\} \in L$,

$$\xi_i(v, L) - \xi_i(v, L\backslash\{i, j\}) = \xi_j(v, L) - \xi_j(v, L\backslash\{i, j\}).$$

[2] A G-value ξ is *component fair* (CF) if, for any cycle-free graph game $\langle v, L\rangle$, for every link $\{i, j\} \in L$,

$$\frac{1}{|(N/L\backslash\{i, j\})_i|} \sum_{t \in (N/L\backslash\{i,j\})_i} (\xi_t(v, L) - \xi_t(v, L\backslash\{i, j\}))$$
$$= \frac{1}{|(N/L\backslash\{i, j\})_j|} \sum_{t \in (N/L\backslash\{i,j\})_j} (\xi_t(v, L) - \xi_t(v, L\backslash\{i, j\}))\,.$$

that for any rooted-tree digraph game $\langle v, L\rangle \in \mathcal{G}_N^{\mathcal{L}}$,

$$t_i(v, L) = \sum_{S \subseteq \bar{S}_L(i)} \lambda_S^{v^L} - \sum_{S \subseteq S_L(j)} \lambda_S^{v^L}, \quad \text{for all } i \in N.$$

Wherefrom and since in the restricted game the dividends of all disconnected coalitions are equal to zero, we easily obtain the validity of

Theorem 3 *The tree value of any rooted-tree digraph game $\langle v, L\rangle$ in terms of the distribution of the Harsanyi dividends is given by*

$$t_i(v, L) = \sum_{\substack{S \subseteq C^L(\bar{S}_L(i)) \\ S \ni i}} \lambda_S^{v^L}, \quad \textit{for all } i \in N. \tag{9}$$

Since in a rooted tree every connected coalition is a rooted tree as well, the next corollary arises directly from (9).

Corollary 1 *The tree value of a rooted-tree digraph game assigns dividend of any connected coalition to its root.*

Similarly, for any sink-tree game $\langle v, L\rangle \in \mathcal{G}_N^{\mathcal{L}}$, for all $i \in N$,

$$s_i(v, L) \overset{(8),(1)}{=} \sum_{S \subseteq \bar{P}_L(i)} \lambda_S^{v^L} - \sum_{S \subseteq P_L(j)} \lambda_S^{v^L}, \quad \text{for all } i \in N,$$

which yields the validity of

Theorem 4 *The sink value of any sink-tree digraph game $\langle v, L\rangle$ in terms of the distribution of the Harsanyi dividends is given by*

$$s_i(v, L) = \sum_{\substack{S \subseteq C^L(\bar{P}_L(i)) \\ S \ni i}} \lambda_S^{v^L}, \quad \textit{for all } i \in N. \tag{10}$$

Corollary 2 *The sink value of a sink-tree digraph game assigns dividend of any connected coalition to its absorbing sink.*

4 Sharing a river with a delta or with multiple sources

Ambec and Sprumont (2002) approach the problem of optimal water allocation for a given river with certain capacity over the agents (cities, countries) located along the river from the game theoretic point of view. Their model assumes that between each pair of neighboring agents there is an additional inflow of water. Each agent, in principal, can use all the inflow between itself and its upstream neighbor. However, this allocation, in general, is not optimal in respect to total welfare. In order to obtain more profitable allocation, it is allowed to allocate more water to downstream agents, which in turn can compensate the extra water obtained by side-payments to upstream ones. The problem of optimal water allocation is approached as the problem of optimal welfare distribution. van den Brink et al. (2007) show that the Ambec–Sprumont river game model can be naturally embedded into the framework of a graph game with line-graph cooperation structure. We extend the line-graph model of a river to the rooted-tree digraph model of a river with a delta and to the sink-tree digraph model of a river with multiple sources.

Let N be a set players (users of water) located along the river from upstream to downstream. Let $e_{ki} \geq 0$, $i \in N$, $k \in O(i)$, be the inflow of water in front of the most upstream player(s) (in this case, $k = 0$) or the inflow of water entering the river between neighboring players in front of the player i. Figure 1 provides a schematic representation of the model.

