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Hierarchies achievable in simple games

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Abstract A previous work by Friedman et al. (Theory and Decision, 61:305–318, 2006) introduces the concept of a hierarchy of a simple voting game and characterizes which hierarchies, induced by the desirability relation, are achievable in linear games. In this paper, we consider the problem of determining all hierarchies, conserving the ordinal equivalence between the Shapley–Shubik and the Penrose–Banzhaf–Coleman power indices, achievable in simple games. It is proved that only four hierarchies are non-achievable in simple games. Moreover, it is also proved that all achievable hierarchies are already obtainable in the class of weakly linear games. Our results prove that given an arbitrary complete pre-ordering defined on a finite set with more than five elements, it is possible to construct a simple game such that the pre-ordering induced by the Shapley–Shubik and the Penrose–Banzhaf–Coleman power indices coincides with the given pre-ordering.

Keywords Simple game · Power index · Desirability · Weak desirability · Linear game · Weakly linear game

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1 Introduction

The concept of a *hierarchy* of a simple game, introduced in Friedman et al. (2006), captures the ordering of the influence held by the voters (or players) in the game. For example, writing that a five-player game G has hierarchy $> = = >$ means that there is one player which has the maximum influence, another one that has the minimum influence and the other three have all the same intermediate influence. A situation where each player has a different amount of influence will be called a strict hierarchy.

Any power index considered in a simple game induces a total ordering on the set of voters, and thus a hierarchy. Two power indices which induce the same hierarchy are said to be *ordinally equivalent*. Previous works by Diffo Lambo and Moulen (2002) and Felsenthal and Machover (1998) show that the Penrose–Banzhaf–Coleman (PBC, henceforth) and the Shapley–Shubik (SS, henceforth) power indices are ordinally equivalent for linear games, i.e., games for which the desirability relation is complete, and that the common induced hierarchy is the one given by the desirability relation.

Carreras and Freixas (2008) introduced weakly linear games, i.e., games for which the weak desirability relation is complete, and they demonstrate that all regular semivalues (see Carreras and Freixas 1999, 2000) are ordinally equivalent for this kind of games and that the common induced hierarchy is the one given by the weak desirability relation. As linear simple games form a subclass of weakly linear simple games, and both the PBC and the SS power indices are regular semivalues, this work extends and generalizes the former ones.

Friedman et al. (2006) characterized all achievable hierarchies in linear simple games (induced by the desirability relation). Precisely, they proved in their main result (Theorem 3 in Friedman et al. (2006) that all hierarchies are achievable in linear games except the types $= = \cdots = = > >$ and $= = \cdots = = > > >$. Furthermore, they proved that all hierarchies achievable in linear games are also achievable in weighted games.

The aim of this paper is two-fold. Firstly, we characterize all achievable hierarchies, induced by the weak desirability relation, in linear simple games and prove that even the hierarchies not achievable in linear simple games are achievable in weakly linear simple games provided that the number of voters is high enough. Secondly, we demonstrate that all hierarchies achievable in simple games are obtainable in weakly linear games. More precisely, we will prove that:

- All hierarchies are achievable in the class of weakly linear games as long as the number of voters is >5.
- All strict hierarchies are achievable in the class of weakly linear games as long as the number of voters is >4.
- Exactly four hierarchies are not achievable in weakly linear games but all of them concern games with <6 voters.
- These four hierarchies are not achievable either in the class of all simple games.

The paper is organized as follows. Basic definitions and preliminary results are included in Sect. 2. Section 3 contains the main theorem of the paper, where it is proved that all hierarchies are achievable in weakly linear games except four of them. In Sect. 4 we prove that all hierarchies obtainable in simple games are obtainable in weakly linear games. Some Conclusions end the paper in Sect. 5.

