HANNU NURMI

A COMPARISON OF SOME DISTANCE-BASED CHOICE RULES IN RANKING ENVIRONMENTS

ABSTRACT. We discuss the relationships between positional rules (such as plurality and approval voting as well as the Borda count), Dodgson's, Kemeny's and Litvak's methods of reaching consensus. The discrepancies between methods are seen as results of different intuitive conceptions of consensus goal states and ways of measuring distances therefrom. Saari's geometric methodology is resorted to in the analysis of the consensus reaching methods.

KEY WORDS: Borda count, Dodgson's method, Geometry of voting, Kemeny's rule, Litvak's median, Positional procedures, Social choice

1. INTRODUCTION

The bulk of modern social choice theory deals with choice functions, i.e. rules resulting in well-defined sets of winning alternatives or candidates under various constellations of voter opinions. In most cases the winners can also be seen as maximal elements of an underlying ranking. The part of the ranking pertaining to elements not chosen is simply suppressed. Yet, the underlying ranking and its relationship to the voter opinions reveals the basic motivation of the methods and is thus of considerable interest in assessing the methods and their suitability for various kinds of choice problems.

In this paper our focus is on a subset of choice methods. The common feature in the methods under scrutiny is their underlying assumption of a distance to a goal state. The latter, in turn, is defined in terms of the observed opinions of the voters. The ranking produced by methods can be interpreted in terms of the distance of various alternatives from the goal state. The differences between methods then boil partly down to differences in

Theory and Decision **57:** 5–24, 2004.

the goal states and partly to different ways of measuring distances from them.

The main aim of this paper is to spell out these distinctions and to provide a comparative analysis of the methods in terms of a few crucial criteria. The criteria pertain to the responses of choice methods to stimuli that take the form of certain type of opinion changes in the electorate. We start with the introduction of the methods and then, using some theoretical examples, proceed to showing their differences. Their analysis in terms of responsiveness criteria follows, and the paper concludes with a few points regarding the applicability of the results.

2. THE METHODS

Borda's method is well-known in social choice literature (DeGrazia, 1953; Young, 1974; Saari 1990). It is based on rank numbers of alternatives in the voters' preference rankings. Each rank is assigned a number starting from 0 (last) to $k-1$ (first), where k is the number of alternatives under consideration. The difference between two consecutive rank numbers is 1. The Borda score of each alternative is the sum of its rank numbers, i.e. of the numbers it receives from each voter. In the case of tie, the same rank number is assigned to all tied alternatives so that each gets a score that equals the average of the scores that these alternatives would get had there been no tie. The rank number of the alternative that follows next the tied ones, is the one that it would have been given if there had not been a tie preceding it. Thus, for example, if the preference order is: $A_1 \succ A_2 \sim A_3 \sim A_4 \succ A_5$, where $\succ$ denotes strict preference and tied alternatives are separated with $\sim$ from each other, the Borda scores are: $A_1 : 4, A_2 = A_3 = A_4 = 2, A_5 = 0$.[1]

Dodgson's method is a Condorcet extension. In other words, it ends up with a Condorcet winner whenever there is one in the preference profile. A Condorcet winner, in turn, is an alternative that would defeat any other alternative by a simple majority in pairwise votes. In the absence of a Condorcet winner, Dodgson's method looks for an alternative that is closest to the Condorcet winner in the following sense: it can be rendered a Condorcet winner after a minimum number of preference changes of voters.

Kemeny's rule is an explicitly consensus geared method (Kemeny, 1959). If each voter has an identical preference ranking, then that is the obvious collective preference as well. In the absence of such a consensus one looks for a preference ranking that could be reached from the expressed rankings after a minimum number of preference changes.

Like Borda's also Litvak's method is based on a scoring rule (Litvak, 1982). Each preference ranking is assigned a vector of k components. The component i indicates how many alternatives are placed ahead of the i'th one in the ranking under consideration. Each such vector thus consists of numbers $0, \ldots, k-1$. Litvak's rule looks for the k-component vector V that is closest to the observed preferences in the sense that the sum of component-wise absolute differences between the reported preference vectors and V is minimal.[2]

For example, in a 5-person preference profile where three persons have the preference $A \succ B \succ C$ and two the preference $B \succ C \succ A$, the Litvak sums of all six rankings are: $A \succ B \succ C : 8$, $A \succ C \succ B : 14$, $B \succ A \succ C : 10$, $B \succ C \succ A : 12$, $C \succ A \succ B : 20$ and $C \succ B \succ A : 16$. Thus, the Litvak ranking is $A \succ B \succ C$.

