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EVALUATING TIME STREAMS OF INCOME: DISCOUNTING WHAT?

ABSTRACT. For decisions whose consequences accrue over time, there are several possible techniques to compute total utility. One is to discount utilities of future consequences at some appropriate rate. The second is to discount per-period certainty equivalents. And the third is to compute net present values (NPVs) of various possible streams and to then apply the utility function to these net present values. We find that the best approach is to first compute NPVs of various possible income streams and then take the utility of such NPVs. We show the drawbacks of other alternative models of evaluating income streams. The article discusses the advantages of the power and logarithmic forms in the modeling of time preference. These are the only forms for which utility of income and utility of consumption are strategically equivalent. Further, these forms permit the flexibility in the choice of a time period (e.g., monthly or quarterly) without modifying the utility function, thus simplifying analysis.

KEY WORDS: discounted utility, indirect utility of income, net present value, time preference

1. INTRODUCTION

Consider the decision problem of an MBA student who is faced with the decision of selecting a job from a set of alternative job offers that he has received. The student evaluates these jobs on the criteria of first year salary and future salary (say 3 years from now), along with other criteria such as location of the job, functional area, travel requirements, etc. For simplicity, we assume that attributes other than the monetary attributes (first year salary and future salary) are fixed at a reference level and a pricing out procedure (Keeney and

Raiffa, 1976) is used to reduce all jobs in terms of equivalent first year salary, x_1, and future salary, x_2. Thus, all jobs have identical values on non-monetary attributes and differ with one another only on equivalent first year and future salaries. Further, suppose that x_1 and x_2 may be uncertain; for example, a part of the salary may be dependent on the performance of the company.

Keeney and Raiffa (1976) note that several techniques are suggested in practice to evaluate alternatives when consequences accrue over time. The first approach is to take expected utilities at each point in time and to discount these expected utilities. The second approach is to take certainty equivalents at each point in time and to discount these certainty equivalents. The third approach is to discount the various possible certain streams, assess a utility for such present values and then weight these utilities by the respective probabilities of the streams. We will define these approaches precisely in the ensuing sections. Our aim is to evaluate these three approaches from a prescriptive/normative standpoint.

For simplicity, we assume that there are only two time periods and that (x_1, x_2) denotes the stream where a consequence x_1 occurs in period 1 (now) and x_2 occurs in period 2 (later). Consequences represent *income streams* (cash flows, earnings or lottery winnings). In general, consequences in a period may be uncertain.

To motivate the study, we begin in Section 2 with a simple example showing that different methods yield different valuations for the same project. In Section 3, we examine the normative appropriateness of the discounted utility model to evaluate income streams. We show that this model is normatively inappropriate. We also discuss Bell's (1974) model, which is a modification of discounted utility for income streams, which may have some normative appeal. Section 4 examines an alternative model that discounts the per-period certainty equivalents. We show the inappropriateness of this method unless used in the framework of the capital asset pricing model (CAPM). Essentially, these models result in a paradoxical preference for more money later to more money now. Instead of discounting utilities or certainty equivalents, we propose that one should first discount the cash flows and then take the utility of these discounted cash flows. In Section 5, we show that the model using utility of NPV has the desired properties. We further show that utility of discounted cash flows can be naturally interpreted as the indirect utility derived from the optimal consumption of the cash flows. Section 6 studies indirect utility, and characterizes the utility forms for which utility for consumption is strategically equivalent to utility for income. Section 7 concludes.

2. AN EXAMPLE

Consider an example from Bell (1974) where a project yields a profit of \$1 million or a loss of \$90,000 with 50/50 chance. The outcomes of the project are realized one year from now. What should be the certainty equivalent for the firm for this project now? Assuming a discount rate $r = 0.1$, the expected NPV of the outcomes is $\frac{10}{11}[0.5 * 1,000,000 - 0.5 * 90,000] = \$413,636$.

The firm may, however, be risk averse with a utility function:

$$u(x) = \ln(x + 100,000).$$

Table I shows the expected present value, together with the certainty equivalent now for this project using the above utility function under the three methods of evaluation.

Expected NPV does not account for the risk attitude of the decision maker. This is easily seen as the other three methods that incorporate a concave utility function provide for a much lower value of certainty equivalent. Each of these three methods of evaluation have been suggested in the literature. Discounted utility of cash receipts is discussed in Bell (1974) and Clemen (1996). The discounting of certainty equivalents is applied in finance (Brealey and Myers, 1991; Grinblatt and Titman, 1998) in the context of the CAPM. Utility of NPV has been commonly used in decision analysis, where NPV is

TABLE I

Certainty equivalents of a risky project under various methods

Method	Certainty equivalent
Expected NPV	\$413, 636
Discounted utility	−\$63, 334
Discounted certainty equivalent	\$4, 437
Utility of NPV	\$35, 451

taken as the payoff associated to a terminal node of the tree (Smith, 1998; Clemen, 1996).

Under some special conditions, the certainty equivalents of any two of these methods may coincide. For example, if we restrict ourselves to simple lotteries in each period (probability p of some outcome, and $1-p$ of zero) and power utility functions, then the discounted certainty equivalent and the utility of NPV yields the same values. In general, however, for fixed discount rates, these methods will provide different answers, as shown in Table I.

Our aim is to clarify the distinction between these three methods and argue that, in decision analysis, utility of NPV is the most appropriate method. Decision analysts often use the utility of NPV method; however, there has been a sprinkling of suggestions that other methods could be used as well. We will reassure that, in teaching and applications, utility of NPV is the only correct approach.