2 Definitions and preliminaries

In the sequel, $N = \{1, 2, \dots, n\}$ denote a fixed but otherwise arbitrary finite set of *players*. Any subset $S \subseteq N$ is a *coalition*. A cooperative game v (in N, omitted hereafter) is a *simple game* (SG, henceforth) if (a) $v(S) = 0$ or 1 for all S,[1] (b) is monotonic, i.e. $v(S) \le v(T)$ whenever $S \subset T$, and (c) $v(N) = 1$. Either the family of *winning* coalitions $\mathcal{W} = \mathcal{W}(v) = \{S \subseteq N : v(S) = 1\}$ or the subfamily of *minimal* winning coalitions $\mathcal{W}^m = \mathcal{W}^m(v) = \{S \in \mathcal{W} : T \subset S \Rightarrow T \notin \mathcal{W}\}$ determines the game. A simple game is *proper* if for any winning coalition, its complement is not winning. A voter $i \in N$ is null in $\mathcal{W}$ if $i \notin S$ for all $S \in \mathcal{W}^m$. $\mathcal{W}_i$ denotes the set of winning coalitions which contain i. Finally, the *null extension* of game $\mathcal{W}$ for a voter $n + 1$ outside N is the game $\mathcal{W}'$ whose voters belong to $N \cup \{n + 1\}$ and $(\mathcal{W}')^m = \mathcal{W}^m$.

2.1 The desirability relation

Definition 2.1 (*Isbell 1958*) Let v be a simple game and $i, j \in N$. Then

$$\begin{aligned} i \succsim_D j \quad &\text{iff } S \cup \{j\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W} \quad \text{for all } S \subseteq N \backslash \{i, j\}, \\ i \succ_D j \quad &\text{iff } i \succsim_D j \quad \text{and} \quad j \not\succsim_D i, \\ i \approx_D j \quad &\text{iff } i \succsim_D j \quad \text{and} \quad j \succsim_D i. \end{aligned}$$

It is not difficult to check that $\succsim_D$ is a pre-ordering. The relation $\succsim_D$ (resp., $\succ_D$) is called the *desirability* (resp., *strict desirability*) relation, and $\approx_D$ is the *equi–desirability* relation.

Definition 2.2 A simple game v is *linear*[2] whenever the desirability relation $\succsim_D$ is complete.

In a LSG, the hierarchy given by the desirability relation coincides with the hierarchy induced by SS and PBC power indices.

Important examples of LSGs are weighted games. A simple game v is a *weighted game* iff there exist nonnegative *weights* $w_1, w_2, \dots, w_n$ allocated to the players and a *quota* $q \in (0, \sum_{i \in N} w_i]$ such that $S \in \mathcal{W}$ iff $\sum_{i \in S} w_i \ge q$. We then write $v \equiv [q; w_1, w_2, \dots, w_n]$. Any weighted game is linear because $w_i \ge w_j$ implies $i \succsim_D j$.

[1] For a detailed discussion of some issues raised by allowing abstentions, see Felsenthal and Machover (1998) and for several levels of approval in input and output, see Freixas and Zwicker (2003), Freixas (2005a, b).

[2] Linear games are also called complete, ordered or directed games in the literature, see Taylor and Zwicker (1999) (henceforth, LSG) for references on these names.

LSGs have been widely studied. Taylor and Zwicker (1993, 1999) gave respective characterizations of weighted games and linear games in terms of trades among players and within coalitions. They stated the following result.

Theorem 2.3 (Taylor and Zwicker 1999) *Let v be a simple game. Then*

(a) *v is a weighted game iff it is trade robust* (Theorem 2.4.2, p. 57).
(b) *v is a linear game iff it is swap robust* (Proposition 3.2.6, p. 90).

In Carreras and Freixas (1996) an existence, uniqueness and classification theorem for LSGs was provided that enables us to enumerate all these games up to isomorphism. This theorem supplies also an alternative efficient way to determine, in terms of a set of inequalities, which LSGs are weighted. For instance, all simple games with $n \leq 3$ players are weighted and hence linear. For $n = 4$ and $n = 5$ there exist non-weighted games, but none of them is linear. Linear games that are not weighted arise only for $n \geq 6$.

