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Valuing future cash flows with non separable discount factors and non additive subjective measures: conditional Choquet capacities on time and on uncertainty

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Abstract We consider future cash flows that are contingent both on dates in time and on uncertain states. The decision maker (DM) values the cash flows according to its decision criterion: Here, the payoffs' expectation with respect to a capacity measure. The subjective measure grasps the DM's behaviour in front of the future, in the spirit of de Finetti's (1930) and of Yaari's (1987) Dual Theory in the case of risk. Decomposition of the criterion into two criteria that represent the DM's preferences on uncertain payoffs and time contingent payoffs are derived from Ghirardato's (1997) results. Conditional Choquet integrals are defined by dynamic consistency (DC) requirements and conditional capacities are derived, under some conditions on information. In contrast with other models referring to DC, ours does not collapse into a linear one because it violates a weak version of consequentialism.

Keywords Capacities · Comonotonicity · Conditional Choquet integrals · Conditional capacities · Discount factors · Information · Product spaces · Subjective measures

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1 Introduction

In order to decide on a public project, a private investment or any bet on future uncertain cash flows[1], a Decision Maker (DM) needs a present certainty equivalent, i.e. a present value, which can be compared to the costs. Such a present equivalent can be justified by a criterion representing the DM's behaviour. As most projects include flexibilities, it must be the case that the criterion takes them into account, and hence that the present value integrates the future option values. An option will be exercised or not depending on information arrivals in the future.[2] However, information may modify the DM's behaviour and hence its future payoffs valuation conditional on information. Consistence of the present value and of the future conditional values is central to the results in this paper (dynamic consistency).

We assume the DM's behaviour is represented by a non additive subjective expected utility (Choquet Expected Utility, Schmeidler (1989)). In the spirit of de Finetti's (1930) subjective measure and of Yaari's (1987) Dual Theory in the case of risk, we concentrate on the special case where the utility is the identity (Chateauneuf (1991), but here we rely on the simpler Diecidue and Wakker (2002) model). In these models the decision criterion is the future payoffs present value, as well as their present certainty equivalent. Furthermore, the DM's behaviour is completely grasped by the subjective measure: here a Choquet capacity. As a first consistency requirement we assume the DM assumes its future behaviours conditional on information arrivals will satisfy the same axioms as the present one (model consistency). The problem is then to condition Choquet integrals and capacities in a way that is consistent with the present value.

Conditioning capacities are problematic: The pioneer's work of Dempster (1967) and Shafer (1967) who presented the first formula (besides Bayes') opened the way to many researches among which we can cite (forgetting many): Fagin and Halpern (1990) who introduced another rule (often called the Full Bayesian Updating Rule) axiomatised by Jaffray (1992), Gilboa and Schmeidler (1993) who compared the Bayesian and the Dempster-Shafer rule, etc.

Denneberg (1994) deduced updating rules (the Full Bayesian Updating rule or the Bayesian rule, depending on the assumptions) from an implicit definition of the conditional Choquet integrals. He followed the usual (implicit) definition of the conditional Lebesgue integrals (Expectations) in classical measure theory handbooks from which Bayes' rule for probabilities is deduced. Let $E^Y(X)$ be the expectation of a measurable function X conditional on information Y, Denneberg chose the definition: $E[E^Y(X) - X] = 0$ that is equivalent in the linear case to the more usual definition: $E(X) = E[E^Y(X)]$. Each of these formulas yields a different (implicit) definition of the conditional Choquet integral, and, depending on the case, different

[1] We assume consequences are expressed in monetary terms in this paper.

[2] The decision maker is not necessarily a person in this paper, it could be any neutral entity, e.g. an administration.

updating rules (see Kast et al. (2007) for a complete presentation of these results). The previous defining equations impose a consistency between conditional integrals and non-conditional ones.

Conversely, in the axiomatic approach we follow in this paper, the consistency condition (an axiom on the DM's preferences) yields an equation that implicitly defines the conditional Choquet integral. The condition is usually called "Dynamic Consistency" (DC), e.g. in Karni and Schmeidler (1991). Another condition was dubbed "Consequentialism" (C) by Hammond (1989). We know that under the general formulations of the two axioms (DC) and (C), the preference representation criterion is linear (see Sarin and Wakker (1998); Machina (1998); Ghirardato (2002); Lapied and Toqueoeuf (2007) and the relevant literature) and the measures degenerate into additive or quasi-additive ones after several iterations. In this paper, we obtain a different result: non additive updated capacities that do not collapse to additive ones. This is because we weaken Consequentialism in the following sense: As usual, counterfactual events are given a zero measure, but payoffs that would be obtained if these events realised still interfere with the valuation because they may modify the payoffs' ranking.[3] Furthermore, this paper departs from others where, even though it is invoked, DC cannot play its full role because the models themselves are not really dynamical. Indeed, in most models, the set of future states and the information that may arrive "later" are left without reference to any real timing. In the practice of managing an investment or a project and calculating its ex-ante value, the timing of decision-making is crucial: Decisions will be taken at some future dates in accordance with the then available information, e.g. information may induce options to be exercised. Obviously, in a cash flow payoffs are contingent on future dates as well as on uncertain events. Both contingencies have to be taken into account by the decision criterion that bears on decision sequences conditional on future information arrivals, i.e. in the DM's present valuation of the cash flow they generate.

Time is indeed relevant for dynamic decision-making. It is economically measured by discount factors: whether market ones when a market for riskless bonds exists, or individual ones (preference for present over future consumption). Koopmans (1972) gave seminal conditions for this valuation to be additive (time separability). Then, Gilboa (1989) extended the model to the non additive case. Notice that Koopman and Gilboa, as well as all their followers (notably Chateauneuf and Rébillé (2004)) exclude uncertainty considerations.

However, in general, future payoffs are also contingent on states (events) of uncertainty. Here again, axioms yielding additive properties to the representation of preferences were extended by Schmeidler (1989) among others to the non additive case (e.g. Choquet Expected Utility).

In order to take the two contingencies into account in this paper, we consider cash flows (contingent on future states in a set Ω) as being contingent on two factors: uncertain states and future dates. Formally, we write: $\Omega = S \times T$, where S stands for the set of uncertain States and T represent the set of dates in the future Time under consideration. Preferences of the DM could be defined on payoffs contingent

[3] Ranking is fundamental to Choquet integrals, e.g. Rank Dependent Expected Utility.

on future states in Ω, but most of the time they are better known on uncertain payoffs (real functions from S) and on date contingent payoffs (real functions from T). With linear value functions (functionals on real functions from S, T and Ω), present value is unambiguously the discounted expected payoffs.[4] However if, as we assume, the decision criterion may be a non-linear value function, it is not clear how to construct the present value of a cash flow $X : \Omega = S \times T \rightarrow R$ from the value of uncertain cash payoffs and the value of time contingent payoffs separately. Clarifying this point is the first step of this paper; the second steps yield dynamically consistent conditional measures on Uncertainty and on Time.

In Sect. 2 of the paper, we specify our model: representation of preferences, information, and the Ghirardato-Fubini theorem (Ghirardato 1997) on product spaces. We concentrate in Sect. 3 on the conditioning of capacities on uncertain states and in Sect. 4 on the conditioning of capacities on dates (discount factors). In both cases, we check that consequentialism is violated and that capacities do not collapse into additive measures.

2 The model

We consider that a payoff is a measurable function X: $\Omega = S \times T \rightarrow R_+$ where $S = \{s_1, \ldots, s_N\}$ represents the set of uncertain states to whom the payoffs are contingent and $T = \{1, \ldots, T\}$ the set of future dates, both with the sets of parts, 2^S and 2^T, as algebras. Obviously, a project may have negative payoffs and uncertainty and time may not be perceived as finite sets, we restrict the problem to this simple case in order to concentrate on the principles of dynamic valuation, i.e. consistency of preferences with information arrivals.

2.1 Representation of preferences

Given we consider finite spaces, we can refer to a simple representation of preferences model, namely the generalisation (for finite sets) of de Finetti's (1930) axioms by Diecidue and Wakker (2002).[5] Notice that in these models (as well as in Yaari's (1987) Dual Theory) the decision criterion is an expected value (Lebesgue or Choquet integral) with respect to a subjective measure that represents the DM's behaviour. The DM's attitude toward the future payoffs is completely grasped by the measure, and its decision criterion is defined by a cash amount (present certain value) such that the DM is indifferent between this present cash amount and the cash flow.

Following Diecidue and Wakker (2002), we need the following definitions.

[4] Or the discounted expected utility in more general models. Notice however that discounted expected utility is not necessary equal to expected utility of discounted payoffs, so that the same problem as the one we address here is posed.

[5] In a more general setting, we could refer to Chateauneuf's (1991) model, for instance.