3. DISCOUNTING UTILITY OF INCOME

Consider the decision problem where the consequences accrue over time. For simplicity, assume that there are only two periods and let (x_1, x_2) denote the stream where a consequence x_1 occurs in period 1 (now) and x_2 occurs in period 2 (later). The consequences x_1 and x_2 could be uncertain. One approach for evaluating such time streams is the discount of utility (DU):

$$U(x_1, x_2) = u(x_1) + \beta u(x_2), \quad 0 < \beta \leq 1, \tag{1}$$

where U is the multi-period utility function, u is the single-period utility function and β is the discount factor for utility. We will later introduce δ as the discount factor for money. In our formulation, it is assumed that the time at which x_2 is received, t, is a variable, and that β is continuously decreasing in t, tending to one as t tends to zero; and tending to zero as t tends to infinity. When x_1 and x_2 are uncertain, $u(x_1)$ and $u(x_2)$ represent expected utilities of these uncertain payoffs. Formula (1) is imposed on all β and $t \geq 0$. Typically, the relation between β and t will take the form $\beta = e^{-rt}$, where r is the relevant (continuous time) discount rate.

In a multi-attribute analysis, the discounted utility model (1) may be subsumed within a more general multi-attribute utility model. For example, in the job selection problem described in Keeney and Raiffa (1976), immediate and future compensation are treated as two attributes along with other attributes such as location, travel requirements and nature of work. An additive value function or utility function over these multiple attributes that includes immediate and future compensation as two separate attributes with a higher weight on immediate compensation and a lower weight on future compensation is essentially a model where the total utility of a job is computed as the sum of the discounted utility of monetary compensation and the utilities derived from other attributes.

We make the following assumptions about the preferences of a decision-maker:

A1 The single-period utility function, u, is monotonically increasing and strictly concave.

A2 A shift of payoff from period 2 (later) to period 1 (now) is preferred, i.e.,

$$(x_1 + \Delta, x_2 - \Delta) \succ (x_1, x_2), \quad \text{for all } x_1, x_2 \text{ and } \Delta > 0.$$

Assumption 2 is equivalent to $(x_1 + \Delta, x_2) \succ (x_1, x_2 + \Delta)$, $\Delta > 0$, and implies $(x, 0) \succ (0, x)$, for all $x > 0$. The appeal of this assumption comes from the observation that money today is preferred to money tomorrow. In descriptive settings people who lack self-control may violate it (Thaler, 1981). Assumption 2 requires, for example, that a lottery winner should opt to receive all of \$50,000 now rather than receiving \$25,000 now and \$25,000 a year from now. This assumption should not be confused with consumption where indeed a level consumption may be more desirable than enormous immediate consumption now and subsistence level consumption later.[1] In Section 6, we discuss the connection between the derived utility of income and the underlying consumption model. Throughout Sections 3–5, consequences are assumed to be income streams.

We now state our first result, which shows that for any increasing, concave utility function u there is a discount factor $\beta < 1$ such that Equation (1) and assumption $A2$ cannot be simultaneously satisfied.

PROPOSITION 1. *Model (1), $A1$ and $A2$ are incompatible.*

Proof. Consider two payoff streams $(x, 0)$ and $(x/2, x/2)$. Set $u(0) = 0$ and $u(x) = 1$. For an increasing, strictly concave u, $u(x/2) = 1/2 + \delta$, for some $0 < \delta < 1/2$. As the time interval, t, between periods 1 and 2 decreases, β increases approaching 1 as $t \to 0$. Now it is easy to see that for $\beta > (1/2 - \delta)/(1/2 + \delta)$, $(1 + \beta)u(x/2) > u(x)$ and $U(x/2, x/2) > U(x, 0)$ thus violating Assumption 2. □

Our observation that the discounting of utilities and risk aversion cannot coexist without violation of Assumption 2 is not an artifact of some extreme case analysis that is obtained when $t \to 0$. For a mildly risk averse person, a violation of assumption $A2$ will appear for a reasonable time interval, t. Consider a case where $0 \leq x \leq 1$ and $t = 1$ year. Further, assume that a decision-maker is endowed with the well known exponential utility function $u(x) = -e^{-x/\rho}$. In this case, $\beta > e^{-1/2\rho}$ will lead to a preference of $(0.5, 0.5)$ over $(1, 0)$. If $\beta = e^{-rt}$, then $t < 1/(2\rho r)$ will produce such a reversal. Similarly, the utility function $u(x)=x^{1-\gamma}$, $0<\gamma<1$, $\beta>2^{1-\gamma}-1$ will lead to a preference of $(0.5, 0.5)$ over $(1, 0)$. In the latter case, $\beta=0.9$ (present value of \$1 received a year from now is 90 cents) and $\gamma>0.074$ (modest relative risk aversion) yields $U(.5, .5)>U(1, 0)$. A γ value greater than 0.074 implies that the certainty equivalent of a 50/50 lottery between \$0 and \$1,000 is less than \$473. Since γ and β are chosen independently, the inconsistency noted above cannot be avoided. Consequently, a decision-maker that has a reasonable discount factor and a reasonable degree of risk aversion may be prescribed a plan that yields more money later compared to a plan that yields more money now. We note that if the additive form of Model (1) is replaced with a multiplicative form Keeney and Raiffa (1976), the paradox remains.

If the function u is differentiable, then we can compute the intertemporal marginal rate of substitution. We recall that an intertemporal marginal rate of substitution bigger than one implies that the decision-maker prefers to reduce today's income a bit in order to increase tomorrow's income a bit, in violation of $A2$. In Model (1),

$$IT_{\mathrm{MRS}}=\frac{\partial U/\partial x_2}{\partial U/\partial x_1}=\beta\frac{u'(x_2)}{u'(x_1)}.$$

The concavity of u readily implies that for $x_1>x_2$, $u'(x_2)/u'(x_1)>1$. Thus, we can always obtain $IT_{\mathrm{MRS}}>1$ by letting β approach 1; but, assumption $A2$ is equivalent to $IT_{\mathrm{MRS}}<1$, which is not generally met by the DU model.

Another assumption that we would like the intertemporal models to satisfy is local substitutability: as the time distance between periods becomes small, the marginal rate of substitution should approach one. Otherwise, a receipt of two dollars, as compared to one dollar now and one dollar a bit later, could produce a utility jump. Formally, the desirable property we want is:

A3 As $\beta\to 1$, $IT_{\mathrm{MRS}}\to 1$.