3. EXAMPLES

Kemeny's and Dodgson's methods are known to be Condorcet extensions or Condorcet completions, i.e. they result in Condorcet winners in those revealed profiles where such an alternative exists. The Borda count, on the other hand, does not have this property. Litvak's method seems to be based on similar considerations as the Borda count, but may still result in different outcome, as shown by the following example (Table I).

The Borda ranking is $C \succ A \succ B$, while Litvak's is $A \succ C \succ B$. In this example, all Condorcet completion methods, i.a. Dodgson's and Kemeny's rules, would obviously result in A since it is the (strong) Condorcet winner.

TABLE I
Borda and Litvak result in different rankings

Seven voters	Four voters
A	C
C	B
B	A

While Litvak's method in this case results in the Condorcet winner being ranked first, this is not the case in general. In other words, Litvak's ranking may coincide with neither Dodgson's nor Kemeny's rankings. An illustration is given in Table II. In this example there is a Condorcet winner, C, which is thus necessarily ranked first by both Dodgson's and Kemeny's methods. Yet, Litvak's method results in the ranking ACB. Both Condorcet completions end up with the CAB ranking which incidentally also happens to coincide with the Borda ranking.

Several results exist pertaining to the relationships between Dodgson's, Kemeny's and Borda's methods as well as the Condorcet winners and losers. Some of these results are:

- While the Borda count does not necessarily choose the Condorcet winner, the latter is never ranked last by the former (Nanson, 1882; Saari, 1995, p. 157).
- The Condorcet loser is never ranked first by the Borda count (Saari, 1995, p. 157).

TABLE II
Litvak's method does not agree with Condorcet completions

Four voters	Three voters	Two voters
A	B	C
C	C	A
B	A	B

- The Kemeny winner is always higher in the Borda ranking than the Kemeny loser. Conversely, the Borda winner is always ranked higher by Kemeny's rule than the Borda loser (Le Breton and Truchon, 1997; Saari and Merlin, 2000).
- The Dodgson winner can occupy any position in the Kemeny ranking (Ratliff, 2001).
- The Dodgson winner can occupy any position in the ranking based on any positional procedure (e.g. Borda count or plurality rule) (Ratliff, 2002).
- There are profiles in which the Kemeny winner is the Dodgson loser (Klamler, 2003a).

Table II shows that Litvak's method may not choose the Condorcet winner. Most pairwise comparison methods aim at not only choosing the Condorcet winner but also excluding the Condorcet loser. The latter aim is also shared by the Borda count. In contradistinction to these, Litvak's method may end up with a Condorcet loser ranked first. This is demonstrated by the following example (Table III).

The Litvak ranking in this example is ABC, but A is ranked last by an absolute majority of voters and is thus the Condorcet loser. Since it is known that the Condorcet loser is always last in the Kemeny ranking, the same example shows that Litvak's method is also in maximal disagreement with Kemeny's rule, i.e. the Kemeny loser can be the Litvak winner.

Dodgson's method does not necessarily rank the Condorcet loser last. For example in Table III setting, the Dodgson ranking is BAC. In fact, the Condorcet loser may become first in the Dodgson ranking, as shown in Table IV profile.

TABLE III

Litvak's method ranks the Condorcet loser first

Five voters	Three voters	Three voters
A	B	C
B	C	B
C	A	A

There are several other features that differentiate the methods. One of them is monotonicity. A procedure is defined as monotonic if an improvement of a winner's ranking, *ceteris paribus*, does not make it a non-winner. Of the procedures discussed above, Dodgson's method is known to be non-monotonic, while the Borda count and Kemeny's rule are monotonic (Fishburn, 1977; Fishburn, 1982). The following example shows that Litvak's method is non-monotonic (Table V).

Ignoring for the moment the numbers in parentheses, we can express the distances between various rankings *vis-à-vis* the voter preferences as in Table VI.