Definition 2.1.1 Two measurable functions X and Y on a set of states E are comonotonic if and only if, for any two states e and e': $[X(e) - X(e')][Y(e) - Y(e')] \geq 0$.

Definition 2.1.2 A comonotonic set of functions is such that all functions in this set are two by two comonotonic (notice that in R^m a comonotonic set is a positive cone generated by m linearly independent comonotonic characteristic functions).

Definition 2.1.3 A Book is a finite sequence of preferences between two measurable functions (bets' or assets' cash flows, for example): (X_i), (Y_i), $i = 1, \ldots, N$, such that each X_i is weakly preferred to the corresponding Y_i.

Definition 2.1.4 A comonotonic Book is formed of functions belonging to the same comonotonic set.

Definition 2.1.5 A Dutch Book is a Book such that: $\sum_{i=1}^{N} X_i < \sum_{i=1}^{N} Y_i$.

A Dutch Book exhibits incoherence between preferences and monotony. Now we can state the three basic axioms that yield the DM's preference representation.

Axiom 1 Preferences define a complete pre-order on the set of measurable functions.

Axiom 2 For any measurable function, there exists a constant number (constant equivalent) for which the DM is indifferent to the function.

Axiom 3 Preferences allow no comonotonic Dutch Books.

Theorem 2.1.1 (Diecidue and Wakker 2002) *For a preference relation on R^m satisfying axioms 1 and 2, for all X in R^m there exists a constant equivalent $CE(X) \in R$ such that the following three statements are equivalent:*

(*i*) *$CE(.)$: $R^m \to R$ is strictly monotonic, additive on comonotonic vectors (but non necessarily additive on non comonotonic vectors).*
(*ii*) *There exists a unique capacity such that CE(X) is the integral of X with respect to this measure.*
(*iii*) *CE(.) is such that axiom 3 is satisfied.*

Axiom 3 can be replaced by the stronger de Finetti's (1930) coherence axiom:

Axiom 3′ Preferences allow no Dutch Books.

Then a special case of Theorem 2.1 obtains with *CE(.)* additive and a probability as an additive (modular) capacity.

The proof of the theorem mainly relies on Diecidue and Wakker's result: the No Comonotonic Dutch Books axiom implies strict monotonicity of the constant equivalent.

The representation theorem yields the three value functions that we need in order to represent preferences over the future: Ω. With X: $\Omega = S \times T \to R_+$, the constant equivalent of X defined by the theorem is:

$$CE(X) \equiv V(X) = \int_{\Omega} X d\Psi \text{ where } \Psi \text{ is a capacity, and we shall note:}$$

$$V(X) = \int_{\Omega} X \; dP \text{ if } P \text{ is additive.}$$

$V(X)$ is a present certainty equivalent of X.

Obviously, Ψ defines two marginal capacities: ν (or μ if it is additive) on R^S and ρ (or π if it is additive) on R^T. From the previous representation theorem, we know that these measures represent the DM's preferences over R^S and R^T that satisfy the same axioms as its preferences on R^{Ω}. The representations yield two constant equivalents.

The certainty equivalent of uncertain payoff (E for expected) is:

$$\forall \zeta\colon R^S \to R_+,\ E(\zeta) = \int_S \zeta(s)\, d\nu(s) \text{ if } \nu \text{ is a capacity,}$$

$$E(\zeta) = \int_S \zeta(s)\, d\mu(s) \text{ if } \mu \text{ is additive.}$$

The present equivalent of date contingent payoffs ($D(.)$ for discounted) is:

$$\forall \xi\colon R^T \to R_+,\ D(\xi) = \int_T \xi(t)\, d\rho(t) \text{ if } \rho \text{ is a capacity,}$$

$$D(\xi) = \int_T \xi(t)\, d\pi(t) \text{ if } \pi \text{ is additive.}$$

In most economic models these two representations are assumed to be known and the problem is to define a representation of preferences over $R^{S\times T}$ that is consistent with the previous ones. Two obvious candidates are: $\forall X\colon \Omega = S \times T \to R_+$,

$$D[E(X)] = \int_T \left[\int_S X(s,t) d\nu(s) \right] d\rho(t) \text{ (Discounted Expectation)}$$

and:

$$E[D(X)] = \int_S \left[\int_T X(s,t) d\rho(t) \right] d\nu(s) \text{ (Expected Discounting).}$$

In the special case where de Finetti's coherence axiom (axiom 3′) is satisfied, the cash flows' valuation representing the DM's preferences is unambiguously the (subjective) present certainty equivalent. Indeed, in this case we have:

$$V(X) = D[E(X)] = \int_T \left[\int_S X(s,t) d\mu(s) \right] d\pi(t)$$
$$= \int_S \left[\int_T X(s,t) d\pi(t) \right] d\mu(s) = E[D(X)].$$

The equalities are obtained because Fubini's theorem applies to Lebesgue integrals with respect to additive measures.

However, it is not the case that the two candidates yield the same result if the measures are not additive because Fubini's theorem does not apply. This is why, in Sect. 2.3, we shall invoke the Ghirardato-Fubini theorem that will allow us to construct V as whether DE or ED and investigate separately the effect of information arrivals on E and on D.

Integrating informational values in the linear valuation of a cash flow is straightforward: If some information arrives at some date τ, it is valued at that date by the conditional valuation, say V^τ, and the original cash flow $X = (X_1, \ldots, X_T)$ is indifferent to the cash flow $(X_1, \ldots, X_{\tau-1}, \ V^\tau \ (X), 0, \ \ldots \ , 0)$. Then, the later cash flow can be discounted under the usual conditions.

The aim of this paper is to extend this result, as far as it can be done, to non-linear valuations.

2.2 Information

Taking into account future flexibilities and options in an investment or a project, amounts to integrate the value of the options into the project's present value. An option is exercised or not according to information arrivals of the type $[Y = i]$, where $i \in I$, here a finite set of information values, and Y is a measurable function on Ω. Indeed, when information $[Y = i]$ is obtained, the DM may anticipate it and will modify its preferences over the project's payoffs and hence its valuation. For instance, its aversion to uncertainty (convex capacity) may be reduced or increased depending on the type of information ("good" or "bad" news). Or its preferences for present consumption may change if it learns it has more wealth available.

In the following we shall concentrate on the usual type of information, i.e. information at a given future date bearing on uncertain states.

Let us consider a filtration on 2^S: $F = \{F_0, \ldots, F_T\}$ with $F_0 = \{\emptyset, S\} \subset F_1 \subset \ldots \subset F_T = 2^S$.

Information at date $t = 1, \ldots, T$ is given by an F_t-measurable function Y_t on S that defines a partition I_t ofF_t with elements $[Y_t = i_t]$. In order to lighten notations, let $M(t) = \#I_t$ and, for $j = 1, \ldots, M(t)$, $i_t^j = \{s \in S;\ Y_t(s) \in i_t^j\}$ so that $I_t = (i_t^1, \ldots, i_t^{M(t)})$. We assume preferences and conditional preferences satisfy the following axiom proposed by Sarin and Wakker (1998):

Axiom 4 (Model Consistency) Preferences on uncertain payoffs and preferences conditional on information satisfy the same axioms. (MC)

In our case: Preferences conditional on information Y_t are represented, respectively, by: ${V^i}_t$, ${E^i}_t$ and ${D^i}_t \equiv D^t$ that are Choquet integrals with respect to capacities: ${\Psi^i}_t$, ${\nu^i}_t$ and ${\rho^i}_t \equiv \rho^t$ on Ω , S and T.

The conditional Choquet integrals (and the corresponding conditional capacities) have to be defined, at least implicitly, from the unconditional ones by some (dynamic) consistency requirements. Consistency between valuations before and after information arrivals can be questioned this way: If, for some i_t, ${V^i}_t(X) \geq {V^i}_t(X')$ can we have: $V(X) < V(X')$? (We drop the time index in what follows).

The answer is yes, there are cases where we could have $V(X) \geq V(X')$. For instance, assume the set $[Y = i]$ excludes the set on which $X < X'$, so that $X \geq X'$ on any set in $\sigma([Y = i])$ the σ-algebra generated by Y^{-1}. Then, if preferences are monotonic we could have a contradiction between unconditional and conditional valuations.[6] However, we need not have one because all the i's are possible and the DM may still take into account payoffs for which $X < X'$ and then not prefer X to X'. If, for all i's, we had $X = X'$ on $[Y = i]^c$, then, consistency with information arrivals would imply that:

$$\forall^i{}_t \in I, {V^i}_t(X) \geq {V^i}_t(X') \Leftrightarrow V(X) \geq V(X').$$

This equivalence (under the condition: $X = X'$ on $[Y = i]^c$) is the way Karni and Schmeidler (1991),[7] for instance, expressed DC (they did it in terms of preferences instead of values as we did and they limited information to a unique value).