It is easy to see that A3 requires that $U(x_1, x_2)$ tends to $U(x_1+x_2, 0)$ when the time interval between periods 1 and

2 is made close to zero. For the DU model, it is clear that $IT_{\text{MRS}} \to u'(x_2)/u'(x_1)$ as $\beta \to 1$. Of course, IT_{MRS}, in general, is different from 1, unless $x_1 = x_2$. To illustrate the problem associated with the failure of local substitutability, consider the receipt of (x, x) v. $(2x, 0)$. According to DU, these two streams yield utilities of $2u(x)$ and $u(2x)$, respectively. But if u is strictly concave, then $2u(x) > u(2x)$, which implies that a small delay produces a utility jump. Hence, almost identical streams receive different evaluations. The lack of perfect local substitution is a problem for both income and consumption streams. See Baucells and Sarin (2007) for a Satiation Model, a modified DU model that exhibits perfect local substitution for consumption streams.

3.1. *Bell's (1974) Model*

Bell (1974) suggests a modification of the DU model when more than one cash flow is to be considered. In the case of two cash flows, Bell's model is given by

$$U(x_1, x_2) = u(x_1) + \beta[u(x_1 + x_2) - u(x_1)], \quad 0 < \beta \leq 1. \tag{2}$$

This model exhibits two nice features. First, for $\beta < 1$, it is always the case that $A2$ holds, as:

$$u(x_1 + \Delta) + \beta[u(x_1 + x_2) - u(x_1 + \Delta)] > u(x_1) + \beta[u(x_1 + x_2) - u(x_1)].$$

Alternatively, we observe that the intertemporal marginal rate of substitution

$$IT_{\text{MRS}} = \frac{\partial U/\partial x_2}{\partial U/\partial x_1} = \frac{\beta u'(x_1 + x_2)}{\beta u'(x_1 + x_2) + (1 - \beta) u'(x_1)} < 1.$$

The second nice feature of Bell's model is that it satisfies $A3$: as the time distance between periods is small ($\beta \to 1$), the intertemporal rate of substitution tends to 1.

However, the difficulty with (2) is in the interpretation. In particular, as we discuss in Section 6 the utility for money is supposed to be derived from an indirect utility for consumption. For this to be true in Bell's model, it has to be the case that the decision maker cannot borrow or lend money, and further his utility for consumption must agree with (2).

To us, the natural interpretation of Bell's model is to account for the preferences of a decision maker who cares about income per se, and enjoys contemplating the amount of savings accumulated so far, which are kept in a piggy bank at zero interest. The cumulative income would then be the right measure of performance. However, impatience and diminishing sensitivity have to be accounted for. Impatience is introduced by discounting utility increments, and diminishing sensitivity is captured by the concavity of u.

Bell's model could be an adequate model for the evaluation of "retention" goods, e.g., the progress made in learning a new language. Here, x_t would be the number of new words learned in period t, and (2) is the utility of the progress made during the first two periods.

A minor modification of the model is required to ensure that it agrees with the utility of NPV as a particular case when we set $\beta = 1$. For clarity, we consider three periods. Let the future value of money $R = 1/\delta$, where δ is the money discount rate. For a 10% interest rate, $R = 1.1$ and $\delta = 10/11$.

$$U(x_1, x_2) = u(x_1) + \beta[u(Rx_1 + x_2) - u(x_1)] + \beta^2[u(R^2 x_1 + Rx_2 + x_3) - u(Rx_1 + x_2)].$$

In this model, the decision maker derives joy from contemplating how the money saved in a bank grows. Now, for $\beta = 1$, the total utility is just the utility of the future value of the cash flows, which will provide the same results as the utility of NPV. As shown in Section 6, for $\beta < 1$, both Bell's model and our modification yield inappropriate results. Bell's modified model could be appropriate for corporate decision making, or, for a decision maker who, as Donald Trump, feels that: "Money was never a big motivation for me, except as a way to keep score. The real excitement is playing the game."

4. DISCOUNTING CERTAINTY EQUIVALENTS

We now examine the following model. Begin by computing the expected utility for each period, then calculate the certainty equivalent for each period, and, finally, discount the certainty equivalents. We recall that if X denotes a random payoff, then the certainty equivalent $CE(X)$ is given by

$$CE(X) = u^{-1}(E[u(X)]).$$

The appeal of this model comes from thinking that the per-period certainty equivalent is the amount of money for which the decision-maker is willing to sell the uncertainty of that given period. The model is then:

$$U(X_1, X_2) = CE(X_1) + \beta \cdot CE(X_2), \quad 0 < \beta \leq 1, \tag{3}$$

where $U(X_1, X_2)$ is the discounted certainty equivalent. We then have that:

PROPOSITION 2. *Model (3),* $A1$ *and* $A2$ *are incompatible.*

Proof. Consider alternatives A and B. In alternative A, the decision-maker receives a deterministic payoff stream of $(x/2, x/2)$. In alternative B, on the contrary, the payoffs are uncertain: with probability 1/2 the decision-maker receives a payoff stream of $(x, 0)$, and with probability 1/2 he receives a payoff stream of $(x/2, x/2)$. Clearly, B dominates A in that with probability 1/2 the decision maker receives the entirety of x in period 1, as opposed to receiving $x/2$ now and $x/2$ later otherwise. By $A2$, B should always be preferred to A. The evaluation of alternative A is immediate:

$$U^A(X_1, X_2) = \frac{x}{2} + \beta\frac{x}{2} = x(1+\beta)/2.$$

For the evaluation of B, note that the strict concavity of u given by $A1$ implies that:

$$Eu(X_1) = \frac{1}{2}u(x) + \frac{1}{2}u(x/2) < u(\frac{1}{2}x + \frac{1}{2}x/2) = u(3x/4), \text{ and}$$
$$Eu(X_2) = \frac{1}{2}u(0) + \frac{1}{2}u(x/2) < u(\frac{1}{2}0 + \frac{1}{2}x/2) = u(x/4),$$

