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Choosers as extension axioms

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Abstract We consider the extension of a (strict) preference over a set to its power set. Elements of the power set are non-resolute outcomes. The final outcome is determined by an "(external) chooser" which is a resolute choice function. The individual whose preference is under consideration confronts a set of resolute choice functions which reflects the possible behaviors of the chooser. Every such set naturally induces an extension axiom (i.e., a rule that determines how an individual with a given preference over alternatives is required to rank certain sets). Our model allows to revisit various extension axioms of the literature. Interestingly, the Gärdenfors (1976) and Kelly (1977) principles are singled-out as the only two extension axioms compatible with the non-resolute outcome interpretation.

Keywords Preferences over sets · Non-resolute outcomes

1 Introduction

It is quite typical that collective decision problems are resolved through the initial choice of a non-resolute set of outcomes which is followed by the final decision of an "external chooser". This two-stage structure is sometimes an explicit part of the social choice rule -hence the external chooser truly exists.[1] But even without an

[1] Such social choice rules are analyzed by Barberà and Coelho (2004) who call them "rules of k names." For a more general treatment of sequential choice procedures, one can see Manzini and Mariotti (2007).

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explicit reference to the "external chooser," a two-stage structure is implicit in the nature of the social choice problem. For, the impossibility of making a resolute choice under desirable axioms is well-known. In fact, as one can see in Moulin (1983), every anonymous and neutral social choice rule must exhibit non–resoluteness, thus leaving the final choice to an "external chooser" —who does not necessarily exist in flesh and bone.

This two-stage nature of collective decision problems raises the question of extending a preference over a set to its power set. This question is typically answered through an *extension axiom* which is a rule that determines how an individual with a given preference over alternatives is required to rank certain sets. As Barberà et al. (2004) beautifully survey, there is a vast literature on extending an order over a set to its power set. To be sure, this literature contains various interpretations of a set, such as being a list of mutually incompatible outcomes[2] or a list of mutually compatible outcomes[3] or a menu from which the individual whose preference under consideration makes a choice[4] or a collection states.[5] All these interpretations have their own axioms. Our consideration is limited to the first interpretation where a set is conceived as an initial non-resolute refinement of outcomes from which a final choice will be made.

We propose a model that underlies this conception of a set. We admit a resolute choice function[6] to be a "(external) chooser" who makes the final decision from any non-resolute outcome. Hence a (non-empty) set $\mathcal{D}$ of resolute choice functions is the list of admissible behaviors that choosers may exhibit. In principle, $\mathcal{D}$ can be anything, ranging from a singleton set to the set of all choice functions. In particular, $\mathcal{D}$ may be determined by well-established axioms of choice theory, such as the weak axiom of revealed preference. After all, any given $\mathcal{D}$ induces an extension axiom in the following natural way: For each possible ordering ρ of alternatives, a set X is required to be ranked above a set Y if and only if the final decision made from X is preferred (according to ρ) to the final decision made from Y, for any chooser belonging to $\mathcal{D}$.

Our model allows to revisit the existing extension axioms of the literature. Among these, two prevalent ones, namely the Gärdenfors (1976) and Kelly (1977) principles, are singled out. For, every "regular" axiom of choice theory determines a domain of admissible choosers which induces either the Gärdenfors (1976) or the Kelly (1977) principle.

Section 2 sets the framework. Section 3 states the results. Section 4 concludes.

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[2] e.g., Gärdenfors (1976), Barberà (1977), Kelly (1977), Feldman (1979), Duggan and Schwartz (2000), Barberà et al. (2001), Benoit (2002), Ching and Zhou (2002), Ozyurt and Sanver (2006).

[3] e.g., Barberà et al. (1991), Ozyurt and Sanver (2007).

[4] e.g., Kreps (1979), Dutta and Sen (1996), Dekel et al. (2001), Gul And Pesendorfer (2001).

[5] e.g., Lainé et al. (1986), Weymark (1997).

[6] A resolute choice function assigns to each non-empty set X a single element of X.

2 Basic notions

We consider a finite set of alternatives A where $\underline{A} = 2^A \backslash \{\emptyset\}$. We let $\#A \geq 3$ and write Π for the set of complete, transitive, and antisymmetric binary relations over A. [7] We write ρ^* for the strict counterpart of $\rho \in \Pi$.[8]

2.1 Extension axioms

An *extension axiom* is a mapping ε which assigns to each $\rho \in \Pi$ a transitive binary relation $\varepsilon(\rho)$ over $\underline{A}$ such that $x\ \rho^*\ y \Leftrightarrow \{x\}\varepsilon(\rho)\{y\}\ \forall x, y \in A$. We interpret $(X, Y) \in \varepsilon(\rho)$ as the requirement of ranking the set X at least as good as the set Y when the ranking of alternatives is ρ. Note that our definition of an extension axiom, perhaps untypically, does not require the antisymmetry of $\varepsilon(\rho)$. Nevertheless, most of the extension axioms we consider turn out to induce antisymmetric binary relations.