We will require a similar but weaker condition, as expressed for example by Nishimura and Ozaki (2003):

Axiom 5 (Dynamic Consistency)

$$\forall\tau = 1, \ldots, T-1, \forall X, X' \text{ such that} : \forall t = 0, \ldots, \ \tau - 1, \forall s \in S, X_t(s) = X'_t(s),$$
$$[\forall^i{}_\tau \in I_\tau, X \underset{\approx_{i_\tau}}{\succ} X'] \Rightarrow X \underset{\approx}{\succ} X'.$$

Or, in terms of values: $[\forall^i{}_\tau, \in I_\tau {V^i}_\tau(X) \geq {V^i}_\tau(X')] \Rightarrow V(X) \geq V(X')$. (DC)

In order to address the problem of consistently conditioning V, D and E when $V = DE$ or $V = ED$, we need the following extension of Fubini's theorem.

2.3 Ghirardato-Fubini theorem

Let us recall that the DM has preferences on R^Ω that are represented by a Choquet integral with respect to a capacity Ψ on 2^Ω: $\forall X \in R^\Omega$, $V(X) = \int_\Omega X d\Psi$.

[6] For instance, it would be the case if the decision maker's preferences satisfied consequentialism.

[7] But see also: Sarin and Wakker (1998); Machina (1998) and Ghirardato (2002).

As $\Omega = S \times T$, Ψ yields two marginal capacity measures, say: ν on 2^S and ρ on 2^T. In turn, these two capacity measures represent the DM's behaviour in front of uncertain states-contingent payoffs and of future dates contingent payoffs. These preferences over R^S and R^T satisfy the same axioms than preferences on R^Ω, so they are represented again by Choquet integrals that define:

– $\forall X \in R^\Omega, \forall t \in T, X_t \approx_t E(X_t) = \int\limits_S X_t d\nu.$

– $\forall X \in R^\Omega, \forall s \in S, X_s \approx_s D(X_s) = \int\limits_T X_s d\rho.$

Mixing up the marginal measures and the value function representing preferences on R^S and R^T and introducing a hierarchy between the two components (that has to be justified), we can define:

– $\forall X \in R^{R\times T} ED(X) = E[D(X)] = \int\limits_S \left[\int\limits_T X(s,t) d\rho(t) \right] d\nu(s).$

– $\forall X \in R^{R\times T} DE(X) = D[E(X)] = \int\limits_T \left[\int\limits_S X(s,t) d\nu(s) \right] d\rho(t).$

These value functions define two orders of preferences that have the same properties as the previous ones and can be represented by: $ED(X) = \int_\Omega X\, d\Psi_1$ and $DE(X) = \int_\Omega X\, d\Psi_2$. In general, Ψ_1, Ψ_2 and Ψ will not coincide except in some particular cases that we shall consider.

As we shall see, the hierarchy between preferences on time and on uncertain states can be justified by some hedging properties. When this is the case, it will be possible to show the coherence of the different preferences on Ω (and of the measures they define).

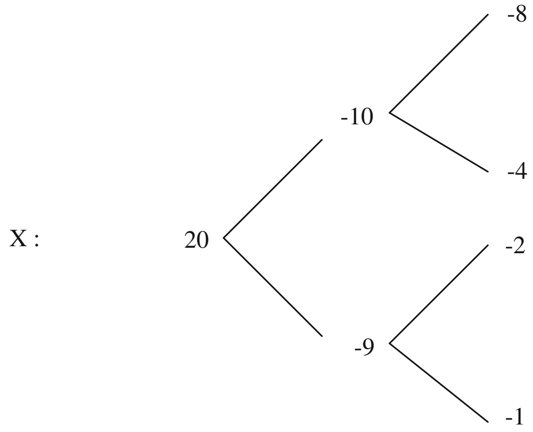
Now, let us recall some definitions introduced by Ghirardato (1997):

Definition 2.3.1 (Slice comonotonicity)

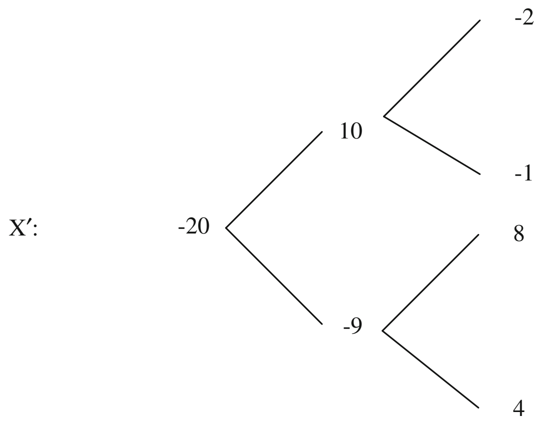
– $X \in R^{S\times T}$ is T-slice (resp. S-slice) comonotonic, if for all t in T, its t-sections on R^S (resp. for all s in S, its s-sections on R^T) are comonotonic.
– $X \in R^{S\times T}$ is slice comonotonic, if all its t-sections and its s-sections are comonotonic.
– A set $F \subset R^{S\times T} = R^\Omega$ is said to be comonotonic if all the s-sections of its characteristic function 1_F are comonotonic, which is equivalent to: all its t-sections are comonotonic, and then to: 1_F is slice comonotonic.

The relevance of this definition for the problem of valuing an investment is related to the notion of hedging future variations, and hence to preferences showing more or less variation aversion. T-slice comonotonicity excludes the possibility that uncertain variations are hedged as time passes. The following example shows a T-slice comonotonic insurance contracts portfolio X: losses at time 1 are not smoothed by losses at time 2 (other examples are in Sects. 3 and 4).

Let $S = \{s_1, \ s_2, \ s_3, \ s_4\}$ (vertical) and $T = \{0, 1, 2\}$ (horizontal, the date 0 is the present, the cash amount can be interpreted as the initial available capital or the investment):



In contrast, the payoffs of project X' at time 1 show a variation that is compensated by the variations at time 2; X'_1 and X'_2 can hedge each other because they are not comonotonic:



Similarly, the DM cannot expect that variations from one date to the other can be hedged by different uncertain trajectories in an S-comonotonic cash flow: X is comonotonic but X' is not S-slice comonotonic because $X'(s_2) = (-20, 10, -1)$ is not comonotonic with $X'(s_3) = (-20, -9, 8)$. Hence, these two cases (and the case where both sections are comonotonic) are relevant for particular investment problems where the DM is more concerned by date variations than by uncertain variations or the converse.

Mathematically, the impact of comonotonicity on linearity is easily understood with the following:

Lemma 2.3.1 *For any T-slice comonotonic X such that $\forall t \in T$, $X_t \in C_k$ where C_k is a comonotonic class, $k \in \{1, \ldots, N!\}$, then capacity ν is represented by a probability measure μ_k and $\forall t \in T$, $E(X_t) = \int_S X_t d\nu = \int_S X_t d\mu_k$.*

For any S-slice comonotonic X such that $\forall s \in S$, $X_s \in C_h$ where C_h is a comonotonic class, $h \in \{1, \ldots, T!\}$, then ρ is represented by a probability measure π_h and

$$\forall s \in S, D(X_s) = \int_T X_s d\rho = \int_T X_s d\rho_h.$$

Proof If X is T-slice comonotonic, $\forall t \in T, X(., t)$ belongs to some comonotonic class, say $C_k, \;\; k = 1, \ldots, N!$ of R^S. A comonotonic class C_k is generated by linearly independent comonotonic characteristic functions of sets: $A_k^1 \subset A_k^2 \subset \ldots \subset A_k^N = R^S$. Then, because we assumed all payoffs to be non negative, if $X_t$is in the comonotonic class C_k:

$$\exists(\alpha_t^1, \ldots, \alpha_t^N) \in R_+^S, X_t = \sum_{i=1}^N \alpha^i{}_t 1_{A_k^i}.$$

Furthermore, we know that a capacity ν on 2^S is additive on each comonotonic class so that:

$$\forall C_k, k = 1, \ldots, N! \; \exists \mu_k \text{ additive,}$$
$$\forall X_t \in C_k, E(X_t) = \int_S X_t d\nu = \sum_{i=1}^N \alpha^i{}_t \nu(A_k^i) = \int_S X_t d\mu_k.$$

Similarly, if X is S-slice comonotonic, we have:

$$\forall C_h, h = 1, \ldots, T! \; \exists \pi_h \text{ additive,}$$
$$\forall X_s \in C_h, D(X_s) = \int_T X_s d\rho = \sum_{i=1}^T \beta_s^i \rho(B_h^i) = \int_T X_s d\pi_h.$$