Following Ambec and Sprumont (2002) and van den Brink et al. (2007), it is assumed that each player $i \in N$ has a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$, where t_i is a monetary compensation to player i, x_i is the amount of water allocated to player i, and $b^i : \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function

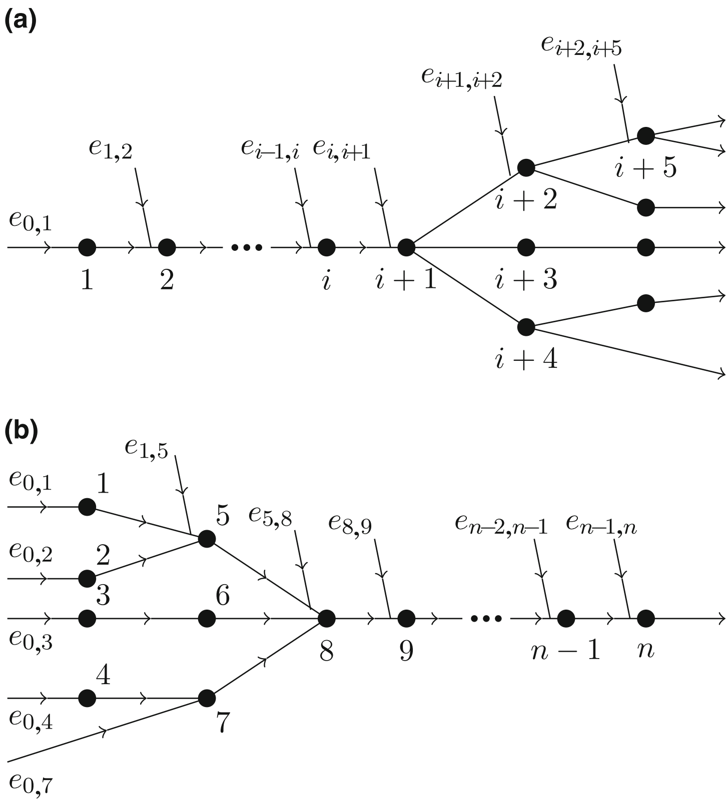

Fig. 1 **a** A river with a delta. **b** A river with multiple sources

providing benefit $b^i(x_i)$ to player i by the consumption of x_i of water. An allocation is a pair $(x,t) \in \mathbb{R}^n_+ \times \mathbb{R}^n$ of water distribution and compensation scheme, satisfying

$$\sum_{i\in N} t_i \leq 0, \quad \text{and} \quad \begin{cases} \sum\limits_{j\in\bar{P}_L(i)} x_j \leq \sum\limits_{j\in\bar{P}_L(i)} \sum\limits_{k\in O(j)} e_{kj}, \\ \sum\limits_{j\in P_L(i)\cup\bar{B}_L(i)} x_j \leq \sum\limits_{j\in P_L(i)\cup\bar{B}_L(i)} \sum\limits_{k\in O(j)} e_{kj}, \end{cases} \quad \text{for all } i\in N,$$

with L being a graph presenting the river structure. The first condition is a budget constraint. While the second condition on the one hand reflects that a player can use only his upstream inflow of water, and on the other it reflects that the total usage of water by any set of all brothers in L cannot exceed the total upstream inflow to all of them. Notice that in the second constraint, the summation over a set of parents $O(j)$ may contain more than one element only in the case of a river with multiple sources. In the second constraint, the second inequality differs from the first one only for a river with a delta. Moreover, in the case of a river with a delta, the second inequality in the second constraint is the same for all brothers. The optimal water distribution $x^* \in \mathbb{R}^n_+$ maximizes the total welfare, i.e., solves the following optimization problem

$$\max_{x\in\mathbb{R}^n_+} \sum_{i\in N} b^i(x_i) \ \ s.t. \begin{cases} \sum\limits_{j\in\bar{P}_L(i)} x_j \leq \sum\limits_{j\in\bar{P}_L(i)} \sum\limits_{k\in O(j)} e_{kj}, \\ \sum\limits_{j\in P_L(i)\cup\bar{B}_L(i)} x_j \leq \sum\limits_{j\in P_L(i)\cup\bar{B}_L(i)} \sum\limits_{k\in O(j)} e_{kj}, \end{cases} \forall i\in N. \tag{11}$$