2.2 The weak desirability relation

Given a simple game v, let us define, for each $i \in N$ and $1 \leq k \leq n$,

$$\mathcal{C}_i = \{S \in \mathcal{W} : S \backslash \{i\} \notin \mathcal{W}\} \quad \text{and} \quad \mathcal{C}_i(k) = \{S \in \mathcal{C}_i : |S| = k\}.$$

$\mathcal{C}_i$ is the set of winning coalitions S for which i is *crucial*, while $\mathcal{C}_i(k)$ is the subset of such coalitions having cardinality k.

Definition 2.4 (*Carreras and Freixas 2008*) Let v be a simple game and $i, j \in N$. Then

$$\begin{array}{lll} i \succsim_d j & \text{iff } |C_i(k)| \geq |C_j(k)| & \text{for all} \quad k = 1, 2, \ldots, n, \\ i \succ_d j & \text{iff } i \succsim_d j \quad \text{and} \quad j \not\succsim_d i, & \\ i \approx_d j & \text{iff } i \succsim_d j \quad \text{and} \quad j \succsim_d i. & \end{array}$$

Then $\succsim_d$ is a pre-ordering called the *weak desirability* relation. The relation $\succ_d$ is the *strict weak desirability* relation and $\approx_d$ is the *weak equi-desirability* relation.

In Diffo Lambo and Moulen (2002) it is proved that the desirability relation is a sub-preordering of the weak desirability relation, that is to say, for any $i, j \in N$, $i \succsim_D j$ implies $i \succsim_d j$ and $i \succ_D j$ implies $i \succ_d j$.

Definition 2.5 (*Carreras and Freixas 2008*) A simple game v is *weakly linear* (WLSG, henceforth) whenever the weak desirability relation $\succsim_d$ is complete.

In a WLSG, the hierarchy given by the weak desirability relation coincides with the hierarchy induced by SS and PBC power indices.

As stated in Carreras and Freixas (2008), the completeness of the desirability relation $\succsim_D$ implies the completeness of the weak desirability relation $\succsim_d$ so that

Table 1 Weakly linear but non-linear simple games of four voters

Game number	Minimal winning coalitions	Hierarchy
1	12, 34	===
2	12, 13, 24, 34	===
3	12, 13, 24	=>=

all linear games are also weakly linear. Moreover, if v is a LSG then v is WLSG and the desirability relation $\succsim_D$ and the weak desirability relation $\succsim_d$ coincide.

There are WLSGs that are not linear. For instance, all simple games with $n \leq 3$ players are WLSGs, but for $n = 4$ there exist 3 non-isomorphic WLSGs, but not LSGs (see Table 1).[3]

2.3 Hierarchies

The last comments in the previous subsection allow us to state that for LSGs the hierarchy induced by the desirability relation and the one induced by the weak desirability relation coincide. But WLSGs form a larger class in which other hierarchies are possible. The notation to describe the different possible hierarchies is stated in the following.

Definition 2.6 A WLSG with $1 \succsim_d 2 \succsim_d \cdots \succsim_d n$ is said to have the hierarchy $r_1 r_2 \ldots r_{n-1}$ if each r_i is either $>$ or $=$ depending on whether $i \succ_d i+1$ or $i \approx_d i+1$ respectively.

The following example shows a WLSG but not LSG.

Example 2.7 Let $N = \{1, 2, 3, 4\}$ and let v be the game defined by

$$\mathcal{W}^m = \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}.$$

This game is not linear because the desirability relation only gives:

$$1 \succ_D 3 \quad \text{and} \quad 2 \succ_D 4,$$

while the weak desirability relation gives:

$$1 \approx_d 2 \succ_d 3 \approx_d 4.$$

Thus, the game is weakly linear and induces the hierarchy $=>=$.

It is straightforward to see that if a WLSG does not satisfy condition $1 \succsim_d 2 \succsim_d \cdots \succsim_d n$ then there is an isomorphic WLSG with this ordering. Thus we only need to consider, hereafter, WLSG s with ordering $1 \succsim_d 2 \succsim_d \cdots \succsim_d n$ as is assumed in Definition 2.6.