□

In order to use some of Ghirardato's (1997) results, we need to introduce a new axiom on preferences that insure that preferences on R^Ω, R^S and R^T are consistent. Consistency of marginal preferences on state or on date contingent payoffs, and global preferences on cash flows can be expressed by: The measures ν, ρ and Ψ are such that Ψ can be reconstructed from ν and ρ. Obviously, this requires too much in general because there are some intertwinements between state and date contingencies that may induce some preferences to be modified when future payoffs are perceived as a whole. However, when no hedging possibilities are available, whether on uncertain payoffs or on certain cash flows, we can require some consistency from the DM's behaviour (notice that the hedging argument is the one used to justify comonotonic additivity, or comonotonic independence, or No comonotonic Dutch books). The axiom could be expressed as: Given a set of comonotonic assets on $R^{S\times T}$, if there are no Dutch Books with their S-sections and no Dutch Books with their T-sections, then there should be no Dutch Books formed with these assets. More precisely in terms of our preference representations:

Axiom 6 (Comonotonic Consistency)
If F is a comonotonic subset of $\Omega = S\times T$ and $\forall t \in T\ F_t = \{s \in S/1_F(s,t) = 1\}$, $\forall s \in S F_s = \{t \in T/1_F(s,t) = 1\}$, then:

$$[\forall t \in T E(1_F(.,t)) = \nu(F_t) \text{ and } \forall s \in S D(1_F(s,.) = \rho(F_s)] \Rightarrow V(1_F) = \Psi(F).$$

Obviously, the axiom is always satisfied by definition of the marginal measures if $F = A \times A'$, $A \subset S$, $A' \subset T$. Assume the axiom is not satisfied by some DM for some comonotonic F that is not a rectangle: For example assume that for some trajectory s, the DM had a measure ρ' on Time, $\rho \neq \rho'$ with ρ' convex while ρ is not. This would mean that the DM is more time-variations averse when confronted to the certain date contingent cash flow trajectory s than it would be if the payoffs were part of a state and date contingent flow. Obviously, that could be acceptable if F were not comonotonic, but, because it is comonotonic, F offers no possibilities for hedging time-variations whatever the trajectory, hence the two different measures are inconsistent.

Axiom 6 yields a result that was imposed as a mathematical condition in Ghirardato (1997) who dubbed it "the Fubini property".

Proposition 2.3.1 *Under Axiom 6, "the Fubini property" is satisfied:*

$$\forall F \in 2^{S\times T}, \Psi(F) = D(\nu[\{s \in S/(s,t) \in F]) = E(\rho[\{t \in T/(s,t) \in F]),\ or:$$
$$\int\limits_{S\times T} 1_F(s,t)d\psi(s,t) = \int\limits_T d\rho(t) \int\limits_S 1_F(s,t)d\nu(s) = \int\limits_S d\nu(s) \int\limits_T 1_F(s,t)d\rho(t).$$

Proof Notice that $\int_S 1_F(s,.)d\nu(s)$ is comonotonic with any of the $1_F(.,t), t \in T$, then there exists an additive probability π_h on T such that:

$$\int\limits_T \left[\int\limits_S 1_F(s,t)d\nu(s)\right] d\rho(t) = \int\limits_T \left[\int\limits_S 1_F(s,t)d\nu(s)\right] d\pi_h(t) \quad \text{and}$$
$$\forall s \in S \int\limits_T 1_F(s,t)d\rho(t) = \int\limits_T 1_F(s,t)d\pi_h(t).$$

But $\int_T 1_F(.,t)d\pi_h(t)$ is comonotonic with any of the $1_F(s,.)$, $s \in S$, hence there exists an additive probability μ_k on S such that:

$$\int_S \left[\int_T 1_F(s,t)d\pi_h(t)\right] d\nu(s) = \int_S \left[\int_T 1_F(s,t)d\pi_h(t)\right] d\mu_k(s).$$

Fubini's theorem applies and $\int_S \left[\int_T 1_F(s,t)d\pi_h(t)\right] d\mu_k(s) = \int_T \left[\int_S 1_F(s,t) d\mu_k(s)\right] d\pi_h(t)$.

This yields the second equality of the lemma.

The first equality obtains because μ_k and π_h define a product probability, say Φ_j on $S \times T$, but Φ_j is an additive representation of Ψ valid on the comonotonic class C_j, $j = 1, \ldots, (N \times T)!$ (in R^Ω) to which 1_F belongs. As this is true for any comonotonic class, the Φ_j's define Ψ. □

Now, Ghirardato's theorem (his Lemma 3 in our simple model) yields the following decomposition of preferences on $R^{S\times T}$ and preferences on R^S and on R^T:

Proposition 2.3.2 (Ghirardato) *Under the comonotonic consistency axiom, if preferences on $R^{S\times T}$ satisfy Axioms 1 to 3 and are represented by V (defined by capacity Ψ) and preferences on R^S by E (capacity ν) and on R^T by D (capacity ρ), we have:*

1. *If X in $R^{S\times T}$ is T-slice comonotonic, then: $V(X) = E[D(X)]$.*

   Furthermore, for any comonotonic class C_k in R^S containing all the comonotonic t-sections of X, there exists a probability distribution μ_k defining an additive representation E_k of preferences on C_k such that for any X' with all its comonotonic t-sections in C_k:

   $$V(X') = E_k[D(X')].$$

2. *If X in $R^{S\times T}$ is S-slice comonotonic, then: $V(X) = D[E(X)]$.*

   Furthermore, for any comonotonic class C_h containing all the comonotonic s-sections of X, there exists a probability distribution π_h defining an additive representation D_h of preferences on C_h such that for any X' with all its comonotonic s-sections in C_h:

   $$V(X') = D_h[E(X')].$$

3. *If X in $R^{S\times T}$ is slice comonotonic, then: $V(X) = E[D(X)] = D[E(X)]$.*

Interpretations: The first two results are Lemma 3 of Ghirardato (1997). The additive representation (valid on one comonotonic class only) is interpreted this way:

For the first one, consider a model consistent with Gilboa's (1989) idea in which time is measured by an non decreasing, non negative and bounded measure (a capacity in our special case). In this model, uncertainty has not been taken into account. Now, if we add it at each date, we obtain our model. However, because all the uncertain variables are comonotonic, comonotonic additivity applies and we only need to know the probability distribution that represents it on each comonotonic class. This can, but need not be, extended to the whole space of uncertain variables, assuming then that de Finetti's coherence axiom applies.

The second formula is the usual discounted expected payoffs (here in the sense of a Choquet integral). Notice that discount factors (mathematically probabilities, here) depend on the comonotonic class in which there are no possibilities for hedging time variations.

The last result is the famous Ghirardato-Fubini theorem applied to our model. In all three cases, we obtain a representation of preferences in terms of some present value (constant equivalent), with the first integral additive.

In the next two sections, we shall use the Ghiraradato-Fubini theorem to address the problem of conditioning the present value expressed in terms of a Choquet integral and derive some results about conditional capacities.

3 Conditional valuation of S-slice comonotonic cash payoffs

In this section, we consider S-slice comonotonic cash payoffs X: $S \times T \to R_{+}$, i.e. payoffs such that their time variations along trajectories all go the same way and hence cannot be hedged. From Ghirardato's theorem (Proposition 2.3.2, part 2) preferences of the DM satisfying Axioms 1 to 3 and 6 are represented for all cash payoffs in a comonotonic class C_h in R^T, by $V = D_h E$ where D_h is linear, expressing the fact that for the cash payoffs at stake the DM decides as if it were time variation neutral. From now on we will drop the h index. In the following, $E^i{}_t$ and $\nu^i{}_t$ as well as $D^i{}_t \equiv D^t$ and $\pi^i{}_t \equiv \pi^t$ will be defined, implicitly, by Axioms 4 and 5.

We assume X is a F-measurable process and we add a "present" for notational convenience as a date 0 that has no other role than defining an eventually non zero initial cash amount. Then, X can be defined as:

$$X = \begin{pmatrix} X_0(s_1) & \dots & X_T(s_1) \\ \dots & & \dots \\ X_0(s_N) & \dots & X_T(s_N) \end{pmatrix} \text{ with, } \forall t \in T, \forall s \in S, X_t(s) \geq 0,$$
$$X_0(s_1) = \dots = X_0(s_N) = x_0 \in R_{+}.$$

Let us introduce the usual notation for a Choquet integral: $\forall t \in T, E_t(X) = \sum_{s \in S} X_t(s)\Delta\nu(s)$ where, if for instance $X(s_1) \leq \dots \leq X(s_N)$, $\Delta\nu(s_n) = \nu(\{s_n, \dots, s_N\}) - \nu(\{s_{n+1}, \dots, s_N\})$ with $\{s_{N+1}\} = \emptyset$ for notational convenience. Then, we have:

$$V(X) = \sum_{t \in T} \left[\sum_{s \in S} X_t(s)\Delta\nu(s)]\pi(t) \right] = \sum_{t \in T} E_t(X)\pi(t).$$

Let: $EC(X) = \begin{pmatrix} E_0(X) & \dots & E_T(X) \\ \dots & & \dots \\ E_0(X) & \dots & E_T(X) \end{pmatrix} \in R^T$, and $\pi(0) = 1$, we have: $V[EC(X)] = V(X)$.