We define below three principal extension axioms that we consider:

- The extension axiom κ, used by Kelly (1977) in his analysis of strategy-proof social choice correspondences, is defined for each $\rho \in \Pi$ as $\kappa(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \backslash \{X\} : x\ \rho\ y\ \forall x \in X\ \forall y \in Y\}$. We refer to κ as the *Kelly principle*.
- The extension axiom γ, used by Gärdenfors (1976) in his analysis of strategy-proof social choice correspondences, is defined for each $\rho \in \Pi$ as $\gamma(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \backslash \{X\} : (x\ \rho^*\ y\ \forall x \in X \backslash Y\ \forall y \in Y)$ and $(x\ \rho^*\ y\ \forall x \in X\ \forall y \in Y \backslash X)\}$. We refer to γ as the *Gärdenfors principle*.
- The extension axiom σ, to which we refer as the *separability principle*, is defined for each $\rho \in \Pi$ as $\sigma(\rho) = \{(X \cup \{x\}, X \cup \{y\}) : X \in 2^A$ and $x\ \rho^*\ y$ for distinct $x, y \in A \backslash X\}$.[9]

The Gärdenfors principle is stronger than the Kelly principle, i.e., $\kappa(\rho) \subsetneq \gamma(\rho)\ \forall \rho \in \Pi$. On the other hand, the separability principle is logically independent of both the Kelly and the Gärdenfors principles. Note that all three extension axioms induce antisymmetric binary relations.

2.2 Choice functions

A (*resolute*) *choice function* is a mapping $C : \underline{A} \rightarrow A$ such that $C(X) \in X,\ \forall X \in \underline{A}$. We write $\mathcal{C}$ for the set of all choice functions and $\mathcal{D} \subseteq \mathcal{C}$ stands for any non-empty subclass of choice functions. We consider axiomatic restrictions over $\mathcal{C}$. The definitions below are quoted from Aizerman and Aleskerov (1995):

[7] So, for any $\rho \in \Pi$ and any $x, y \in A$, by completeness, we have $x\ \rho\ y$ or $y\ \rho\ x$. This implies reflexivity, i.e., $x\ \rho\ x\ \forall x \in A$. Note that by antisymmetry, $x\ \rho\ y \Longrightarrow$ not $y \rho x$ when x and y are distinct. Finally, transitivity ensures $x\ \rho\ y$ and $y \rho z \Longrightarrow x \rho z\ \forall x, y, z \in A$.

[8] So, for any $\rho \in \Pi$ and any $x, y \in A$, we have $x\ \rho^*\ y$ whenever $x\ \rho\ y$ holds and $y \rho x$ fails. As ρ is antisymmetric, when x and y are distinct, we have either $x\ \rho^*\ y$ or $y\ \rho^*\ x$.

[9] The separability principle, which is a modified version of the monotonicity axiom of Kannai and Peleg (1984), is used by Roth and Sotomayor (1990) in their manipulation analysis of many-to-one matching rules.

- A choice function C satisfies the *Weak Axiom of Revealed Preference* (*WARP*) iff $C(Y) \in X$ and $C(X) \in Y \implies C(X) = C(Y)\ \forall X, Y \in \underline{A}$.[10] We write $\mathcal{C}^{WARP}$ for the set of (resolute) choice functions that satisfy WARP.[11] It is to be noted that, defining at each $\tau \in \Pi$, the choice function $C_\tau(X)\tau x\ \forall x \in X,\ \forall X \in \underline{A}$, we have $\mathcal{C}^{WARP} = \{C_\tau\}_{\tau \in \Pi}$.[12]
- A choice function C satisfies *Concordance* iff $C(X) = C(Y) \implies C(X) = C(X \cup Y)\ \forall X, Y \in \underline{A}$. We write $\mathcal{C}^{CONC}$ for the set of (resolute) choice functions that satisfy concordance.
- A choice function C satisfies *direct Condorcet* iff $x = \bigcap_{y \in X} C(\{x, y\}) \implies x = C(X)\ \forall X \in \underline{A},\ \forall x \in A$. We write $\mathcal{C}^{DC}$ for the set of (resolute) choice functions that satisfy direct Condorcet.