[3] For the sake of simplicity we have omitted commas and brackets in the set of minimal winning coalitions in Tables 1 and 2.

3 Achievable hierarchies in weakly linear games

A hierarchy is said to be achievable in a WLSG if there exist a game of this type which has this hierarchy. The aim of this section is to show that for n large enough all hierarchies are achievable in a WLSG. Precisely, we prove that for $n > 5$ all hierarchies are achievable.

Theorem 3.1 *All hierarchies are achievable in a WLSG except*:

$$>>, \quad >>>, \quad =>> \quad \textit{and} \quad =>>>.$$

Proof Theorem 3 in Friedman et al. (2006) guarantees that all hierarchies except $== \cdots ==>>$ and $== \cdots ==>>>$ are achievable in LSGs and because every LSG is a WLSG, all hierarchies achievable in LSGs are also achievable in WLSGs. Hence, we only need to show now the existence of WLSGs with hierarchies: $== \cdots ==>>$ and $== \cdots ==>>>$.

Note that if the hierarchy $== \cdots ==>>$ is achieved in a WLSG with n players without null voters then the hierarchy $== \cdots ==>>>$ is achieved in the *null extension* of this game for a voter $n+1$, because that voter $n+1$ is strictly smaller than any other player by the weak desirability relation.

Now, let us construct a (proper) game G_n for every $n > 4$ with hierarchy $== \cdots ==>>$ and without null voters. The minimal winning coalitions for G_n are defined as follows:

$$\begin{aligned} S_i &= N \setminus \{n-i, n\} && \text{for } i = 1, 2, \ldots, \lfloor \tfrac{n}{2} \rfloor + 1, \\ S_i &= N \setminus \{n-i, n-1\} && \text{for } i = \lfloor \tfrac{n}{2} \rfloor + 2, \ldots, n-1. \end{aligned}$$

To check that G_n is, in fact, a WLSG we need to consider the n^2 numbers $|\mathcal{C}_i(k)|$ for any voter i and any positive integer $k \leq n$. We will distinguish several cases:

(i) Assume $k < n-2$.
Then all coalition with $|S| = k$ is losing so that $|\mathcal{C}_i(k)| = 0$ for all voter i.

(ii) Assume $k = n$.
The grand coalition, N, is winning but any coalition of cardinality $k-1$ is also winning. Thus, $|\mathcal{C}_i(n)| = 0$ for all voter i.

(iii) Assume $k = n-2$.
Then for all voter i, $|\mathcal{C}_i(n-2)| = |\mathcal{W}_i^m|$ because all coalitions with cardinality $n-3$ are losing in G_n. We need to count

$$|\mathcal{C}_i(n-2)| = |\mathcal{W}_i^m| = \begin{cases} n-2, & \text{if } i = 1, 2, \ldots, n-2; \\ \lfloor \frac{n}{2} \rfloor, & \text{if } i = n-1; \\ \lfloor \frac{n-1}{2} \rfloor - 1, & \text{if } i = n. \end{cases}$$

which implies:

$$|\mathcal{C}_1(n-2)| = |\mathcal{C}_2(n-2)| = \cdots = |\mathcal{C}_{n-2}(n-2)| > |\mathcal{C}_{n-1}(n-2)| > |\mathcal{C}_n(n-2)|$$

(iv) Assume $k = n - 1$.
First, let us consider the desirability relation for G_n. The strict relation arises only for:

$$i \succ_D n-1, \quad i \succ_D n$$

where $i = 1, 2, \ldots, n-2$. Further, the equi-desirability relation

$$i \approx_D j$$

only arises for $i \le \lfloor \frac{n-3}{2} \rfloor$ and $j \le \lfloor \frac{n-3}{2} \rfloor$ or for $\lfloor \frac{n-3}{2} \rfloor < i < n-1$ and $\lfloor \frac{n-3}{2} \rfloor < j < n-1$. That is to say, game G_n has four equi-desirability classes:

$$N_1 = \left\{1, \ldots, \lfloor \frac{n-3}{2} \rfloor \right\}, N_2 = \left\{ \lfloor \frac{n-3}{2} \rfloor + 1, \ldots, n-2 \right\},$$
$$N_3 = \{n-1\}, N_4 = \{n\}.$$