Therefore, $EC(X)$ is a certainty equivalent process of X.

From Model Consistency (Axiom 4), we have the same type of value functions for a given information i_τ at some date τ.

$$\forall \tau \in T, \quad \forall i_\tau \in I_\tau, V^{i_\tau}(X) = \sum_{t \in T} \left[\sum_{s \in S} X_t(s) \Delta \nu^{i_\tau}(s) \right] \pi^\tau(t) = \sum_{t \in T} E_t^{i_\tau}(X) \pi^\tau(t),$$

where: $\forall t = 0, \ldots, \tau - 1, \pi^\tau(t) = 0, \pi^\tau(\tau) = 1, \forall t = \tau, \ldots, T,$

$$E_t^{i_\tau}(X) = \sum_{s \in S} X_t(s) \Delta \nu^{i_\tau}(s).$$

If we write: $EC^{i_\tau}(X) = \begin{pmatrix} E_\tau^{i_\tau}(X) & \ldots & E_T^{i_\tau}(X) \\ \ldots & & \ldots \\ E_\tau^{i_\tau}(X) & \ldots & E_T^{i_\tau}(X) \end{pmatrix}$, we have:

$V^{i_\tau}[EC^{i_\tau}(X)] = V^{i_\tau}(X)$.

3.1 Dynamic consistency

In Sect. 2, we introduced a weak definition of DC (Axiom 5) that yields a link between unconditional and conditional valuations. We weaken it again thanks to:

Proposition 3.1.1 *Axiom* 5 (*DC*) *implies:*

$$\forall \tau = 1, \ldots, T-1, \forall t = \tau, \ldots, T, \sum_{i_\tau \in I_\tau} \left[\sum_{s \in S} X_t(s) \Delta \nu^{i_\tau}(s) \right] \Delta \nu(i_\tau)$$
$$= \sum_{s \in S} X_t(s) \Delta \nu(s) \tag{3.1}$$

Proof $\forall \tau = 1, \ldots, T-1$, define Z^τ as follows: $\forall t = 0, \ldots, \tau - 1$, $\forall s \in S$, $X_t(s) = Z_t^\tau(s)$. Then:

$$\forall\, t = \tau, \ldots, T, \forall\, s \in S, \exists l \in \{1, \ldots, M(\tau)\}, \text{such that } s \in i_\tau^l \text{ and } Z_t^\tau(s) = E_t^{i_\tau^l}(X).$$

$$\forall\, i_\tau \in I_\tau, V^{i_\tau}(Z^\tau) = \sum_{t=\tau}^{T} \left[\sum_{i \in I_\tau} E_t^i(X) \Delta \nu^{i_\tau}(i) \right] \pi^\tau(t).$$

Suppose w.l.o.g. that, for an information at date τ: $i_\tau = i_\tau^{m(\tau)}$ and for a date $t = \tau, \ldots, T$ we have $E_t^{i_\tau^1}(X) \le \ldots \le E_t^{i_\tau^{m(\tau)}}(X) \le \ldots \le E_t^{i_\tau^{M(\tau)}}(X)$, then:

$$\sum_{i \in I_\tau} E_t^i(X) \Delta \nu^{i_\tau}(i) = \sum_{l=1}^{M(\tau)} E_t^{i_\tau^l}(X) \left[\nu^{i_\tau^{m(\tau)}} \left(i_\tau^1 \cup \ldots \cup i_\tau^l \right) - \nu^{i_\tau^{m(\tau)}} \left(i_\tau^1 \cup \ldots \cup i_\tau^{l-1} \right) \right]$$

$$= \sum_{l=1}^{m(\tau)-1} E_t^{i_\tau^l}(X)\left[\nu^{i_\tau^{m(\tau)}}\left(i_\tau^1 \cup \ldots \cup i_\tau^l\right) - \nu^{i_\tau^{m(\tau)}}\left(i_\tau^1 \cup \ldots \cup i_\tau^{l-1}\right)\right]$$
$$+E_t^{i_\tau^{m(\tau)}}(X)\left[\nu^{i_\tau^{m(\tau)}}\left(i_\tau^1 \cup \ldots \cup i_\tau^{m(\tau)}\right) - \nu^{i_\tau^{m(\tau)}}\left(i_\tau^1 \cup \ldots \cup i_\tau^{m(\tau)-1}\right)\right]$$
$$+ \sum_{l=m(\tau)+1}^{M(\tau)} E_t^{i_\tau^l}(X)\left[\nu^{i_\tau^{m(\tau)}}\left(i_\tau^1 \cup \ldots \cup i_\tau^l\right) - \nu^{i_\tau^{m(\tau)}}\left(i_\tau^1 \cup \ldots \cup i_\tau^{l-1}\right)\right].$$

With the following normalisation of conditional capacities: $i \subset A \Rightarrow \nu^i(A) = 1$, $i \cap A = \emptyset \Rightarrow \nu^i(A) = 0$, we have: $\sum_{i \in I_\tau} E_t^i(X)\Delta\nu^{i_\tau}(i) = E_t^{i_\tau^{m(\tau)}}(X) = E_t^{i_\tau}(X)$.

It follows that:

$$\forall\, i_\tau \in I_\tau,\ V^{i_\tau}(Z^\tau) = \sum_{t=\tau}^{T} E_t^{i_\tau}(X)\pi^\tau(t) = V^{i_\tau}(X).$$

Therefore, under Axiom 5 (DC), we have: $V(Z^\tau) = V(X)$, which implies:

$$\sum_{t=0}^{\tau-1}\left[\sum_{s\in S} X_t(s)\Delta\nu(s)\right]\pi(t) + \sum_{t=\tau}^{T}\left[\sum_{i_\tau\in I_\tau} E_t^{i_\tau}(X)\Delta\nu(i_\tau)\right]\pi(t)$$
$$= \sum_{t\in T}\left[\sum_{s\in S} X_t(s)\Delta\nu(s)\right]\pi(t).$$

Then:

$$\sum_{t=\tau}^{T}\left[\sum_{i_\tau\in I_\tau} E_t^{i_\tau}(X)\Delta\nu(i_\tau)\right]\pi(t) = \sum_{t=\tau}^{T}\left[\sum_{s\in S} X_t(s)\Delta\nu(s)\right]\pi(t).$$

This equality is satisfied for any X, and then it must be true at each date t:

$$\sum_{i_\tau\in I_\tau} E_t^{i_\tau}(X)\Delta\nu(i_\tau) = \sum_{i_\tau\in I_\tau}\left[\sum_{s\in S} X_t(s)\Delta\nu^{i_\tau}(s)\right]\Delta\nu(i_\tau) = \sum_{s\in S} X_t(s)\Delta\nu(s) \quad (3.1)$$

□

We shall refer to (3.1) in the following as the DC linking unconditional and conditional valuations.

3.2 Updating capacities

Relation (3.1) is a condition on the DM's preferences representation that yields an implicit definition of conditional Choquet expectation. We apply it to characteristic functions in order to derive updating rules for conditional capacities.

Proposition 3.2.1[8] *Under relation* (3.1)*, for any* $i \in I_\tau$*, the conditional capacity of a set* $A \in F_t$*,* $t > \tau$*, is given by*:

(*i*) *If* $A \subset i$, $\nu^i(A) = \dfrac{\nu(A \cap i)}{\nu(i)}$ (*Bayes updating rule*).

(*ii*) *If* $A^C \subset i$, $\nu^i(A) = \dfrac{\nu(A \cup i^C) - \nu(i^C)}{1 - \nu(i^C)}$ (*Dempster-Schafer updating rule*).