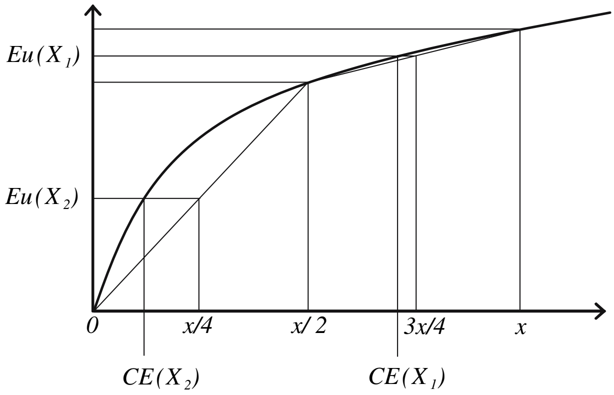

Figure 1. Illustration of the discounting of certainty equivalents.

and because $CE(x)$ is an increasing function we conclude that $CE(X_1) = u^{-1}[Eu(X_1)] < 3x/4$ and $CE(X_2) = u^{-1}[Eu(X_2)] < x/4$ (see Figure 1).

Thus, for some $\delta > 0$, $U^B(X_1, X_2) = CE(X_1) + \beta \cdot CE(X_2) = (3 + \beta)x/4 - \delta$ (δ is the discounted risk premium: if $RP(X_i) = E(X_i) - ce(X_i)$ is the per-period risk premium, then $\delta = RP(X_1) + \alpha RP(X_2)$). As $\beta \to 1$, $U^A(X_1, X_2) \to x$ and $U^B(X_1, X_2) \to x - \delta$. Since $\delta > 0$, $U^A(X_1, X_2) > U^B(X_1, X_2)$ for β close to 1 (small t) is a contradiction. □

In applications of finance, where the decision maker is a corporation with diversified investors, then the discounting of certainty equivalents can be approximately correct, provided one operates in a way consistent with the CAPM framework. In particular, the certainty equivalent cannot be calculated using some given utility function of the decision maker (the decision maker we have in mind is an entrepreneur, and not a well diversified financial investor with mean-variance preferences). Instead, for the CE Method to be correct, it has to be calculated according to the mean-variance approach, in which the risk premium is proportional to the covariance of the cash flow with the market portfolio. Formally, $CE(X_1) = EX_1 - (r_m - r_f)\mathrm{Cov}(X_1, X_m)/\sigma_m^2$, where r_m is the market return, r_f is the risk-free rate and σ_m is the standard deviation of the random variable market return

(X_m). The covariance is linear in the cash flows, that is, covariance of discounted cash flows with the market is the same as the discounted sum of covariances of each period's cash flow with the market. This equality is easily seen as $\mathrm{Cov}(X_1 + \beta X_2, X_m) = \mathrm{Cov}(X_1, X_m) + \beta \mathrm{Cov}(X_2, X_m)$. It follows that, under the assumptions of CAPM, the certainty equivalent of the NPV is the NPV of the certainty equivalents.

For certain corporations and well-diversified investors, the assumptions of the CAPM model might be adequate. In such a case, there is no need to introduce a decision maker's utility function, and market data could be used to evaluate income streams.

In decision analysis, discounting certainty equivalents is not correct even for exponential utilities with a risk tolerance and independent, normally distributed cash flows. Under this assumption, the present value of cash flows is normally distributed. In this case $CE(x) = E(x) - \sigma^2(x)/2\rho$, so that

$$CE(x_1 + x_2) = E(x_1) + \delta E(x_2) - \frac{\sigma^2(x_1) + \delta^2\sigma^2(x_2)}{2\rho} > CE(x_1) + \delta CE(x_2).$$

5. UTILITY OF DISCOUNTED PAYOFFS

We now propose a modification of Equation (1)

$$U(x_1, x_2) = u(x_1 + \delta x_2), \quad 0 < \delta \leq 1. \tag{4}$$

In model (4) above, first payoffs are discounted and then the single period utility function is applied. Also, $\delta = 1/(1+r)$ is the discount factor and r is the lending and borrowing rate. The expectation of U is used to evaluate lotteries over payoff streams.

PROPOSITION 3. *Model (4) A1, and A2 are compatible.*

Proof. Consider two streams:

$$(x_1 + \Delta, x_2 - \Delta) \quad \text{and} \quad (x_1, x_2), \quad \text{where } \Delta > 0.$$

Since u is monotonically increasing, by model (4) we obtain:

$$u(x_1+\delta x_2+\Delta(1-\delta))>u(x_1+\delta x_2).$$

Thus, $(x_1+\Delta, x_2-\Delta)$ is always preferred to (x_1, x_2) for any x_1, x_2 and $\Delta>0$: Assumption $A2$ is satisfied. □

The intertemporal marginal rate of substitution in model (4) is given by

$$IT_{\text{MRS}}=\frac{\partial U/\partial x_2}{\partial U/\partial x_1}=\delta,$$

which is clearly less than 1 by assumption, i.e., the decision maker always prefers the reception of an incremental income in period 1 than in period 2. Moreover, as $\delta\to 1$, the local rate of substitution for money now versus money an instant later tends to one, so that model (4) satisfies $A3$.

Model (4) assumes that consumption decisions are made after all uncertainties about cash flows are resolved. Thus, utility of NPV reflects the utility derived from an optimal consumption plan that is feasible with respect to the realized NPV. Under this assumption, the timing of the resolution of uncertainty is irrelevant and two decisions are equivalent so long as they generate the same probability distribution over NPVs.