Remark 2.1 As one can see in Aizerman and Aleskerov (1995), we have $\mathcal{C}^{WARP} \subsetneq \mathcal{C}^{CONC} \subsetneq \mathcal{C}^{DC} \subsetneq \mathcal{C}$.

3 Inducing extension axioms through choice functions

Any non-empty $\mathcal{D} \subseteq \mathcal{C}$ induces an extension axiom $\varepsilon^{\mathcal{D}}$ as follows: At each $\rho \in \Pi$, for all distinct $X, Y \in \underline{A}$, we have $(X, Y) \in \varepsilon^{\mathcal{D}}(\rho) \iff C(X)\rho C(Y)\ \forall C \in \mathcal{D}$. We interpret $\mathcal{D}$ as the set of possible "behaviors" of the external chooser. So an individual with preference ρ has to view X at least as good as Y if and only if under any possible "behavior" of the external chooser, the final decision made from X is at least as good as (according to ρ) the final decision made from Y. Note that $\varepsilon^{\mathcal{D}}(\rho)$ is antisymmetric if and only if $\mathcal{D}$ satisfies the following *richness* condition: Given any distinct $X, Y \in \underline{A}$, there exists $C \in \mathcal{D}$ such that $C(X) \neq C(Y)$.

Observe that given any two domains $\mathcal{D}_1, \mathcal{D}_2$ of resolute choice functions, $\mathcal{D}_1 \subseteq \mathcal{D}_2 \implies \varepsilon^{\mathcal{D}_2}(\rho) \subseteq \varepsilon^{\mathcal{D}_1}(\rho)\ \forall \rho \in \Pi$ follows for the definition of $\varepsilon^{\mathcal{D}}$. This observation conjoined with Remark 2.1 leads to the following proposition:

Proposition 3.1 $\varepsilon^{\mathcal{C}}(\rho) \subseteq \varepsilon^{\mathcal{C}^{DC}}(\rho) \subseteq \varepsilon^{\mathcal{C}^{CONC}}(\rho) \subseteq \varepsilon^{\mathcal{C}^{WARP}}(\rho)\ \forall \rho \in \Pi$.

Although the set inclusions stated by Remark 2.1 are proper, those announced by Proposition 3.1 need not be so, as we show soon.

[10] For resolute choice functions, the version of WARP we use and the definition given by Aizerman and Aleskerov (1995) are equivalent.

[11] A variety of conditions which differ from WARP over the class of choice correspondences turn out to be equivalent to WARP over the class of resolute choice functions. Among these, we have

(i) *postulate 4* of Chernoff (1954) (called *axiom C2* by Arrow (1959), *condition alpha* by Sen (1974), *upper semi-fidelity* by Sertel and van der Bellen (1979), *heredity* by Aizerman and Aleskerov (1995));
(ii) the independence of irrelevant alternatives condition of Nash (1950) (called *postulate 5** by Chernoff (1954), *axiom 2* by Sanver and Zwicker (2007), *outcast* by Aizerman and Aleskerov (1995) and *absorbance* by Sertel and van der Bellen (1979));
(iii) *postulate 6* of Chernoff (1954) (called *axiom C4* by Arrow (1959) and *constancy* by Aizerman and Aleskerov (1995));
(iv) The *inverse Condorcet* condition of Aizerman and Aleskerov (1995).

[12] What we note follows from many results of the literature, e.g., Theorem 2.10 of Aizerman and Aleskerov (1995).

We first establish the equivalence between the Kelly principle and the extension axiom induced by allowing all logically possible choice functions.

Theorem 3.1 $\varepsilon^{\mathcal{C}}(\rho) = \kappa(\rho)\ \forall\rho \in \Pi$.

Proof Take any $\rho \in \Pi$. To see $\varepsilon^{\mathcal{C}}(\rho) \subseteq \kappa(\rho)$, pick any $(X, Y) \in \varepsilon^{\mathcal{C}}(\rho)$. So, $C(X)\rho C(Y)\ \forall C \in \mathcal{C}$. Now, consider a choice function C_0 with $x\rho C_0(X)\ \forall x \in X$ and $C_0(Y)\rho y\ \forall y \in Y$. Clearly, $C_0 \in \mathcal{C}$. Thus, $C_0(X)\rho C_0(Y)$ which, by the choice of C_0, implies $x\rho y\ \forall x \in X,\ \forall y \in Y$, hence establishing $(X, Y) \in \kappa(\rho)$. To see $\kappa(\rho) \subseteq \varepsilon^{\mathcal{C}}(\rho)$, pick any $(X, Y) \in \kappa(\rho)$. Let $x_0 \in X$ be such that $x\rho x_0\ \forall x \in X$ and $y_0 \in Y$ be such that $y_0\rho y\ \forall y \in Y$. As $(X, Y) \in \kappa(\rho)$, we have $x_0\rho y_0$. Now, take any $C \in \mathcal{C}$. By the choice of x_0 and y_0, we have $C(X)\rho x_0$ and $y_0\rho C(Y)$ which implies $C(X)\rho C(Y)$, establishing $(X, Y) \in \varepsilon^{\mathcal{C}}(\rho)$. □