Proof Relation (3.1): $\sum_{i \in I_\tau} \left[\sum_{s \in S} X_t(s) \Delta \nu^i(s)\right] \Delta \nu(i) = \sum_{s \in S} X_t(s) \Delta \nu(s)$ can be applied to characteristic functions. For

$$A \in F_t, \text{let } X_t = 1_A, \text{ then: } \nu(A) = \sum_{i=i^1}^{i^{M(\tau)}} \nu^i(A) \Delta \nu(i). \tag{3.2}$$

The conditional capacity $\nu^i(A)$ can be calculated in two cases only:

(i) When $A \subset i$, the "comonotonic" case (because 1_A and 1_i are comonotonic uncertain variables).
In this case, $\nu^i(A) \geq 0$ and $\nu^j(A) = 0$, for $j \in I_\tau$, $j \neq i$. Relation (3.2) implies: $\nu(A) = \nu^i(A)\nu(i)$, and then $\nu^i(A) = \dfrac{\nu(A)}{\nu(i)} = \dfrac{\nu(A \cap i)}{\nu(i)}$, which is Bayes formula.

(ii) When $A^C \subset i$, the "antimonotonic" case (because 1_A and 1_i are anticomonotonic, i.e. 1_A and -1_i are comonotonic uncertain variables).

In this case, $\nu^i(A) \leq 1$ and $\nu^j(A) = 1$, for $j \in I_\tau$, $j \neq i$. Relation (3.2) implies: $\nu(A) = \nu^i(A) + [1 - \nu^i(A)]\, \nu(i^C)$, and then $\nu^i(A) = \dfrac{\nu(A) - \nu(i^C)}{1 - \nu(i^C)} = \dfrac{\nu(A \cup i^C) - \nu(i^C)}{1 - \nu(i^C)}$, which is the Dempster-Shafer formula. □

The two rules we obtain result from the ranking of values after information obtains, and it depends on the type of information (comonotonic or antimonotonic with payoffs). The type of information can be interpreted as a "good" or "bad" news (with respect to what was expected). The fact that these rules integrate values that could not be obtained after information is in contradiction with consequentialism as we confirm below.

3.3 Consequentialism

Another familiar consistency condition known as consequentialism (Hammond (1989)) is usually imposed as an axiom on preferences. It is well known however (see, for instance Sarin and Wakker (1998); Machina (1998); Karni and Schmeidler (1991); Ghirardato (2002) and Lapied and Toqueoeuf (2007)) that Model consistency, DC and Consequentialism imply additive (or quasi always additive) models. Ours is not, under

[8] Notice that the same results were obtained by Chateauneuf et al. (2001) with another preference representation model in the case of uncertain payoffs in a static setting and under different assumptions.

the two first assumptions, hence it must be that Consequentialism is not satisfied, as we show below.

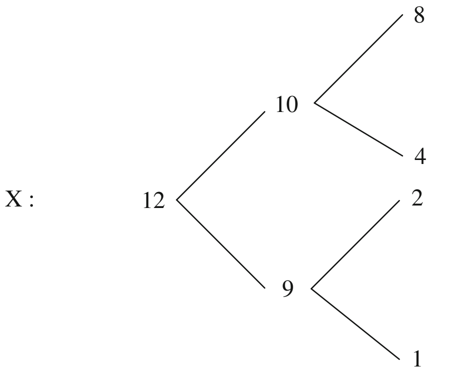
Definition 3.3.1 (Consequentialism in a dynamic setting)

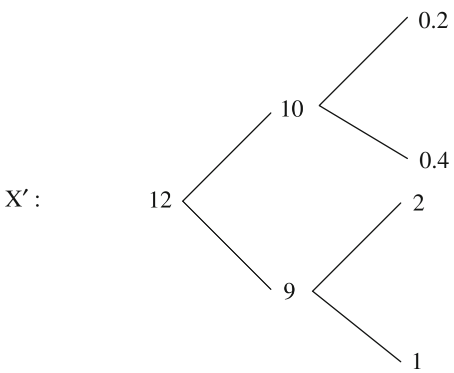
$$\forall \tau = 0, \ldots, T, \forall i_\tau \in I_\tau, [\forall t = \tau, \ldots, T, \forall s \in i_\tau, X_t(s) = X'_t(s)] \Rightarrow X \approx_{i_\tau} X' \quad \text{(C)}.$$

Proposition 3.3.1 *Preferences satisfying Axioms 1 to 6 on the subset of S-slice comonotonic cash payoffs do not satisfy* (*C*).

Proof Let us consider $S = \{s_1, s_2, s_3, s_4\}$, $I_1 = \{i^1, i^2\}$, $i^1 = \{s_1, s_2\}$, $i^2 = \{s_3, s_4\}$, $T = \{0, 1, 2\}$ and two cash flows X, X', with the following payoffs:

$$X_0 = X'_0 = 12,$$
$$X_1(\{s_1\}) = X_1(\{s_2\}) = X'_1(\{s_1\}) = X'_1(\{s_2\}) = 10,$$
$$X_1(\{s_3\}) = X_1(\{s_4\}) = X'_1(\{s_3\}) = X'_1(\{s_4\}) = 9,$$
$$X_2(\{s_1\}) = 8, X_2(\{s_2\}) = 4, X'_2(\{s_1\}) = 0.2, X'_2(\{s_2\}) = 0.4,$$
$$X_2(\{s_3\}) = X'_2(\{s_3\}) = 2, X_2(\{s_4\}) = X'_2(\{s_4\}) = 1.$$





Because X and X' are S-slice comonotonic cash flows, we can apply DE valuation. For these cash flows, Consequentialism, implies that $V_1^{i^2}(X) = V_1^{i^2}(X')$. We have:

$$V_1^{i^2}(X) = 9 + \pi^1(2)\{1 \times \left[1 - \nu^{i^2}(\{s_1, s_2, s_3\})\right] + 2 \times \nu^{i^2}(\{s_1, s_2, s_3\})\},$$
$$V_1^{i^2}(X') = 9 + \pi^1(2)\left\{1 \times \left[1 - \nu^{i^2}(\{s_3\})\right] + 2 \times \nu^{i^2}(\{s_3\})\right\}.$$

From Proposition 3.2.1:

– Because $\{s_3\} \subset i^2$ its conditional capacity is given by Bayes updating rule:

$$\nu^{i^2}(\{s_3\}) = \frac{\nu(\{s_3\})}{\nu(i^2)}.$$

– Because $\{s_1, s_2, s_3\}^C = \{s_4\} \subset i^2$ its conditional capacity is given by Dempster-Schafer updating rule:

$$\nu^{i^2}(\{s_1, s_2, s_3\}) = \frac{\nu(\{s_1, s_2, s_3\}) - \nu(i^1)}{1 - \nu(i^1)}.$$

Let $\pi^1(2) = 0.9$, and ν be a convex (non additive) capacity with:

$$\nu(\{s_3\}) = 0.3, \nu(i^1) = 0.5, \nu(i^2) = 0.4, \nu(\{s_1, s_2, s_3\}) = 0.9.$$

We obtain: $V_1^{i^2}(X) = 10.62 > V_1^{i^2}(X') = 10.575$, which is in contradiction with Consequentialism. □

Proposition 3.3.2 *$V = DE$ does not collapse into discounted expected cash flows.*

Proof Consider the example used in Proposition 3.3.1. We have:

$$V_1^{i^1}(X) = 10 + \pi^1(2)\{4 \times [1 - \nu^{i^1}(\{s_1\})] + 8 \times \nu^{i^1}(\{s_1\})\},$$
$$V_1^{i^1}(X') = 10 + \pi^1(2)\{0.2 \times [1 - \nu^{i^1}(\{s_2, s_3, s_4\})] + 0.4 \times \nu^{i^1}(\{s_2, s_3, s_4\})\}.$$

From Proposition 3.2.1:

– Because $\{s_1\} \subset i^1$ its conditional capacity is given by Bayes updating rule:

$$\nu^{i^1}(\{s_1\}) = \frac{\nu(\{s_1\})}{\nu(i^1)}.$$

– Because $\{s_2, s_3, s_4\}^C = \{s_1\} \subset i^1$ its conditional capacity is given by Dempster-Schafer updating rule:

$$\nu^{i^1}(\{s_2, s_3, s_4\}) = \frac{\nu(\{s_2, s_3, s_4\}) - \nu(i^2)}{1 - \nu(i^2)}.$$

Let us complete the definition of ν and set $\nu(\{s_1\}) = 0.3$, $\nu(\{s_2, s_3, s_4\}) = 0.6$, we obtain:

$$\sum_{s\in S} X_2(s)\Delta\nu^{i^1}(s) = 6.4, \quad \sum_{s\in S} X_2(s)\Delta\nu^{i^2}(s) = 1.8, \quad \text{and}$$
$$\sum_{s\in S} X'_2(s)\Delta\nu^{i^1}(s) = \frac{0.8}{3}, \quad \sum_{s\in S} X'_2(s)\Delta\nu^{i^2}(s) = 1.75.$$

Relation (3.1) is trivially satisfied for $\tau = t = 1$, we only have to consider the case where $\tau = 1$ and $t = 2$:

$$\sum_{i\in I_1}\left[\sum_{s\in S} X_2(s)\Delta\nu^{i}(s)\right]\Delta\nu(i) = 4.1 = \sum_{s\in S} X_2(s)\Delta\nu(s),$$
$$\sum_{i\in I_1}\left[\sum_{s\in S} X'_2(s)\Delta\nu^{i}(s)\right]\Delta\nu(i) = 0.86 = \sum_{s\in S} X'_2(s)\Delta\nu(s).$$

Therefore, relation (3.1) is consistent with the (non additive) capacity ν and with the conditional capacities defined by Proposition 3.2.1. □

The counter example used to prove the proposition uses the alternative two updating rules, other models based on DC rely on one rule only.