The Litvak ranking is $B \succ C \succ A$. Consider now an improvement in the winner B's position, *ceteris paribus*, so that the four voters with ranking $A \succ C \succ B$, raise B before C. With no other changes, the profile then becomes one indicated by the numbers in parentheses. In this modified profile the Litvak

TABLE IV
Dodgson winner is the Condorcet loser

Ten voters	Eight voters	Seven voters	Four voters
D	B	C	D
A	C	A	C
B	A	B	A
C	D	D	B

TABLE V
Litvak's method is non-monotonic

4(0) voters	3(7) voters	5(5) voters	3(3) voters
A	A	B	C
C	B	C	B
B	C	A	A

TABLE VI
Computing Litvak's ranking

Ranking	3(7) voters	4(0) voters	0(0) voters	5(5) voters	0(0) voters	3(3) voters
ABC	0	2	2	4	4	4
ACB	2	0	4	4	2	4
BAC	2	4	0	2	4	4
BCA	4	4	2	0	4	2
CAB	4	2	4	4	0	2
CBA	4	4	4	2	2	0

ranking is $A \succ B \succ C$. Improving the Litvak winners positions, thus, can make it non-winner. Hence, monotonicity is violated.

Another property that differentiates the methods under discussion here is vulnerability to reversal bias. Reversal bias occurs whenever the same alternative is ranked first both under a given profile and its reversal. In other words, if an alternative, say x, is chosen when the voters have expressed a given distribution of preferences and x would also be chosen had every voter expressed completely reversed preference ranking, then the procedure used is vulnerable to reversal bias. This notion is discussed by Saari and Barney (2003). It takes on degrees of severity, e.g. when the winner is the same under a profile and its reversal or when the winner and the runner-up remain the same etc. Of the methods discussed here, Saari and Barney show that Kemeny's rule is invulnerable to the reversal bias, while Dodgson's procedure can exhibit it. Of positional methods, only the Borda count is invulnerable to this bias. The following example devised by Tommi Meskanen (2004) shows that Litvak's method may lead to a reversal bias (Table VII).

Applying Litvak's method leads to the Litvak ranking $A \succ B \succ C$ both under the profile of Table VII and its reversal. Note, however, that the individual preferences in the example are not strict. With strict preferences it seems that Litvak's method is immune to reversal bias.

TABLE VII
Litvak's method and reversal bias

One voter	One voter	One voter	One voter
AB	C	A	C
C	AB	B	B
		C	A

Of the methods considered above, only one, viz. Dodgson's, is not homogeneous. Homogeneity is a property that dictates the invariance of the collective choice under multiplication of voters. More specifically, if the number of voters having an identical ranking of alternatives is multiplied by a constant to form a new profile, then the voting outcomes should remain unchanged. Fishburn (1977, p. 477) has shown by way of a counterexample that Dodgson's method is not homogeneous (see also Ratliff, 2001, p. 84).[3] Inhomogeneous systems pose a major challenge to representative arrangements since the outcomes ensuing from the representative body depend not only on the correspondence between the voters' and their representatives' views but also on the size of the representative body.

The above examples show that despite similar in spirit, the methods are not only non-equivalent, but sometimes downright contradictory in their results under identical profiles. The following section aims at explaining these discrepancies.

4. MEANS AND ENDS

Despite their discrepancies under some preference profiles the methods discussed above end up with identical rankings under unanimity. In other words, if all voters have identical preferences over all alternatives, then this shared preference ranking coincides with the Borda, Dodgson, Kemeny and Litvak ranking. In fact, each method can be viewed as minimization of a distance between a goal-state and the prevailing profile, i.e. each method minimizes the number of changes needed to reach the goal-state from the expressed preference profile. What differentiates the methods is the definition of the goal-state and the notion of what constitutes the unit of change.

Let us define the distance function over a set X as any function $d : X \times X \rightarrow R^+$, where R^+ is the set of non-negative real numbers. A distance function d_m is called a metric if the following conditions are met for all elements x, y, z of X:

1. $d_m(x, y) = 0$,
2. if $x \neq y$, then $d_m(x, y) > 0$,
3. $d_m(x, y) = d_m(y, x)$,
4. $d_m(x, z) \leq d_m(x, y) + d_m(y, z)$.