Remark 3.1 The antisymmetry of $\varepsilon^{\mathcal{C}}$ can be deduced from the antisymmetry of κ as well as from the richness of $\mathcal{C}$.

Remark 3.2 For any $\mathcal{D}$, we have $\kappa(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)\ \forall\rho \in \Pi$. In other words, the Kelly principle is the weakest extension axiom that can be conceived in our environment.

We now show that restricting the set of admissible choice functions to those which satisfy the concordance axiom does not induce an extension axiom stronger than the Kelly principle.

Theorem 3.2 $\varepsilon^{\mathcal{C}^{CONC}}(\rho) = \kappa(\rho)\ \forall\rho \in \Pi$.

Proof Take any $\rho \in \Pi$. The inclusion $\kappa(\rho) \subseteq \varepsilon^{\mathcal{C}^{CONC}}(\rho)$ follows from Remark 3.2. To see $\varepsilon^{\mathcal{C}^{CONC}}(\rho) \subseteq \kappa(\rho)$, pick some $(X, Y) \notin \kappa(\rho)$. So, $\exists\overline{y} \in Y$ and $\exists\overline{x} \in X\backslash\{\overline{y}\}$ such that $\overline{y}\rho^*\overline{x}$. First, consider the first case where $\overline{y} \notin X$. Pick some $\tau \in \Pi$ with $\overline{y}\tau\overline{x}\tau z\ \forall z \in A\backslash\{\overline{x}, \overline{y}\}$. Note that $C_\tau \in \mathcal{C}^{WARP} \subsetneq \mathcal{C}^{CONC}$. As $\overline{y} \notin X$, we have $C_\tau(X) = \overline{x}$ and $C_\tau(Y) = \overline{y}$, thus $C_\tau(X)\rho C_\tau(Y)$ fails, establishing $(X, Y) \notin \varepsilon^{\mathcal{C}^{CONC}}(\rho)$. Next, consider the case where $\overline{x} \notin Y$. Pick some $\tau \in \Pi$ with $\overline{x}\tau\overline{y}\tau z\ \forall z \in A\backslash\{\overline{x}, \overline{y}\}$. Note that $C_\tau \in \mathcal{C}^{CONC}$. As $\overline{x} \notin Y$, we have $C_\tau(Y) = \overline{y}$ and $C_\tau(X) = \overline{x}$, thus $C_\tau(X)\rho C_\tau(Y)$ fails, establishing $(X, Y) \notin \varepsilon^{\mathcal{C}^{CONC}}(\rho)$. Finally, consider the case where $\overline{y} \in X$ and $\overline{x} \in Y$. Pick some $\tau \in \Pi$ with $z\tau\overline{x}\tau\overline{y}\ \forall z \in A\backslash\{\overline{x}, \overline{y}\}$. Consider the choice function $\overline{C}$ defined as $\overline{C}(X) = \overline{x}$, $\overline{C}(Y) = \overline{y}$ and $\overline{C}(Z) = C_\tau(Z)\ \forall Z \in \underline{A}\backslash\{X, Y\}$. Note that $\overline{C}(X)\rho\overline{C}(Y)$ fails. So we complete the proof by showing $\overline{C} \in \mathcal{C}^{CONC}$. To see this, take any distinct $S, T \in \underline{A}$ with $\overline{C}(S) = \overline{C}(T)$. Note that $S, T \in \{X, Y\}$ cannot hold, by construction of $\overline{C}$. Now, consider the following three exhaustive cases:

Case 1 $X \in \{S, T\}$, say $S = X$ without loss of generality. So $\overline{C}(T) = \overline{x}$, which implies $T \in \{\{\overline{x}, \overline{y}\}, \{\overline{x}\}\}$, which in turn implies $S \cup T = S$, establishing $\overline{C}(S \cup T) = \overline{C}(S)$.

Case 2 $Y \in \{S, T\}$, say $S = Y$ without loss of generality. So $\overline{C}(T) = \overline{y}$, which implies $T = \{\overline{y}\}$, which in turn implies $S \cup T = S$, establishing $\overline{C}(S \cup T) = \overline{C}(S)$.