Hence, hereafter a single representative for each class can be taken. We choose voter 1 for N_1 and voter $n-2$ for N_2. We can compute the sets

$$\begin{aligned}
\mathcal{C}_1(n-1) &= \{S \in \mathcal{W}^m \ : \ |S| = n-1, \ 1 \in S, \ S \neq N \setminus \{n-1\}\} \\
\mathcal{C}_{n-2}(n-1) &= \{S \in \mathcal{W}^m \ : \ |S| = n-1, \ n-1 \in S, \ S \neq N \setminus \{n\}\} \\
\mathcal{C}_{n-1}(n-1) &= \{S \in \mathcal{W}^m \ : \ S \cup \{n\}, \ S \in \mathcal{C}_{n-1}(n-2)\} \\
\mathcal{C}_n(n-1) &= \{S \in \mathcal{W}^m \ : \ S \cup \{n-1\}, \ S \in \mathcal{C}_n(n-2)\}
\end{aligned}$$

and thus $|\mathcal{C}_1(n-1)| = |\mathcal{C}_{n-2}(n-1)| = n-2$, $|\mathcal{C}_{n-1}(n-1)| = |\mathcal{C}_{n-1}(n-2)|$, $|\mathcal{C}_n (n-1)| = |\mathcal{C}_n(n-2)|$. Therefore,

$$|\mathcal{C}_i(n-1)| = |\mathcal{C}_i(n-2)| = |\mathcal{W}_i^m| = \begin{cases} n-2, & \text{if } i = 1, 2, \ldots, n-2; \\ \lfloor \frac{n}{2} \rfloor, & \text{if } i = n-1; \\ \lfloor \frac{n-1}{2} \rfloor - 1, & \text{if } i = n. \end{cases}$$

which implies:

$$|\mathcal{C}_1(n-1)| = |\mathcal{C}_2(n-1)| = \cdots = |\mathcal{C}_{n-2}(n-1)| > |\mathcal{C}_{n-1}(n-1)| > |\mathcal{C}_n(n-1)|$$

Thus, it has been proved that $1 \approx_d 2 \approx_d \cdots \approx_d n-2 \succ_d n-1 \succ_d n$, that is to say, the hierarchy for G_n is $== \cdots =>>$.

To conclude the proof we need to show that the four hierarchies:

$$>>, \quad >>>, \quad =>> \quad \text{and} \quad =>>>$$

are not achievable. The smallness of the number of non-isomorphic simple games for 3, 4 or 5 voters allows us to prove that these hierarchies never appear, by making an exhaustive checking.

Indeed, for $n = 3$ there are 8 simple games (see Table 2 for a full classification of simple games with 3 voters) which are all weighted, and thus linear and weakly linear, but none of them has hierarchy $>>$. For $n = 4$ there are 28 simple games (see Table 3 for a full classification of simple games with 4 voters), 25 of which are weighted, and so linear and weakly linear, and the 3 remaining ones (see Table 1) are weakly linear but not linear. None of these games has either the hierarchy $>>>$ or $=>>$. Finally, there are 202 simple games for $n = 5$ (see Table 4 for a full classification of simple games with 5 voters). These games can be classified into three mutually excluding groups: 117 of them are weighted, and so linear and weakly linear, 68 are weakly linear but not linear, and 17 are not weakly linear. None of these 68 non-linear but weakly linear games have the hierarchy $=>>>$. This concludes the proof. □

4 Achievable hierarchies for simple games

The last question we study in this paper is: Which hierarchies are achievable in the class of all simple games?

The class of simple games considered so far is the class of weakly linear simple games. In these games, the SS and PBC power indices are ordinally equivalent, and Theorem 3.1 shows all possible hierarchies they induce on the set of voters.