A welfare distribution distributes the total benefits $\sum_{i\in N} b^i(x_i^*)$ of optimal water distribution x^* among the players. In the case of a river with a delta, it is also assumed that if a splitting of the river into branches happens to occur after a certain player, then this player takes, besides his own quota, also the responsibility to split the rest of the water flow to the branches such as to guarantee the realization of the water distribution plan x^* to his successors.

Approaching the optimization problem (11) the same way as in van den Brink et al. (2007) in the case of line-graphs, it turns out that the problem of finding an optimal welfare distribution among the users of water located along a river with a delta or with multiple sources can be modeled as a rooted-tree or sink-tree digraph game correspondingly. For any pair of players, the water inflow entering the river before the upstream player can only be allocated to the downstream player if all players between them cooperate, otherwise any player between them can take this water for his own use. Hence, only coalitions of consecutive players are admissible. In order to define the characteristic function v, put $v(N) = \sum_{i\in N} b^i(x_i^*)$ with x^* being a solution of (11). For any connected coalition S, put $v(S) = \sum_{i\in N} b^i(x_i^S)$, where $x^S \in \mathbb{R}^s$ solves

$$\max_{x\in\mathbb{R}^s_+}\sum_{i\in S} b^i(x_i)\ s.t. \begin{cases} \displaystyle\sum_{j\in\bar{P}_L(i)\cap S} x_j \leq \sum_{j\in\bar{P}_L(i)\cap S}\ \sum_{k\in O(j)} e_{kj}, \\ \displaystyle\sum_{j\in(P_L(i)\cup\bar{B}_L(i))\cap S} x_j \leq \sum_{j\in(P_L(i)\cup\bar{B}_L(i))\cap S}\ \sum_{k\in O(j)} e_{kj}, \end{cases} \forall i \in S. \quad (12)$$

For any disconnected coalition $S \subset N$, $v(S) = \sum_{T\in C^L(S)} v(T)$, and so, the restricted game v^L is equal to v. Depending on the river graph structure L, we refer to this game as the *river game with a delta* or the *river game with multiple sources*, or shortly *river game* similarly as in van den Brink et al. (2007), if no ambiguity appears.

It is easy to see that the river game is superadditive. Hence both, the tree value in the river game with a delta and the sink value in the river game with multiple sources, are core selectors. The tree value of the river game with a delta is a natural extension of the lower equivalent solution for the line-graph river game studied in van den Brink et al. (2007). The tree value assigns the total dividend of any connected coalition to its most upstream player, its root. This seems to be reasonable, since the most upstream player of any connected coalition keeps control over creating this coalition. The sink value of the river game with multiple sources can be considered as an extension of the upper equivalent solution for a line-graph river game (van den Brink et al. 2007), which in turn coincides with solution for the river game proposed by Ambec and Sprumont (2002). The sink value assigns the full dividend of any connected coalition to its most downstream player, its absorbing sink. Since the creation of any connected coalition is fully up to the upstream players, the sink value, even being a core selector, is contradictious from the perspective of the distribution of Harsanyi dividends.

5 Concluding remarks

This article opens a few interesting and important questions concerning the values for digraph games which respect the given graph orientation. Which component efficient value for a sink-graph game supplies a quota of the dividend of a connected coalition to its most upstream player? Are there any component efficient values for digraph games with graph structure given by merging of a sink tree and a rooted tree that, in particular, can solve the river game with both multiple sources and a delta? How to solve a digraph game with general cycle-free digraph presenting flow situation when some links may merge while others split into several separate ones?

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