Case 3 $X, Y \notin \{S, T\}$. Let $z = \overline{C}(S) = \overline{C}(T)$. So $z\tau s\ \forall s \in S$ and $z\tau t\ \forall t \in T$, thus $z\tau u\ \forall u \in S \cup T$, implying $z = \overline{C}(S \cup T)$.

Therefore, $\overline{C} \in \mathcal{C}^{CONC}$, hence $(X, Y) \notin \varepsilon^{\mathcal{C}^{CONC}}(\rho)$. □

Remark 3.3 The antisymmetry of $\varepsilon^{\mathcal{C}^{CONC}}$ follows from the antisymmetry of κ as well as from the richness of $\mathcal{C}^{CONC}$.

The following result is a corollary to Theorem 3.1 and Theorem 3.2.

Theorem 3.3 *Given any* $\mathcal{D} \supseteq \mathcal{C}^{CONC}$ *we have* $\varepsilon^{\mathcal{D}}(\rho) = \kappa(\rho)\ \forall \rho \in \Pi$.

Note that Theorem 3.3 covers the particular case where $\mathcal{D} = \mathcal{C}^{DC}$. Our next result shows that by further restricting the set of admissible choice functions through WARP, we fall into the Gärdenfors principle.[13]

Theorem 3.4 $\varepsilon^{\mathcal{C}^{WARP}}(\rho) = \gamma(\rho)\ \forall \rho \in \Pi$.

Proof Take any $\rho \in \Pi$. To see $\varepsilon^{\mathcal{C}^{WARP}}(\rho) \subseteq \gamma(\rho)$, pick some $(X, Y) \notin \gamma(\rho)$. So $\exists \overline{y} \in Y, \exists \overline{x} \in X \backslash Y$ with $\overline{y}\, \rho^* \,\overline{x}$ or $\exists \overline{y} \in Y \backslash X, \exists \overline{x} \in X$ with $\overline{y}\, \rho^* \,\overline{x}$. In the former case, pick some $\tau \in \Pi$ with $\overline{x} \tau \overline{y} \tau z\ \ \forall z \in A \backslash \{\overline{x}, \overline{y}\}$, thus $C_\tau(X) = \overline{x}$ and $C_\tau(Y) = \overline{y}$, implying the failure of $C_\tau(X)\, \rho C_\tau(Y)$ while $C_\tau \in \mathcal{C}^{WARP}$, hence establishing $(X, Y) \notin \varepsilon^{\mathcal{C}^{WARP}}(\rho)$. In the latter case, pick some $\tau \in \Pi$ with $\overline{y} \tau \overline{x} \tau z\ \forall z \in A \backslash \{\overline{x}, \overline{y}\}$, thus $C_\tau(X) = \overline{x}$ and $C_\tau(Y) = \overline{y}$, implying the failure of $C_\tau(X)\, \rho C_\tau(Y)$ while $C_\tau \in \mathcal{C}^{WARP}$, hence establishing $(X, Y) \notin \varepsilon^{\mathcal{C}^{WARP}}(\rho)$.

To see $\gamma(\rho) \subseteq \varepsilon^{\mathcal{C}^{WARP}}(\rho)$, pick any $(X, Y) \in \gamma(\rho)$. So we have $(x\, \rho^*\, y\ \forall x \in X \backslash Y,\ \forall y \in Y)$ and $(x\, \rho^*\, y\ \forall x \in X,\ \forall y \in Y \backslash X)\}$. In particular, $C(X \backslash Y) \rho^* C(Y)$ $\forall C \in \mathcal{C}$ whenever $X \backslash Y \neq \emptyset$ and $C(X) \rho^* C(Y \backslash X)\ \forall C \in \mathcal{C}$ whenever $Y \backslash X \neq \emptyset$. Note that X and Y are distinct, thus $X \backslash Y$ and $Y \backslash X$ cannot be both empty. Let, without loss of generality, $X \backslash Y \neq \emptyset$. Take any $C \in \mathcal{C}^{WARP}$. First, consider the case where $C(X) \in X \backslash Y$. Since $X \backslash Y \subseteq X$ and $C \in \mathcal{C}^{WARP}$, we have $C(X) = C(X \backslash Y)$. Thus, $C(X) \rho^* C(Y)$. Now, consider the case where $C(X) \notin X \backslash Y$. So $C(X) \in X \cap Y$. Since $X \cap Y \subseteq X$ and $C \in \mathcal{C}^{WARP}$, we have $C(X) = C(X \cap Y)$. If $C(Y) \in X \cap Y$ then $C(Y) = C(X \cap Y)$ follows by $C \in \mathcal{C}^{WARP}$, establishing $C(X) \rho^* C(Y)$. If $C(Y) \notin X \cap Y$, then $C(Y) \in Y \backslash X$, and we get $C(Y) = C(Y \backslash X)$ by $C \in \mathcal{C}^{WARP}$, implying $C(X) \rho^* C(Y)$. Thus $(X, Y) \in \varepsilon^{\mathcal{C}^{WARP}}(\rho)$ and $\gamma(\rho) \subseteq \varepsilon^{\mathcal{C}^{WARP}}(\rho)$. □