Let us consider now the largest class of simple games in which the concept of hierarchy still makes sense, i.e., the class formed by simple games with ordinally equivalent SS and PBC power indices.

Definition 4.1 A simple game v is *coherent* (CSG, henceforth) whenever its power indices SS and PBC are ordinally equivalent.

Table 2 Frequency of each hierarchy for three voters

Hierarchy	Number of LSGs
$>>$	0
$>=$	3
$=>$	2
$==$	3
Total	8

Table 3 Frequency of each hierarchy for four voters

Hierarchy	Number of LSGs	Number of WLSGs
$>>>$	0	0
$>>=$	2	2
$>=>$	4	4
$=>>$	0	0
$>==$	6	6
$=>=$	6	7
$==>$	3	3
$===$	4	6
Totals	25	28

Table 4 Frequency of each hierarchy for five voters

Hierarchy	Number of LSGs	Number of WLSGs	Number of CSGs
>>>>	2	4	4
>>>=	6	8	8
>>=>	12	14	15
>=>>	6	8	10
=>>>	0	0	0
>>==	8	14	16
>=>=	19	25	29
>==>	11	13	15
=>>=	4	10	11
=>=>	8	19	19
==>>	0	2	2
>===	10	20	23
=>==	13	17	17
==>=	9	11	13
===>	4	13	13
====	5	7	7
Totals	117	185	202

Table 5 Coherent but non–weakly linear simple games of five voters

Game number	Minimal winning coalitions	Hierarchy
1	12, 13, 14, 235, 245, 345	>===
2	12, 13, 14, 235, 245	>>==
3	12, 13, 14, 235	>=>=
4	12, 13, 145, 234, 245, 345	>==>
5	12, 13, 145, 234, 245	>>=>
6	12, 13, 145, 245, 345	>===
7	12, 13, 145, 245	>>==
8	12, 13, 234, 235, 245, 345	==>=
9	12, 13, 234, 235, 245	=>>=
10	12, 14, 15, 23, 345	>=>=
11	12, 14, 23, 135, 345	>=>>
12	12, 134, 135, 145, 234, 235, 345	>=>=
13	12, 134, 135, 145, 234, 345	>==>
14	12, 134, 135, 145, 345	>===
15	12, 134, 135, 234, 235, 345	==>=
16	12, 134, 135, 234, 345	>=>>
17	12, 134, 135, 345	>=>=

LSGs, and WLSGs are coherent, but there are CSGs which are neither linear nor weakly linear (see Table 5).

Definition 4.2 Let v be a simple game, $\varphi[v]$ and $\beta[v]$ be its SS and PBC power indices, respectively, and $i, j \in N$. Then

$$\begin{aligned} i \succsim_\delta j \quad &\text{iff } \varphi_i[v] \geq \varphi_j[v] \quad \text{and} \quad \beta_i[v] \geq \beta_j[v], \quad \text{for all} \quad S \subseteq N \setminus \{i, j\}, \\ i \succ_\delta j \quad &\text{iff } \varphi_i[v] > \varphi_j[v] \quad \text{and} \quad \beta_i[v] > \beta_j[v], \\ i \approx_\delta j \quad &\text{iff } \varphi_i[v] = \varphi_j[v] \quad \text{and} \quad \beta_i[v] = \beta_j[v]. \end{aligned}$$

It is not difficult to check that $\succsim_\delta$ is a pre-ordering. The relation $\succsim_\delta$ (resp., $\succ_\delta$) is called the *weakest* (resp., *strict weakest*) desirability relation, and $\approx_\delta$ is the *weakest equi-desirability* relation.

Note that a simple game is coherent if and only if the weakest desirability relation is complete. In this case, the hierarchy induced by the weakest desirability relation coincides with the hierarchy induced by the SS and PBC power indices.

Theorem 4.3 *A hierarchy is achievable in a CSG if and only if it is achievable in a WLSG.*

Proof Because every WLSG is a CSG, all hierarchies achievable in a WLSG are also achievable in a CSG.