6. INDIRECT UTILITY

Suppose x_1 and x_2 are monetary payoffs in periods 1 and 2, respectively. If δ is the discount factor, then (x_1, x_2) is indifferent to $(x_1+\delta x_2, 0)$. We make the reasonable assumption that the decision maker operates in an economic environment where borrowing and lending is possible at some approximately equal rate. Hence, the discount factor δ is a function of the market interest rate, and it is independent of cash flows (x_1, x_2). For a given project (x_1, x_2), the NPV, $x=x_1+\delta x_2$, is the total income available to finance consumption. We now assume that money is a means to an end, and that the utility for money is indirectly derived from the utility of the consumption that can be purchased with that money. The utility of optimal consumption at each date that can be purchased with x, may be called indirect utility of consumption. In decision analysis, however, utility of income or money is often assessed assuming that the decision maker is implicitly solving the consumption problem. Hence, we define the indirect utility for money, $\hat{V}(x)$, as the solution to

$$\begin{aligned} \hat{V}(x) = &\max V(c_1, c_2) \\ &\text{s.t. } c_1 + \delta c_2 \le x_1 + \delta x_2. \end{aligned} \tag{5}$$

Here, $V(c_1, c_2)$ is the utility for consumption that gives rise to $\hat{V}(x)$. This constraint reflects that the present value of money allocated to consumption must not exceed the present value of earnings. In this setup, all uncertainty is resolved before consumption begins. Thus, all projects whose NPV is the same, induce identical optimal consumption plans and, consequently, they all have identical utilities. Lotteries over cash flows can now be converted into lotteries over discounted payoffs, which can be evaluated using a vNM utility. It follows that two projects with uncertain cash flows that yield identical risk profiles (probability distributions) over discounted payoffs are deemed indifferent.[2]

The connection between the utility for consumption and the indirect utility for money, in general, is non-trivial except that $\hat{V}(x)$ is non-decreasing in x. However, in some cases there is a nice link. We first assume that $V(c) = V(c_1, c_2, \ldots, c_T)$ takes the separable DU form:

$$V(c) = v(c_1) + \beta v(c_2), \quad 0 < \beta \le 1. \tag{6}$$

Here, $v(c)$ is the per-period utility function for consumption. The DU model for consumption was axiomatized by Koopmans (1960) and Koopmans et al. (1964), and is commonly used in economics. The indirect utility for money is then the maximization of (6) subject to the budget constraint and taking into account the possibilities for borrowing and lending.

In principle, given the utility for consumption, $u(c)$, there is a corresponding utility for money, $\hat{V}(x)$. For a general $u(c)$ (e.g., $u(c) = a_1c^3 + a_2c^2 + a_3c + a_4$) the corresponding utility for money may be complex and requires a somewhat painful iterative solution of the optimization problem by varying the budget level. The simplest case arises when one can choose amongst uncertain income streams by simply maximizing the expected utility of NPV by assuming a simple utility function for money such as $\ln x$. Clearly, for consistency, it needs to be justified that the underlying consumption problem has been implicitly solved and the chosen income stream will indeed maximize the expected utility of consumption. If the consumption utility $u(c)$ and the money utility $\hat{V}(x)$ are strategically equivalent (positive linear transformation of each other), then a decision analyst can work with income streams directly without having to explicitly solve the underlying, but more basic, consumption problem.

6.1. *Strategic equivalence*

We now show that if the per-period utility exhibits constant relative risk aversion (power or logarithm), then the indirect utility for NPV also exhibits constant relative risk aversion with the same exponent as the per-period utility. Hence, the indirect utility for money is an affine transformation of the per-period utility for consumption. We then say that $\hat{V}(x)$ and $v(c)$ are *strategically equivalent*. In such a case, the elicitation of the utility curve can be done either using money lotteries or consumption lotteries.

The key observation to see this is that in the power and logarithmic case, the optimal consumption in period t, c_t, is always proportional to the NPV, x (Mossin, 1968). Therefore, for some θ_t that depends on δ and β, but not on x, the optimal consumption level in period t is $c_t = \theta_t x$. Under the power and logarithmic form, $v(c) = c^{1-\gamma}$, $\gamma \neq 0$, or $v(c) = \ln c$, $\gamma = 0$, respectively, so that:

$$\hat{V}(x) = \sum_{t=1}^{T} \beta^{t-1} v(\theta_t x) = x^{1-\gamma} \sum_{t=1}^{T} \beta^{t-1} \theta_t^{1-\gamma}, \quad \gamma \neq 1,$$

$$\hat{V}(x) = \ln x \sum_{t=1}^{T} \beta^{t-1} + \sum_{t=1}^{T} \beta^{t-1} \ln \theta_t, \quad \gamma = 1,$$

showing that, in either case, $\hat{V}(x)$ is an affine transformation of $v(c)$.

We now show that power and log are the only forms that exhibit this property.

PROPOSITION 4. *The indirect utility for money, $\hat{V}(x)$, is strategically equivalent to the per-period utility for consumption, $v(c)$, if and only if $v(c)$ is of power or logarithmic form.*

Proof. We have already proved sufficiency. To prove necessity, it suffices to consider the particular case with $\delta = \beta = 1$. The fact that u is concave yields an optimal consumption per period equal to x/T. By the definition of u in (5), $\hat{V}(x) = Tv(x/T)$, which, combined with the strategically equivalence condition $\hat{V}(x) = g(T)v(x) + h(T)$, yields:

$$Tv(x/T) = g(T)v(x) + h(T). \tag{7}$$

By evaluating (7) using $x + \varepsilon$ and substracting (7), we obtain:

$$Tv\left(\frac{x+\varepsilon}{T}\right) - Tv\left(\frac{x}{T}\right) = g(T)[v(x+\varepsilon) - v(x)]$$

Dividing by ε and taking limits produces:

$$v'(x/T) = g(T)v'(x)$$

Letting $y = 1/T$, we obtain from (Aczél, 1966, p. 144) that $v'(xy) = \hat{g}(y)v'(x)$ necessarily implies $v'(x) = ax^b$ and $\hat{g}(y) = \hat{a}y^b$. Hence, $v(x) = \int v'(s)ds = ax^{b+1}/(b+1) + k$ if $b \neq -1$ or $v(x) = a\ln(x) + k$ if $b = -1$. □

In decision analysis, utility functions for money, profit, payoffs, cash position, etc. are directly elicited. The project with the highest expected utility is then chosen. It is often unclear how a particular utility function for money follows from the optimal utilization of that budget to maximize some utility function over consumption. Our result shows that if log or power form are used, then the consistency of the utility of income with the basic consumption problem is easily achieved.