Substituting preference relations for elements in the above conditions we can extend the concept of distance function to preference relations. But these conditions leave open the way in which the distance between two relations is measured. Kemeny's (1959) proposal is the following (see also Baigent, 1987a, b). Let R and R$'$ be two preference relations. Then their distance is:

$$d_K(R, R') = |(R \setminus R') \cup (R' \setminus R)|.$$

The distance is thus counted as the number pairwise preference inversions needed to transform one ranking into another. This is called inversion metric. The distance between a ranking and a preference profile is, then, the sum of the distances between the former and each ranking in the latter. i.e.

$$d(R, P) = \sum_{P_i \in P} d_K(R, P_i),$$

where $P = P_1, \ldots, P_n$ is a profile of rankings and R a ranking.

Kemeny's goal state is that of unanimous preference relation. The method proceeds by considering each possible preference ranking over the alternatives and determining which one of them is closest to the reported preferences in the sense of minimizing the sum of distances between the latter and the ranking under consideration.

The Borda count can be viewed in a similar fashion, i.e. by means of a goal-state and distance minimization (see Nitzan, 1981). The latter is the same as the one used in Kemeny's ranking, but the former, the goal-state, is different. Given a profile P, consider for each alternative $x \in X$ the profile $U(x)$ obtained from P by putting x in the top position of every individual ranking, *ceteris paribus*. Compute now for each x

$$S(x) = \sum_{i \in N} d_K(P_i, U_i(x)).$$

Here $U_i(x)$ (P_i, respectively) is individual i's preference ranking in $U(x)$ (P). $S(x)$ obviously counts the sum of the number of inversions in individual pairwise preference rankings that are needed to make x unanimously first ranked. $S(x)$ is related to the Borda score of x, denoted by $B(x)$, as follows:

$$B(x) = \sum_{i \in N} ((k-1) - d_K(P_i, U_i(x)) = n(k-1) - S(x),$$

where k is the number of alternatives. Thus, maximizing $B(x)$ amounts to minimizing $S(x)$ and the Borda ranking coincides with inverse of the ranking in terms of $S(x)$ scores. We see that Kemeny's and Borda's rules differ, not in terms of metrics, but in terms of the goal states from which the distances are measured.

Turning now to Dodgson's method, its goal state is one where there is a Condorcet winner. Since typically a large number of such states exist, the Dodgson goal state is the one that is closest to the expressed preference profile. The search for the Dodgson ranking can be viewed as defining a goal state for each alternative and then forming the ranking on the basis of the closeness of these goal states to the expressed profile. The closeness, in turn, is measured by the inversion metric.

Litvak's method, in turn, is based on a similar goal state as Kemeny's rule, but uses a different distance measure. This can be seen e.g. comparing the distance of rankings $A \succ B \succ C \succ D$ and $D \succ A \succ B \succ C$, on the one hand, and $A \succ B \succ C \succ D$

and $A \succ C \succ D \succ B$, on the other. In Kemeny's sense these two pairs are equally far apart, but in Litvak's sense the latter distance is strictly smaller.

5. PROFILE COMPONENTS

The above examples demonstrate that the methods analyzed are, indeed, different and that these discrepancies may in some profiles be wide.[4] Further insight into the methods can be gained by utilizing the geometric methodology of Donald Saari (1995, 1999, 2000, 2001b, 2002). In three-alternative cases a convenient expositional device is the representation triangle. Any preference profile over three alternatives can be represented as a triangle where the vertices stand for alternatives. They represent points in three dimensional space. Drawing all median lines results in division of the triangle into six small triangles each of which can be interpreted as a preference ranking so that the more median lines one has to cross when proceeding from any point in a small triangle to each of the vertices, the lower down is the alternative that the vertex stands for in the ranking represented by the small triangle. Profiles can then be concisely described by inserting in each small triangle the number of voters whose preferences coincide with the ranking that the triangle stands for. For example, Figure 1 gives the representation triangle corresponding Table II.