Remark 3.4 The antisymmetry of $\varepsilon^{\mathcal{C}^{WARP}}$ can be deduced from the antisymmetry of γ as well as from the richness of $\mathcal{C}^{WARP}$.

We summarize below our findings up to now.

Corollary 3.1 $\kappa(\rho) = \varepsilon^{\mathcal{C}}(\rho) = \varepsilon^{\mathcal{C}^{DC}}(\rho) = \varepsilon^{\mathcal{C}^{CONC}}(\rho) \subsetneq \varepsilon^{\mathcal{C}^{WARP}}(\rho) = \gamma(\rho)$ $\forall \rho \in \Pi$.

Remark that a rich variety of choice axioms[14] single out the Kelly and Gärdenfors principles. As an interesting observation, the separability principle has not been induced by any of the choice axioms we considered. In fact, as we show below, there

[13] Sanver and Zwicker (2007) consider various monotonicity and manipulability properties of irresolute social choice rules. Among other things, they show that certain monotonicity conditions turn out to be equivalent, independent of whether the irresolute social choice rule is refined through a total order or preferences over alternatives are extended over sets through the Gärdenfors principle. In fact, it is the result announced by Theorem 3.4 which underlies this equivalence.

[14] Recall the remark made by Footnote 11.

exists no class of admissible choice functions that induces the separability principle. Before proving this, we state a lemma.

Lemma 3.1 *Let $\mathcal{D} \subseteq \mathcal{C}$ ensure $\sigma(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ $\forall \rho \in \Pi$. Given any $C \in \mathcal{D}$ and any $X, Y \in \underline{A}$ with $\#X = \#Y = 2$ and $\#(X \cap Y) = 1$, we have $C(X) = X \cap Y \implies C(Y) = X \cap Y$.*

Proof Let $\mathcal{D}$ be as in the statement of the lemma. Take any $C \in \mathcal{D}$. Let $X = \{x, y\}$ and $Y = \{x, z\}$ for some distinct $x, y, z \in A$. Take any $\rho \in \Pi$with $y\rho^* z\rho^* x$. Suppose $C(X) = x$ and $C(Y) = z$. So $C(X)\rho C(Y)$ fails, hence $(X, Y) \notin \varepsilon^{\mathcal{D}}(\rho)$ while $(X, Y) \in \sigma(\rho)$, contradicting the choice of $\mathcal{D}$. □

Theorem 3.5 *$\nexists \mathcal{D} \subseteq \mathcal{C}$ which ensures $\sigma(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ $\forall \rho \in \Pi$.*

Proof Let, for a contradiction, $\mathcal{D} \subseteq \mathcal{C}$ ensure $\sigma(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ $\forall \rho \in \Pi$. Take any $C \in \mathcal{D}$ and any distinct $x, y, z \in A$. Let, without loss of generality, $C(\{x, y\}) = x$. By Lemma 3.1, we have $C(\{x, z\}) = x$ and $C(\{y, z\}) = z$. However, again by Lemma 3.1, $C(\{x, z\}) = x$ implies $C(\{y, z\}\} = y$, giving the desired contradiction. □

The impossibility announced by Theorem 3.5 prevails for any variant of Kannai and Peleg (1984) monotonicity, which is stronger than separability.

We close the section by a remark regarding the strengths of the extension axioms that are conceivable in our environment. As noted by Remark 3.2, the Kelly principle is the weakest among all conceivable extension axioms. On the other hand, although the Gärdenfors principle is the strongest extension axiom we encountered, we cannot claim it to be the strongest among all conceivable extension axioms. For, although WARP is a fairly demanding condition, the set of admissible choice functions can be further reduced. In fact, at the extreme, $\mathcal{D}$ can be assumed to contain only one choice function. Actually, the strongest conceivable extension axioms will be those that are induced by singleton sets of admissible choice functions. In fact, any $\mathcal{D} = \{C\}$ with $C \in \mathcal{C}$ induces a complete and transitive binary relation $\varepsilon^{\mathcal{D}}(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} : C(X)\rho C(Y)\}$ at each $\rho \in \Pi$.[15] Nevertheless, as we note below, it is not possible to speak about "the strongest" extension axiom.