4 Conditional valuation of T-slice comonotonic cash payoffs

In this section, we concentrate on cash payoffs with all their t-sections in the same comonotonic class in R^S, say C_k, hence their uncertain variations can not be hedged. As a consequence of Ghirardato's theorem (Proposition 2.3.2, part 1), preferences on $R^{S\times T}$ are represented by the valuation function $V = E_k D$, where E_k is a Lebesgue integral with respect to a probability distribution μ_k on 2^S and D a Choquet integral with respect to capacity ρ. In the following, we drop the index k.

As information only bears on 2^S, its influence on preferences over R^T is only related to the date at which it obtains. Otherwise stated: $D^i{}_t \equiv D^t$ is a Choquet integral with respect to a capacity ρ^t that is contingent on date t only, while $E^i{}_t$ is the usual conditional expectation and $\mu^i{}_t$ is obtained by the probabilistic Bayes'rule.

With the notation we introduced in Sect. 2 for Choquet integrals (here on R^T) the valuation formula becomes:

$$V(X) = \sum_{s\in S}\left[\sum_{t\in T} X_t(s)\Delta\rho(t)\right]\mu(s) = \sum_{s\in S} D_s(X)\mu(s), \text{ where:}$$
$$\forall s \in S, D_s(X) = \sum_{t\in T} X_t(s)\Delta\rho(t).$$

Let us define:

$$ET(X) = \begin{pmatrix} D_{s_1}(X) & 0 & \ldots & 0 \\ \ldots & \ldots & & \ldots \\ D_{s_N}(X) & 0 & \ldots & 0 \end{pmatrix} \in R^S, \text{ with } \rho(0) = 1.$$

Then: $V[ET(X)] = V(X)$ and $ET(X)$ is an uncertain present equivalent of X.

Under Model Consistency, Axiom 4, we have the same type of value functions conditional to a given information:

$$\forall \tau \in T, \forall i_\tau \in I_\tau, V^{i_\tau}(X) = \sum_{s \in S} \left[\sum_{t \in T} X_t(s) \Delta\rho^\tau(t) \right] \mu^{i_\tau}(s) = \sum_{s \in S} D_s^\tau(X) \mu^{i_\tau}(s),$$

where:

$$\forall s \in i_\tau, D_s^\tau(X) = \sum_{t \in T} X_t(s) \Delta\rho^\tau(t).$$

Let us note:

$$ET^\tau(X) = \begin{pmatrix} D_{s_1}^\tau(X) & 0 & \ldots & 0 \\ \ldots & \ldots & & \ldots \\ D_{s_N}^\tau(X) & 0 & \ldots & 0 \end{pmatrix} \in R^S, \text{ with } \rho^\tau(\tau) = 1, \text{ we have:}$$

$$V^{i_\tau}[ET^\tau(X)] = V^{i_\tau}(X).$$

Discounting must give zero weight to the payoffs before information obtains: $\forall F \subset T$, $\rho^\tau(F) = \rho^\tau(F \cap \tau^+)$, where $\tau^+ = \{\tau, \ldots, T\}$.

4.1 Dynamic consistency

As in the previous section, DC (Axiom 5) yields a link between unconditional and conditional valuations.

Proposition 4.1.1 *Axiom* 5 (*DC*) *implies:*

$$\forall \tau = 1, \ldots, T-1, \forall s \in S, \sum_{t \in \tau^-} X_t^\tau(s) \Delta\rho(t) = \sum_{t \in T} X_t(s) \Delta\rho(t) \qquad (4.1)$$

where: $\tau^- = \{0, \ldots, \tau\}$, $X_t^\tau(s) = X_t(s)$, *if* $t \in \tau^- - \{\tau\}$, $X_t^\tau(s) = D_s^\tau(X)$, *if* $t = \tau$, *and* $\forall s \in S$, $D_s^\tau(X) = \sum_{t \in T} X_t(s) \Delta\rho^\tau(t)$.

Proof $\forall \tau = 1, \ldots, T-1$, define:

$$Z^\tau = \begin{pmatrix} X_0 \ldots X_{\tau-1}(s_1) & D_{s_1}^\tau(X) & 0 & \ldots & 0 \\ \ldots & \ldots & & \ldots & \ldots \\ X_0 \ldots X_{\tau-1}(s_N) & D_{s_N}^\tau(X) & 0 & \ldots & 0 \end{pmatrix}.$$

We have: $\forall\tau = 1, \ldots, T-1, \forall t = 0, \ldots, \tau - 1, \forall s \in S, X_t(s) = Z_t^\tau(s)$.

For any $s \in S$, consider a permutation of the dates $t = 0, \ldots, \tau - 1$ such that

$$0 \leq X_{(0)}(s) \leq \ldots \leq X_{(k)}(s) \leq D_s^\tau(X) \leq X_{(k+1)}(s) \leq \ldots \leq X_{(\tau-1)}(s).$$

Then:

$$\begin{aligned}\sum_{t\in T} Z_t^\tau \Delta\rho^\tau(t) = &\sum_{(t)=(0)}^{(k)} X_{(t)}(s)\{\rho^\tau[(t), \ldots, (\tau-1), \tau] - \rho^\tau[(t+1), \ldots, (\tau-1), \tau]\} \\ &+ D_s^\tau(X)\{\rho^\tau[(k+1), \ldots, (\tau-1), \tau] - \rho^\tau[(k+1), \ldots, (\tau-1)]\} \\ &+ \sum_{(t)=(k+1)}^{(\tau-1)} X_{(t)}(s)\{\rho^\tau[(t), \ldots, (\tau-1)] - \rho^\tau[(t+1), \ldots, (\tau-1)]\} \\ = &\, D_s^\tau(X)\rho^\tau(\tau) = D_s^\tau(X).\end{aligned}$$

$$\forall i_\tau \in I_\tau, V^{i_\tau}(Z^\tau) = \sum_{s\in S}\left[\sum_{t\in T} Z_t^\tau \Delta\rho^\tau(t)\right]\mu^{i_\tau}(s) = \sum_{s\in S} D_s^\tau(X)\mu^{i_\tau}(s) = V^{i_\tau}(X).$$

Therefore, under Axiom 5 (DC), we have: $V(Z^\tau) = V(X)$, which implies:

$$\sum_{s\in S}\left[\sum_{t\in\tau^-} X_t^\tau(s)\Delta\rho(t)\right]\mu(s) = \sum_{s\in S}\left[\sum_{t\in T} X_t(s)\Delta\rho(t)\right]\mu(s),$$

where $X_t^\tau(s) = X_t(s)$, *if* $t \in \tau^- - \{\tau\}$, $X_t^\tau(s) = D_s^\tau(X)$, *if* $t = \tau$. This equality is satisfied for any X, and then it should be true for each state s:

$$\forall\tau = 1, \ldots, T-1, \forall s \in S, \sum_{t\,\in\,\tau^-} X_t^\tau(s)\Delta\rho(t) = \sum_{t\in T} X_t(s)\Delta\rho(t) \tag{4.1}$$

□

4.2 "Upstating" capacities on time

If updating means that we modify the measure of uncertainty according to information at some date, then we dubb "upstating" the fact that we modify the measure of time according to information (on the set of states) at the date at which it is obtained. Given that the DM's preferences satisfy Axiom 5 (DC) and hence relation (4.1) we have:

Proposition 4.2.1 *Under relation* (4.1)*, for $F \subset T$, with $\tau^- = \{0, \ldots, \tau\}$, and $\tau^+ = \{\tau, \ldots, T\}$, the "upstated" discount factors are given by:*

(*i*) *If* $\rho(F) \geq \rho[(F \cap \tau^-) \cup \{\tau\}]$*:*

$$\rho^\tau(F \cap \tau^+) = \frac{\rho(F) - \rho[(F \cap \tau^-) \cup \{\tau\}] + \rho(\{\tau\})}{\rho(\{\tau\})}.$$