As the triplets of numbers next to alternative names suggest, the large triangle can be viewed as a three-dimensional simplex. In other words, the coordinates of each point in the triangle are non-negative real numbers between 0 and 1 so that they sum to

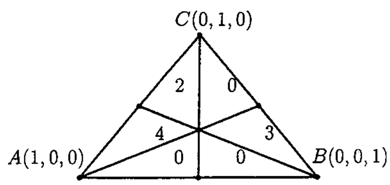

Figure 1. Triangle representing Table II profile.

unity. Thus, they can be given the interpretation of vote shares. So, each point in the triangle represents a distribution of votes expressed as shares of the vote total.

Consider now any positional voting procedure, i.e. $\mathbf{w_s} = (1, s, 0)$, where $\mathbf{w_s}$ is the normalized voting vector in which the first component indicates the normalized weight given to the first ranked alternative, $s(0 \leq s \leq 1)$ the normalized weight of the second ranked one and 0 of the last ranked one (see Saari, 2001a, pp. 43–53). Saari shows that, given any profile over three alternatives, the outcomes resulting from all positional voting procedures can be depicted as a line segment connecting two points located in the representational triangle. One of the points corresponds to the plurality and the other to the anti-plurality voting outcome. The line segment is called the procedure line. The anti-plurality procedure is one where each voter casts a vote for the two highest ranked alternatives, i.e. $\mathbf{w_{ap}} = (1, 1, 0)$. The plurality voting, in turn, is characterized by $\mathbf{w_p} = (1, 0, 0)$. Given any distribution of voters over three alternatives, the outcomes of these two procedures can be plotted in the representational triangle, whereupon the procedure line can be drawn.

The procedure line contains information about the intuitive contestability of electoral outcomes under any given profile. The more median lines the procedure line crosses, the more collective rankings are conceivable under different positional rules. Obviously, by virtue of being a line, the procedure line cannot go through all six small triangles. The maximum number of different strict collective rankings is four. When this is the case, the procedure dependence of electoral outcomes is dramatic: depending on the positional procedure one can point to a given ranking and its complete reversal as possible outcomes in the same profile (see Tabarrok, 2001; Nurmi and Suojanen, 2003 for empirical examples).

Figure 2 gives the triangle representation of the profile presented in Table VIII as well as the procedure line corresponding to this profile. Depending on the positional procedure adopted we could then have one of the following outcomes:

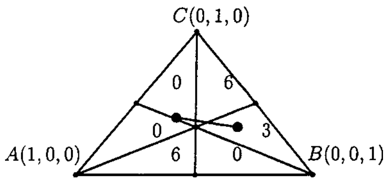

Figure 2. Triangle and procedure line representing Table VIII profile.

TABLE VIII
A 15-voter profile

Six voters	Six voters	Three voters
A	C	B
B	B	C
C	A	A

- $A \sim C \succ B$
- $C \succ A \succ B$
- $B \sim C \succ A$
- $B \succ C \succ A$
- $C \succ A \succ B$
- $C \succ A \sim B$

In fact, only two strict rankings are excluded, viz. $A \succ B \succ C$ and $B \succ A \succ C$.

One of Saari's (1999) results on profiles with three alternatives states that any such profile can be uniquely decomposed. To illustrate, let **p** be a profile expressed as a vector:

$$\mathbf{p} = (p_1, \ldots, p_6),$$

where p_i denotes the number of voters having the preference ranking represented by the i'th small triangle in the representational triangle. In the Table II profile $\mathbf{p} = (0, 4, 2, 0, 3, 0)$. The decomposition then takes the following general form:

$$\mathbf{p} = a\mathbf{K} + \mathbf{p}_{Bas} + \mathbf{p}_R + c\mathbf{C}.$$

Of the components **K** is the neutral profile in which there is one voter for each ranking. The constant a allows for increasing the electorate to get rid of negative numbers of voters that appear in the following components. $\mathbf{P}_{Bas}$ is the basic component constructed by adding one voter for each ranking where a given alternative is ranked first and removing one voter (or adding -1 voters) for each ranking where the same alternative is ranked last. $\mathbf{p}_R$ is the reversal component where one voter is assigned to each ranking where a given alternative is ranked either first or last and -2 voters to each ranking where this alternative is ranked in the middle position. $\mathbf{C}$, in turn, is the Condorcet component where one voter is assigned to each ranking of the Condorcet cycle, i.e. one voter to $A \succ B \succ C$ one to $C \succ A \succ B$ and one to $B \succ C \succ A$ and minus one voter for each of the remaining three rankings.