Proposition 3.2 *Given any $\mathcal{D} = \{C\}$ and $\mathcal{D}' = \{C'\}$ with distinct $C, C' \in \mathcal{C}$, both $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\mathcal{D}'}(\rho)$ and $\varepsilon^{\mathcal{D}'}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ fail at every $\rho \in \Pi$.*

Proof Take any $\mathcal{D} = \{C\}$ and $\mathcal{D}' = \{C'\}$ with distinct $C, C' \in \mathcal{C}$. So, there exists $X \in \underline{A}$ such that $C(X) \neq C'(X)$. Note that $\#X \geq 2$. Take any $\rho \in \Pi$. Consider the first case where $C'(X)\rho^* C(X)$. Note that $(\{C(X)\}, X) \in \varepsilon^{\mathcal{D}}(\rho)$ but $(\{C(X)\}, X) \notin \varepsilon^{\mathcal{D}'}(\rho)$. Moreover $\left(X, \left\{C'(X)\right\}\right) \in \varepsilon^{\mathcal{D}'}(\rho)$ but $\left(X, \left\{C'(X)\right\}\right) \notin \varepsilon^{\mathcal{D}}(\rho)$. Hence, neither $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\mathcal{D}'}(\rho)$ nor $\varepsilon^{\mathcal{D}'}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ holds. Now, consider the case where $C(X)\rho^* C'(X)$. Note that $(X, \{C(X)\}) \in \varepsilon^{\mathcal{D}}(\rho)$ but $(X, \{C(X)\}) \notin \varepsilon^{\mathcal{D}'}(\rho)$. Moreover $\left(\left\{C'(X)\right\}, X\right) \in \varepsilon^{\mathcal{D}'}(\rho)$ but $\left(\left\{C'(X)\right\}, X\right) \notin \varepsilon^{\mathcal{D}}(\rho)$. Hence, neither $\varepsilon^{\mathcal{D}}(\rho) \subseteq \varepsilon^{\mathcal{D}'}(\rho)$ nor $\varepsilon^{\mathcal{D}'}(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$ holds. □

[15] Remark that no $\mathcal{D} = \{C\}$ is rich; hence the corresponding complete preorder $\varepsilon^{\mathcal{D}}(\rho)$ is not antisymmetric.

As a case of particular interest, we have $\mathcal{D} = \{C\}$ for $C \in \mathcal{C}^{WARP}$. Let $\beta_\rho(X) \in X$ denote the best element of $X \in \underline{A}$ at $\rho \in \Pi$, i.e., $\beta_\rho(X)\rho x\ \forall x \in X$. The *leximax extension* is the extension axiom λ^+ defined for each $\rho \in \Pi$ as $\lambda^+(\rho) = \{(X, Y) \in \underline{A} \times \underline{A}\backslash\{X\} : \beta_\rho(X)\rho\beta_\rho(Y)\}$. Similarly, let $\omega_\rho(X) \in X$ satisfy $x\rho\omega_\rho(X)\ \forall x \in X$. The *leximin extension* is the extension axiom λ^- defined for each $\rho \in \Pi$ as $\lambda^-(\rho) = \{(X, Y) \in \underline{A} \times \underline{A}\backslash\{X\} : \omega_\rho(X)\rho\omega_\rho(Y)\}$.[16]

Proposition 3.3 *Given any $\mathcal{D}$ and any $\rho \in \Pi$, we have*

(i) $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)$ *if and only if* $\mathcal{D} = \{C_\rho\}$.

(ii) $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)$ *if and only if* $\mathcal{D} = \{C_\tau\}$ *for* $\tau \in \Pi$ *with* $x\tau y \iff y\rho x\ \forall x, y \in A$.

Proof Take any $\mathcal{D}$ and any $\rho \in \Pi$.