(*ii*) *If* $\rho(F) \leq \rho[(F \cap \tau^-) \cup \{\tau\}]$*:*

$$\rho^\tau(F \cap \tau^+) = \frac{\rho(F) - \rho(F \cap \tau^-)}{\rho[(F \cap \tau^-) \cup \{\tau\}] - \rho(F \cap \tau^-)}.$$

Proof We drop the reference to state s in relation (4.1) w.l.o.g: $\sum_{t\in\tau^-} X_t^\tau \Delta\rho(t) = \sum_{t\in T} X_t \Delta\rho(t)$ (4.2)

For $F \subset T$ and $X = 1_F$, we have:

$$\sum_{t\in T} X_t \Delta\rho(t) = \rho(F), \text{ and } D^\tau(X) = \sum_{t\in\tau^+} X_t \Delta\rho^\tau(t) = \rho^\tau(F \cap \tau^+).$$

We have to consider two cases:

(i) $\rho^\tau(F \cap \tau^+) \geq 1$, then:

$$\sum_{t\in\tau^-} X_\tau^t \Delta\rho(t) = \rho[(F \cap \tau^-) \cup \{\tau\}] + \rho(\{\tau\})[\rho^\tau(F \cap \tau^+) - 1] = \rho(F),$$

(ii) $\rho^\tau(F \cap \tau^+) \leq 1$, then:

$$\sum_{t\in\tau^-} X_\tau^t \Delta\rho(t) = \rho[(F \cap \tau^-) \cup \{\tau\}]\rho^\tau(F \cap \tau^+) + \rho(F \cap \tau^-)[1 - \rho^\tau(F \cap \tau^+)] = \rho(F).$$

These relations yield the "upstating" formulas under the equivalent conditions given in the proposition. □

In the more familiar case where $F = \{0, \ldots, T\}$ we have the following:

Corollary 4.2.1 *Under relation* (4.1)*, for* $F = \{0, \ldots, T\}$*, the "upstated" discount factors are given by*:

$$\rho^\tau(\{\tau, \ldots, T\}) = \frac{\rho(\{0, \ldots, T\}) - \rho(\{0, \ldots, \tau\}) + \rho(\{\tau\})}{\rho(\{\tau\})}.$$

As in the previous section, the different formulas come from the ranking of payoffs, but this time it is the payoffs before information obtains that make the difference.

The interpretations of these "upstating" formula are not straightforward. We can however propose the following: Given we deal with T-slice comonotonic payoffs, the important variations for the DM are the ones due to time. Hence, the timing of decisions is the most relevant feature for the valuation problem (and not the subsets of states that may be obtained after information). The weights given to the payoffs after information is obtained depend on the weights given in the past because these enter into the payoffs' ranking. As a result, aversion to time variations, say, may be modified depending on the relative importance of future versus past payoffs. The important point to note, is that the value of the past does count. This is in contrast with the additive case where the usual compound discount factors formula would yield: $\pi^{\tau}(\{\tau, \ldots, T\}) = \frac{\pi(\{\tau, \ldots, T\})}{\pi(\{\tau\})}$. We shall come back to this in Sect. 4.3 where we will see that this is what violates consequentialism in our model. Here, in contrast with the case of conditioning uncertainty, comonotonicity only plays a role in the ranking of values obtained before and after information is revealed at date τ.

4.3 Consequentialism

With the definition of Consequentialism (C) given in Sect. 3, we have:

Proposition 4.3.1 *Preferences satisfying Axioms* 1 *to* 6 *on the subset of S-slice co-monotonic cash payoffs do not satisfy* (*C*).

Proof Let us consider the following certain payoffs X and X':

$X_0 = 1, X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1$, or: $X = 1_F$, $F = \{0, 3, 4\}$,

$X'_0 = 0, X'_1 = 1, X'_2 = 0, X'_3 = 1, X'_4 = 1$, $X' = 1_H$, $H = \{1, 3, 4\}$.

t	0	1	2	3	4
X :	1	0	0	1	1
X' :	0	1	0	1	1

Let ρ be a capacity such that:

$$\begin{aligned}\rho(F) &= \rho(0,3,4) > \rho(0,2) = \rho[(F \cap \tau^-) \cup \{\tau\}], \text{ and} \\ \rho(H) &= \rho(1,3,4) < \rho(1,2) = \rho[(F \cap \tau^-) \cup \{\tau\}].\end{aligned}$$

From Proposition 4.2.1,

$$\rho^2(F \cap \tau^+) = \rho^2(3,4) = \frac{\rho(0,3,4) - \rho(0,2) + \rho(2)}{\rho(2)} > 1 \qquad \text{case (i)}$$

$$\rho^2(H \cap \tau^+) = \rho^2(3,4) = \frac{\rho(1,3,4) - \rho(1)}{\rho(1,2) - \rho(1)} < 1 \qquad \text{case (ii)}$$

For these payoffs, Consequentialism implies that $V_2(X) = V_2(X')$. We have:

$$\begin{aligned}
V_2(X) &= \sum_{t=0}^{4} X_t \Delta\rho^2(t) = D^2(X) = \rho^2(F) \\
&= \rho^2(F \cap \tau^+) = \rho^2(3,4) = \frac{\rho(0,3,4) - \rho(0,2) + \rho(2)}{\rho(2)} > 1, \\
V_2(X') &= \sum_{t=0}^{4} X'_t \Delta\rho^2(t) = D^2(X') = \rho^2(H) \\
&= \rho^2(H \cap \tau^+) = \rho^2(3,4) = \frac{\rho(1,3,4) - \rho(1)}{\rho(1,2) - \rho(1)} < 1,
\end{aligned}$$

We obtain: $V_2(X) > V_2(X')$, which is in contradiction with consequentialism. □

Proposition 4.3.2 *$V = ED$ does not collapse to expected additively discounted cash flows.*

Proof We consider the same example as in Proposition 4.3.1. We have:

$$\begin{aligned}
\sum_{t=0}^{2} X_t^2 \Delta\rho(t) &= 1 \times [\rho(0,2) - \rho(2)] + D^2(X) \times \rho(2) \\
&= \rho(0,3,4) = \sum_{t=0}^{4} X_t \Delta\rho(t), \\
\sum_{t=0}^{2} X_t'^2 \Delta\rho(t) &= D^2(X') \times [\rho(1,2) - \rho(1)] + 1 \times \rho(1) \\
&= \rho(1,3,4) = \sum_{t=0}^{4} X'_t \Delta\rho(t).
\end{aligned}$$

Hence relation (4.1) is satisfied for $\tau = 2$.

The same result holds for $\tau = 1$ and $\tau = 3$.

Relation (4.1) is then consistent with a (non additive) capacity ρ and with the conditional capacities defined by Proposition 4.2.1. □

5 Conclusions

In our explicitly dynamic model, consistency of ex-ante and ex-post valuations plays a central role in the definition of conditional Choquet integrals and then of conditional capacities. We can derive updating and "upstating" capacities in the case where information is comonotonic (or antimonotonic) with payoffs. The role of comonotonicity is crucial in our model, it has an interpretation in terms of information: information comonotonic can be interpreted in terms of "good" or "bad" news (depending if future payoffs are greater or lower than past ones). More importantly, comonotonicity has a financial meaning: when two payoffs are comonotonic they cannot hedge each other. This property is central to the Ghirardato–Fubini theorem that yields two hierarchies between preferences on uncertain payoffs and preferences on date contingent payoffs. In practice, comonotonicity is too strong a condition to satisfy and must be understood as a reference for applications to valuation problems where whether the timing or the uncertainty is the most relevant feature. We considered investments where time variations cannot be hedged so that we can apply the criterion *DE* (discounted expectation) over future payoffs in Sect. 3 and situations where uncertain variations cannot be hedged so that we can use the criterion *ED* (expected discounting) in Sect. 4. In both cases, the criterion is a double integral with the first one linear. As can be seen in the examples we referred to, there are cash flows where uncertainty is the primary concern of the DM, others where it is time that is more relevant. In a situation where the timing of decisions is crucial (for instance for a public project such as: when shall we launch a preventive campaign against an epidemics?) the DM may want to concentrate on the conditional discount factors. As we have seen in Sect. 4, the payoffs before information arrives do influence the conditional discount factors: for instance a lot of cash before information strikes may lower the discount factors used for later dates.

Conversely, in many investment problems, it is the uncertain variations of payoffs that are the main concern, for instance, because the DM refers to market (additive) discount factors. Then, the DM needs to know the conditional measure on uncertain states in order to include option values in the present value. As we have seen, this measure depends on the type of information: if it is good news (information goes in the same way than future payoffs), Bayes rule is used; if it is bad news, it is the contrary of information that is taken into account in the updating rule (Dempster-Shafer). In a different model (Chateauneuf et al. 2001) it was shown that in both cases, a DM with uncertainty aversion (convex capacity) is diffident with information.

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