Saari (1999) gives a general method of decomposing any given profile over three alternatives. It consists of multiplying the profile vector by the following matrix T:

$$T = \frac{1}{6}\begin{pmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In the Table II profile we get:

$$T(0,4,2,0,3,0) = \frac{1}{6}(-1,-5,5,7,1,9).$$

The first two components refer to the basic profile with the additional convention that only A's and B's numbers are shown and C's is assumed to be 0. The order of the two explicit and one implicit numbers indicates the ranking of A, B and C. In this case it is, thus, $C \succ A \succ B$. The next two components $5/6$ and $7/6$ indicate the reversal ranking of $B \succ A \succ C$ (assuming again that C's value is 0). The penultimate component $1/6$ expresses the size of the Condorcet component, while the last entry gives the the number of voters in the neutral profile. Table II profile can thus be seen to consist of a basic profile which suggests the ranking $C \succ A \succ B$. This outcome is "perturbed" by a reversal component which favors B and A. What is not shown in this example is that the Condorcet component may change the Condorcet winner.

In assessing the methods an important question to ask is how various profile components affect the outcomes resulting from the methods. The above definitions of the components suggest the answer:

- The basic profile component is one where all positional procedures are in agreement with the pairwise voting winner.
- The reversal component consists of voters whose preferences are exact opposites of each other (e.g. one voter has $A \succ B \succ C$ and the other $C \succ B \succ A$). Adding reversal components may change the collective rankings of all positional voting procedures except the Borda count which is immune to this type of profile modification.
- The Condorcet component consists of voters whose preferences form a majority cycle with identical margins of pairwise votes. This component may change the Condorcet winner.

Of the procedures discussed above, Kemeny's and Dodgson's rules are Condorcet extensions. Thus, their collective rankings can be affected by adding or removing Condorcet cycles from profiles. An example is given in Figure 3 where the entries marked with "x" constitute a Condorcet cycle. Setting $x = 0$ gives us a profile with a strong Condorcet winner, A. It is thus both Dodgson and Kemeny winner. Now, give x the value 4. This renders B the Condorcet and consequently Dodgson and Kemeny winner.

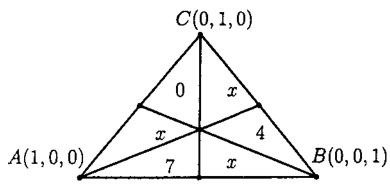

Figure 3. Triangle showing the effect of a Condorcet cycle.

In Figure 3 with $x = 0$ A is obviously the plurality winner, while B is the winner of all positional systems w_s with $s > 3/7$. Adding a Condorcet cycle with one voter in each ranking adds the score of each alternative by $1 + s$. Thus, the relative rankings remain the same under adding or removing Condorcet cycles when positional systems are used.

All positional methods except the Borda count, in turn, can be affected by adding to or removing from profiles voters whose ballots cancel out each other in the sense of being reversals of one another. Figure 4 gives an example. The entries marked with "y" and "z" define two reversal components. With $y = 0$ and $z = 0$ the plurality winner is A, while the Borda winner is B. To make B the plurality winner we need to find values for y and z that would satisfy the following:

$$z + 4 > 7 + y,$$

$$z + 4 > y + z.$$

These expressions yield:

$$z > y + 3,$$

$$y < 4.$$

Thus, for example, values $y = 2$ and $z = 6$ will do. Hence, adding two groups (i) one in which 2 voters have preference $A \succ B \succ C$ and 2 voters having the reversed preference, and (ii) another one with six voters having the preference $B \succ A \succ C$ and six voters the reversal of it, we are able to render B the positional winner.

The Borda exceptionalism is due to the fact that adding a Condorcet cycle increases the Borda score of each alternative by an equal amount, $3x$ if x voters are added to each of the

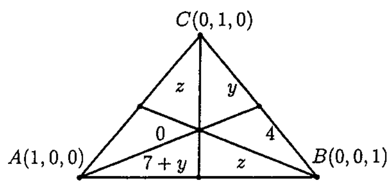

Figure 4. Triangle showing the effect of a reversal component.