We prove (i). To establish the "if" part, let $\mathcal{D} = \{C_\rho\}$. To see $\varepsilon^{\mathcal{D}}(\rho) \subseteq \lambda^+(\rho)$, take some $(X, Y) \in \varepsilon^{\mathcal{D}}(\rho)$. So $C_\rho\,(X)\,\rho C_\rho\,(Y)$. Moreover, by the definition of C_ρ, we have $C_\rho\,(X) = \beta_\rho(X)$ and $C_\rho\,(X) = \beta_\rho(Y)$, thus, $\beta_\rho(X)\rho\beta_\rho(X)$, showing $(X, Y) \in \lambda^+(\rho)$. To see $\lambda^+(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$, pick some $(X, Y) \in \lambda^+(\rho)$. So $\beta_\rho(X)\rho\beta_\rho(Y)$, thus $C_\rho\,(X)\,\rho C_\rho\,(Y)$, showing $(X, Y) \in \varepsilon^{\mathcal{D}}(\rho)$. To establish the "only if" part, assume $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)$and suppose $\exists C \in \mathcal{D}$ with $C \neq C_\rho$. So, $C(X) \neq C_\rho(X)$ for some $X \in \underline{A}$. Check that $\left(X, \{C_\rho(X)\}\right) \in \lambda^+(\rho)$ but $\left(X, \{C_\rho(X)\}\right) \notin \varepsilon^{\mathcal{D}}(\rho)$, contradicting $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)$.

We prove (ii). To establish the "if" part, let $\mathcal{D} = \{C_\tau\}$ for $\tau \in \Pi$ with $x\tau y \iff y\rho x\ \forall x, y \in A$. To see $\varepsilon^{\mathcal{D}}(\rho) \subseteq \lambda^-(\rho)$, take some $(X, Y) \in \varepsilon^{\mathcal{D}}(\rho)$. So $C_\tau\,(X)\,\rho C_\tau\,(Y)$. Moreover, by the choice of τ, we have $C_\tau\,(X) = \omega_\rho(X)$ and $C_\tau\,(Y) = \omega_\rho(Y)$, thus $\omega_\rho(X)\rho\omega_\rho(Y)$, showing $(X, Y) \in \lambda^-(\rho)$. To see $\lambda^-(\rho) \subseteq \varepsilon^{\mathcal{D}}(\rho)$, pick some $(X, Y) \in \lambda^-(\rho)$. So $\omega_\rho(X)\rho\omega_\rho(Y)$, thus $C_\tau\,(X)\,\rho C_\tau\,(Y)$, showing $(X, Y) \in \varepsilon^{\mathcal{D}}(\rho)$. To establish the "only if" part, assume $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)$ and suppose $\exists C \in \mathcal{D}$ with $C \neq C_\tau$. So, $C(X) \neq C_\tau(X)$ for some $X \in \underline{A}$. Check that $(\{C_\tau(X)\}, X) \in \lambda^-(\rho)$ but $(\{C_\tau(X)\}, X) \notin \varepsilon^{\mathcal{D}}(\rho)$, contradicting $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)$. □

So at a given ρ the leximax ordering $\lambda^+(\rho)$ is induced if and only if $\mathcal{D} = \{C_\rho\}$. Similarly, at a given ρ the leximin ordering $\lambda^-(\rho)$ is induced if and only if $\mathcal{D} = \{C_\tau\}$ such that τ is the opposite ranking of ρ. As a corollary which we state below, there exists no $\mathcal{D}$ which induces leximax (or leximin) orderings at every ρ.

Theorem 3.6 *There exists no $\mathcal{D}$ such that*

(i) $\varepsilon^{\mathcal{D}}(\rho) = \lambda^+(\rho)\quad \forall \rho \in \Pi$ *or*

(ii) $\varepsilon^{\mathcal{D}}(\rho) = \lambda^-(\rho)\quad \forall \rho \in \Pi$.

4 Conclusion

As Barberà et al. (2004) eloquently survey, the literature on extending an order over a set to its power set admits a plethora of extension axioms. Nevertheless, the

[16] Pattanaik and Peleg (1984), Bossert (1995), Campbell and Kelly (2002), Kaymak and Sanver (2003), Dogan and Sanver (2007) explore lexicographic extensions under a variety of definitions.

appropriateness of an extension axiom depends on how elements of the power set are interpreted. We propose a model which incorporates the "non-resolute outcome" interpretation. We show that among the plethora of extension axioms of the literature, two of them—namely the Gärdenfors (1976) and Kelly (1977) principles—arise as the appropriate ones. This observation does not necessarily exclude the use of extension axioms based on "expected utility consistency," as these are essentially equivalent to either the Gärdenfors (1976) or the Kelly (1977) principle, depending on the precise meaning attributed to "expected utility consistency."[17] On the other hand, Theorem 3.5 sets an obstacle in using the separability principle when sets are conceived as non-resolute outcomes.[18]

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[17] One can see Can et al. (2007) for a detailed exploration of this matter.

[18] To be sure, this does not criticize Roth and Sotomayor (1990) who use separability in their manipulation analysis of many-to-one matching rules, as their environments conceive sets as lists of mutually compatible outcomes.

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