Consider an example with $V(c_1, c_2) = \ln(c_1) + \beta \ln(c_2)$. Suppose, $x_1 = \$300$ and x_2 is a 50/50 lottery with outcomes \$100 and −\$50. For simplicity, let $\delta = \beta = 1$. The decision maker therefore faces a lottery that yields \$400 present value with a 1/2 chance and \$250 with a 1/2 chance. Using utility of NPV approach, the certainty equivalent is calculated as:

$$\ln(CE) = 0.5 \ln(400) + 0.5 \ln(250), \quad CE = \$316.$$

We now verify that the same CE would be obtained if we were to solve the optimal consumption problem. As assumed throughout the article, consumption takes place after the uncertainty is resolved. Hence, we solve (5) twice: once using the income stream (\$300, \$100) and again using (\$300, −\$50). The solution is, of course, a consumption of $c_1 = c_2 = 200$ for $x = \$400$, and of $c_1 = c_2 = 125$ for $x = \$250$. The certainty equivalent for this scenario is the NPV that, consumed optimally, would yield a utility equal to the expected utility. In this case, a consumption of (158, 158) would yield the same expected utility, so that the certainty equivalent is \$316. As discussed above, the indirect utility for NPV in the logarithmic case is $\hat{V}(x) = \ln(x)$. The usefulness of the indirect utility is that it allows us to find the certainty equivalent of this prospect without having to solve the optimization problem.

6.2. *Linear risk tolerance*

A situation where there is a simple relation between the indirect utility for money and the per-period utility for consumption is the exponential case. In this case, $v(c_t) = 1 - e^{-c_t/\rho_t}$, and $\hat{V}(x) = 1 - e^{-x/R}$, where $R = \rho_1 + \delta\rho_2$. So even though u and $\hat{v}$ are not strategically equivalent as was the case in the power and the log forms, these belong to the same exponential class, where the risk tolerance for money is the weighted sum of the risk tolerance of consumption in each period. More generally, we consider the HARA utility class. For a utility function in the HARA class, the risk tolerance associated with the utility of consumption is linear in consumption, c. Therefore, for a

TABLE II
Different forms of the HARA class of linear risk tolerance

Utility of consumption $v(c)$	Risk tolerance $R_v(c)$	Risk tolerance of income $R_{\hat{V}}(x)$	Utility of income $\hat{V}(x)$
$-\exp(-c/\rho)$	ρ	$\rho+\delta\rho$	$-\exp(\frac{-x}{\rho+\delta\rho})$
$(\gamma c+\rho)^{1-\frac{1}{\gamma}}, \ \gamma\neq 1$	$\rho+\gamma c$	$\rho+\delta\rho+\gamma x$	$(\gamma x+\rho+\delta\rho)^{1-\frac{1}{\gamma}}$
$\ln(c+\rho), \ \gamma=1$	$\rho+c$	$\rho+\delta\rho+x$	$\ln(x+\rho+\delta\rho)$

utility function u in the HARA class:

$$R_u(c)=\frac{u'(c)}{u''(c)}=\rho+\gamma c,$$

where ρ and γ are independent of c. Note that for $\rho=0$, one obtains the log or the power form discussed in the previous section. Table II gives the different forms of the HARA class, which include the quadratic for $\gamma=-1$. We now show that if the utility of consumption v_t's are HARA, all sharing the same relative term γ, then the utility of income $\hat{V}(x)$ is also in the HARA class, having the same relative term γ, and an absolute term ρ which is the discount of the absolute terms ρ_t of the per-period risk tolerances. The converse is also true, and the proposition holds for any number of periods.

PROPOSITION 5. *Let risk tolerance of consumption* $R_{v_t}(c_t)=-v_t'(c_t)/v_t''(c_t)$*,* $t=1,2$*, and risk tolerance of income* $R_{\hat{V}}(x)=-\hat{V}'(x)/\hat{V}''(x)$.

$$R_v(c_t)=\rho+\gamma c_t, \quad t=1,2 \Leftrightarrow R_{\hat{V}}(x)=\rho+\delta\rho+\gamma x.$$

Proof. ($\Rightarrow$) From the first order conditions and the budget constraint of (5) one can obtain (Gollier, 2004):

$$\begin{aligned}R_{\hat{V}}(x)&=R_v(c_1)+\delta R_v(c_2)=\rho+\gamma c_1+\delta(\rho+\gamma c_2)\\&=\rho+\delta\rho+\gamma(c_1+\delta c_2)=\rho+\delta\rho+\gamma x.\end{aligned}$$

($\Leftarrow$) Consider the case $\beta=\delta=1$, so that the optimal consumption is $x/2$ in each period. Then,

$$R_{\hat{V}}(x)=R_v(x/2)+R_v(x/2)=\rho+\gamma x,$$

so that $R_v(c)=\rho/2+\gamma c$. □

The above result shows that, if consumption utility is in the HARA class, a decision analyst could legitimately use the utility for money in project evaluation provided that the risk tolerance is appropriately estimated by ensuring consistency with the consumption model. Thus, if the risk tolerance of the per-period utility of consumption is exponential with risk tolerance ρ, then, for the n period time horizon, the risk tolerance for the utility of money should be $\rho+\delta\rho+\cdots+\delta^{n-1}\rho$. The advantage of the log and the power forms in the previous section now becomes vivid as once it is established that the consumption utility is one of the two forms, then the money utility if the same and requires no adjustments.