By Theorem 3.1 any hierarchy is achievable in a WLSG except:

$$>>, \quad >>>, \quad =>> \quad \text{and} \quad =>>>.$$

The first three hierarchies are not achievable in a CSG because all simple games of three or four voters are weakly linear. The $=>>>$ hierarchy is not achievable in a CSG either, because there are 17 games of 5 voters which are coherent but not weakly linear, and none of them has this hierarchy, as is shown in Table 5. □

5 Conclusion

Our paper complements and enforces the hierarchy theory for simple games initiated by Friedman et al. (2006) and continued by Bean et al. (2007). Indeed, in Friedman et al. (2006) it is proved that LSGs and, particularly, weighted simple games show many different hierarchies, although two sequences of hierarchies are never available. Moreover, in Bean et al. (2007), it is proved that simple majority weighted games are not enough to get all the achievable hierarchies, even though a modification on them is sharp.

In this paper it is proved that *all* hierarchies are achieved in a WLSG as long as the number of voters is greater or equal than 6. For <6 voters, only four hierarchies are not achieved in this class of games, and these are:

$$>>, \quad >>>, \quad =>> \quad \text{and} \quad =>>>.$$

But none of these four hierarchies is achieved either in any other kind of simple games. Thus, we may assert that all hierarchies achievable in a simple game are achievable in a WLSG.

As a consequence of these results we can state that, given any complete pre-ordering defined on a finite set (with more than five elements), it is possible to construct a simple game such that the pre-orderings induced by the SS and the PBC power indices coincide with the given pre-ordering.

The results obtained in this paper suggest the following question for future research. Given an arbitrary partial pre-ordering defined on a finite set, is it possible to construct a simple game on this set such that the weak desirability relation coincide with the given pre-ordering?

It would be worth solving the above problem so that we encourage future research about the formulated question.

Appendix

The latest part in the proof of Theorem 3.1 is based on the exhaustive calculation of the hierarchies considered in this paper for less than six voters. We make in this appendix some comments on the way we obtained the tables. Notice that the frequency of each hierarchy, shown in Tables 2, 3 and 4, is the number of non-isomorphic games that have this hierarchy.

In the case of LSGs, a classification theorem given in (Carreras and Freixas 1996, Theorem 4.1) allows to generate and count up to isomorphism the number of LSGs for small values of the number of voters n. A vector $\overline{n}$ with t positive integer components representing the cardinalities of the equi-desirability classes ordered from the strongest class to the weakest one, and a matrix $\mathcal{M}$ with non-negative integer entries is associated to every LSG. In this way, a vector like for example $\overline{n} = (3, 4, 2)$ would lead to the hierarchy $==>===>=$. The four conditions stated for vector $\overline{n}$ and matrix $\mathcal{M}$ in that theorem allow to get all achievable hierarchies for LSGs. We used that theorem for determining the frequencies for every hierarchy, for LSGs.

To get the results for WLSGs and for CSGs we need two steps. Firstly, we generate all non-isomorphic simple games. Secondly, for all of them, we count the n^2 numbers $|\mathcal{C}_i(k)|$ for each $i \in N$ and $1 \leq k \leq n$, and we check whether the simple game is weakly linear. If it is not weakly linear, then we check whether this simple game is coherent.

Looking at the results on the tables it is worth summarizing the following:

- All SG with less than four voters are weighted and thus LSG, WLSG and CSG (see Table 2).
- For $n = 4$ there are three non–isomorphic simple games which are not linear, but all of them are weakly linear and, thus, coherent (see Table 3). These three simple games are shown in Table 1, defined by its set of minimal winning coalitions. None of these three games shows the hierarchies $>>>$ or $=>>$.
- For $n = 5$ there are seventeen non–isomorphic simple games which are not weakly linear, but all of them are coherent (see Table 4). However, none of these 17 games (shown in Table 5) has the hierarchy: $=>>>$.

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