6.3. *Smith's (1998) model*

In this article, we make the assumption that all uncertainties are resolved before consumption planning begins. In general, consumption will take place before and after some uncertainties are resolved. To illustrate, consider the example in Section 6.1, but now assume that the lottery x_1 is resolved in period 1, but that lottery x_2 is resolved in period 2. The decision maker knows the results of x_1 in choosing the consumption c_1, but does not know the results of x_2; however, he must commit to c_1 without knowing whether x_2 will be high or low. The choice of c_2 will be made after the results of x_2 become known. In order to decide c_1, the decision maker has to consider which values of c_2^h and c_2^ℓ will be used in each scenario. Instead of (5), the problem now is to choose c_1, c_2^ℓ, and c_2^h so as to:

$$\begin{aligned}&\max_{(c_1,c_2^\ell,c_2^h)} \ln(c_1)+\beta[p\ln(c_2^h)+(1-p)\ln(c_2^\ell)]\\&\text{s.t. } c_1+\delta c_2^\ell\le 250;\ \ c_1+\delta c_2^h\le 400.\end{aligned} \tag{8}$$

The solution is $c_1 = 147$, $c_2^{\ell} = 103$, and $c_2^h = 253$; and the certainty equivalent is \$308, with an associated consumption of $c_1 = c_2 = 154$. Notice that when all uncertainties are resolved in the beginning, the certainty equivalent for the project was \$316. The delayed resolution makes the project less attractive with a concomitant reduction in the certainty equivalent. In the case of the delayed resolution of uncertainty, the indirect utility approach is complicated because the indirect utility depends on both $x^{\ell} = 250$ and $x^h = 400$. As the number of periods and the complexity of the timing of resolution increases, one needs to solve a consumption problem of the type (8) to find the utility associated with each terminal branch of the tree.

Surprisingly, there is one case in which the indirect utility approach can be implemented, with some modifications, while accounting for the timing of resolution of uncertainty. In the case when the per-period utilities for consumption are exponential, Smith (1998) finds a procedure to evaluate the cash flow streams without having to solve the consumption problem. The valuation model assumes the ability to borrow and lend money, and favors the early resolution of uncertainty because it improves the planning of consumption. Smith gives a recursive procedure to calculate an effective certainty equivalent of the NPV of a cash flow in each chance node. The idea is to adjust the risk tolerance in each case. In Smith's (1998) procedure the certainty equivalents are always derived from the NPV of the terminal branch of the decision tree, and these certainty equivalents are never discounted. Hence, Smith's model is consistent with the proposed (4) model, and not with the discounting of certainty equivalents. Recall that model (3) computes the certainty equivalents of *per-period* cash flows. More to the point, if one applies Smith's procedure to the case where consumption decisions occur after all uncertainties are resolved, then Smith's procedure particularizes into model (4).

6.4. *Fluctuation Aversion v. risk aversion*

So far, we have assumed that V is a vNM utility function. However, the concave v of the DU model is meant to account for aversion to consumption fluctuation. Thus, $R_v(c_t)$

responds to a consumption *fluctuation tolerance*, rather than to a risk tolerance (Gollier, 2004). However, $V(c) = v(c_1) + \delta v(c_2)$ in itself is silent with regard to risk aversion, and it is better to view $V(c)$ as a multi-attribute value function, where the consumption in different time periods are treated as different attributes. Now consider two situations. In situation A, one receives consumption stream $(1, 1)$ with probability 1/2 and stream $(0, 0)$ with probability 1/2. In situation B, one receives consumption stream $(1, 0)$ with probability 1/2 and consumption stream $(0, 1)$ with probability 1/2. For simplicity, assume $\beta = \delta = 1$. Thus, the expected discounted utility is the same, namely, $v(1)$. However, it is conceivable that the decision maker strictly prefers situation B, because it guarantees some consumption. Rather than introducing a multiplicative factor to account for such interactions (Keeney and Raiffa, 1976), it is simpler to model risk aversion separately. To do so, one could consider a vNM utility function U, whose argument is the value function $V(c)$. The vNM utility of c is then $U(V(c))$, where $U: \mathbb{R} \to \mathbb{R}$ accounts for risk aversion (Dyer and Sarin, 1982; Matheson and Abbas, 2006). In this example, the concavity of U would imply that $U(v(1)) > [U(2v(1)) + U(2v(0))]/2$.

In the context of decisions under uncertainty, the utility of a consumption stream can be directly assessed by a lottery method (i.e., if (c_1, c_2) is indifferent to a 1/2 chance of $(\bar{c}, \bar{c})$ and a 1/2 chance of (c°, c°), where $\bar{c} > c > c^\circ$; then $v(c_1, c_2) = 1/2 v(\bar{c}, \bar{c}) + 1/2 v(c^\circ, c^\circ)$). Thus, aversion to fluctuations in consumption levels from period to period and aversion to risk are both accounted for in the assessed utility v. In some cases, however, assessments and interpretation may be simpler if $V(c_1, c_2)$ is assessed by tradeoff methods capturing fluctuation aversion and then U is assessed over the value function V to account for risk aversion.

6.5. *Choosing time periods*

In applications of the DU model, time periods (monthly, quarterly or yearly) are specified exogenously based on the level of detail desired in an application. The interval between adjacent time periods, Δ, has an obvious influence on the discount factor. The appropriate discount factor for utility and money for an interval Δ is β^{Δ} and δ^{Δ}, respectively. Thus, if β refers to monthly data, then β^3 should be used for quarterly data ($\Delta = 3$). We examine whether per-period utility $v(c)$ needs to be modified as one changes the length of the time period. The evaluation of a consumption stream should remain invariant with respect to a change in the length of a time interval. Thus, if there are T periods with monthly data; then, there will be $T/3$ periods with quarterly data. Hence, β, δ and $v(c)$ must be modified so that the preference ordering among consumption streams (more generally, lotteries over consumption streams) does not change if the analysis is done with monthly data or with quarterly data (i.e., one-third as many periods, each with three times the duration).

Consider a monthly model with $\Delta = 1$ month, $T = 12$ periods and a consumption stream $c = (c_1, \dots, c_{12})$. The evaluation of this consumption stream $V(c)$ is given by $(\sum_{t=0}^{11} \delta^t v(c_t))$. Now consider a quarterly model ($n = 3$), having $\bar{T} = T/n = 4$ periods, each lasting for $\bar{\Delta} = n\Delta = 3$ months. Given $V(c)$ for the monthly model, we need to construct $\bar{V}(\bar{c})$ for the quarterly model. Note that c is a T-dimensional outcome and $\bar{c}$ is a $\bar{T}$-dimensional outcome. For a hypothetical outcome in the modified problem, say $\bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$, $\bar{c}_1$ can be thought of as a budget constraint for the first three months of the original $T = 12$ period model. Similarly, $\bar{c}_2$ can be thought as a budget constraint for the months 4, 5, and 6. Thus, given $\bar{c}_1$ and the separability of $V(c)$, $\bar{v}(\bar{c}_1)$ can be constructed by solving:

$$\begin{aligned} \bar{v}(\bar{c}_1) &= \max_{(c_1, c_2, c_3)} v(c_1) + v(c_2) + v(c_3) \\ \text{s.t.} &\sum_{k=1}^{n} c_1 + \delta c_2 + \delta^2 c_3 \leq \bar{c}_1. \end{aligned} \tag{9}$$

In a similar manner, we can find $\bar{v}(\bar{c}_j)$, $j = 1, \dots, \bar{T}$, and obtain $\bar{V}(\bar{c}) = \bar{v}(\bar{c}_1) + \bar{v}(\bar{c}_2) + \bar{v}(\bar{c}_3) + \bar{v}(\bar{c}_4)$. Essentially, program (9) ensures that, in the quarterly model, the utility evaluation of $(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$ is derived from the original monthly evaluation model in such a way that the quantity available in a quarter is optimally consumed throughout the 3 months of that quarter. The natural interpretation is to assume that the monthly model is behaviorally more accurate, but due to information constraints, the modeler is restricted to use of a quarterly model. Thus, the modeler is interested in deriving the indirect utility on the object $(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$ that is given by the optimal consumption of this total amount over each sub-period.

Noticing that (9) is equivalent to (5), we can now apply the previous results to this framework. In particular, if the per-period utilities for consumption in a quarterly model are HARA, then so are the per-period utilities for consumption in a quarterly model. In the latter case, the relative term γ will remain unaltered, and the absolute term $\rho_{\text{quarterly}}$ will be equal to $\rho_{\text{monthly}}(1 + \delta + \delta^2)$. Hence, the HARA class is very convenient, since the econometrician does not need to change the parametric form as he considers different lengths of periods. Of course, the estimation has to be done each time, to ensure that the absolute term ρ has the appropriate units.

An additional simplification is possible if the per-period utility is power or logarithmic. In this case, $\bar{V}(\bar{c})$ is also power or logarithmic with the same exponent. The same argument given in the proof of Proposition 4 shows that the power and logarithmic forms are the only forms for which $\bar{v}$ and v remain strategically equivalent. The advantage in this case is that the modeler can change the length of the time intervals without having to redo the estimation of the parameter γ.

To highlight the interest of this result, we observe that when the per-period utility is power or logarithmic, the modeler can ignore the "true" length of the time interval (e.g., 1 month) and choose a more convenient length (e.g., a quarter or a year) while retaining the power or logarithmic form. If the per-period utility is of a different form, then the modeler is required to use the "true" or actual time interval associated with the decision-maker's time preference. Alternatively, a modeler could utilize a more convenient time interval, but only after modifying the form of the per-period utility.

Since the "true" length is rarely known, our result renders the power and logarithmic forms appealing for use in project evaluation. In other words, one can estimate the parameter γ without knowing the length of time periods and time span of the consumption model that the decision maker has in mind.

7. CONCLUSIONS

In this article, we consider the problem where the consequences of a decision accrue over time and are uncertain. We examine the case where one element of the decision is an income stream. This case arises in decision analysis or multiattribute utility analysis problems such as job selection where immediate and future compensation serve as a proxy attributes for economic well-being.

Our main result is that risk aversion and discounting of utilities of income cannot coexist without violating the principle that money now is preferable to money later. Specifically, we show that the discounted utility model when applied to income streams or cash flows leads to an undesirable result: postponing income increases utility for an individual with concave utility. The paradox remains if, instead of using discounted utility, one discounts the certainty equivalents in each period. We show that discounting cash flows first and then applying utility to NPVs leads to desirable results.

We also highlight the important modeling advantages of the power and log form to model time preferences. In particular, the exponent of these forms is the same for both the per-period utility for consumption and the utility for money; it is independent of the time horizon of consumption, and independent of the duration of the time period. This makes the elicitation or the econometric estimation of relative risk aversion very robust, since the modeler does not need to know the time horizon or the separation of time periods that the decision making is using in his implicit consumption model.

In contrast, the risk tolerance of the exponential form depends on the time horizon. If the decision maker inadvertently considers that the money received in a lottery has to be consumed within some exogenous time period, then he may give the risk tolerance associated with that time period. The modeling advantages of the power and log forms in multi-period portfolio selection are well known (Mossin, 1968). We have shown some additional advantages of these forms in the broad context of evaluating income streams in decision analysis.

NOTES

1. In a competitive market setting, Fisher's separation theorem provides a justification for the NPV rule for project selection and then maximizing utility of consumption through borrowing and lending. In Decision Analysis, a decision maker's risk attitudes are relevant as markets (e.g., for loans) may be imperfect.
2. If $\beta < 1$, then Bell's model does not satisfy this property. We set $\delta = 1$ (both the original and the modified model are the same), and consider three streams having the same NPV, namely $(x + y, 0)$, (x, y) and $(0, x + y)$. Clearly, the u(NPV) model finds the three streams indifferent. In contrast, Bell's model yields a decreasing preference:

$$u(x+y) > u(x) + \beta[u(x+y) - u(x)] > \beta u(x+y).$$

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