TABLE IX
A 7-voter profile

Three voters	Three voters	One voter
A	C	B
B	B	C
C	A	A

three ranking that form the cycle. Thus, the score differences are unaffected by Condorcet cycles. The same is true of reversal components: adding a reversal component consisting of x voters ranking alternatives in a specific way and x voters having the opposite preference increases the Borda score of each alternative by $2x$.

As we have seen, Litvak's procedure is not a Condorcet extension. Adding a Condorcet cycle consisting of one voter in each of the three ranking adds the difference between each logically possible ranking and the preference profile by 8. Thus, the Litvak ranking remains the same after the addition (or removal) of a Condorcet cycle.

Adding a reversal component may change the Litvak ranking. An example is provided by Table IX where the Litvak ranking is $C \succ B \succ A$. Adding now three voters with ranking $A \succ C \succ B$ and three voters with the opposite ranking $B \succ C \succ A$ changes the Litvak ranking into $B \succ C \succ A$.

6. CONCLUSION

The main area of application of the above methods is in committee decision making. Litvak's rule is specifically intended for expert group decision making and this field of application seems well suited for the other systems as well. In these settings it is often important to find out not only the expert choice set or ranking but also how much disagreement exists in the group with regard to the result. Distance-based methods allow for a precise expression of this disagreement. In

Borda count, the difference between the Borda scores allows for this interpretation, but that is also the case in Kemeny's, Litvak's and Dodgson's rules. As we have seen, however, the methods differ both in terms of the state from which the distances to the prevailing profile are measured, and in terms of the distance function used. Kemeny's and Litvak's goal states are identical, viz. ones in which unanimity exists concerning the ranking of each alternative. In Borda's and Dodgson's rules, in contrast, the goal state is either one in which unanimity prevails concerning the alternative ranked first or concerning the Condorcet winner alternative. All rules except Litvak's are based on inversion metric.

Saari's geometric approach sheds new light on the distance-based methods. Thus, for example, we observe that Kemeny's and Dodgson's methods, in virtue of being Condorcet completions, are vulnerable to adding and subtracting Condorcet cycles. Litvak's ranking, in turn, appears to be vulnerable to adding or subtracting groups of voters whose preferences cancel out each other. Only the Borda ranking remains robust under these types of transformations.

ACKNOWLEDGEMENTS

The author thanks Manfred J. Holler, Christian Klamler, Tommi Meskanen and anonymous referees for important comments on an earlier version.

NOTES

1. In the tables that follow the rankings are expressed as vertical sequences with more preferred alternatives above less preferred ones.
2. Bury and Wagner (2003) give a concise discussion on Litvak's method and provide computational algorithms for determining the Litvak median.
3. Due to the limit device used in Fishburn's interpretation of Dodgson's rule, it is homogeneous, but the counterexample shows that the more widely used definition (which does not refer to limit behavior) makes the rule inhomogeneous.

4. It is worth mentioning in passing that the notion of discrepancy between methods is not without ambiguity. At least two nonequivalent senses can be distinguished: (i) discrepancy as the distance of rankings stemming from methods under identical profiles (no matter how infrequent), and (ii) the frequency of the occurrence of nonidentical (no matter how slightly different) results by the methods under identical profiles. Klamler (2003b) resorts to the former interpretation, while Nurmi (1988) represents the latter view.

REFERENCES

Baigent, N. (1987a), Preference proximity and anonymous social choice, *The Quarterly Journal of Economics* 102, 161–169.

Baigent, N. (1987b), Metric rationalization of social choice functions according to principles of social choice, *Mathematical Social Sciences* 13, 59–65.

Bury, H. and Wagner, D. (2003), Use of preference vectors in group judgement: The median of Litvak, in Kacprzyk J. and Wagner D. (eds), *Group Decisions and Voting* (Exit, Warszawa).

DeGrazia, A. (1953), Mathematical derivation of an election system, *Isis* 44, 42–51.

Fishburn, P.C. (1977), Condorcet social choice functions, *SIAM Journal of Applied Mathematics* 33, 469–489.

Fishburn, P.C. (1982), Monotonicity paradoxes in the theory of voting, *Discrete Applied Mathematics* 4, 119–134.

Kemeny, J. (1959), Mathematics without numbers, *Daedalus* 88, 571–591.

Klamler, Ch. (2003a), The Dodgson ranking and its relation to Kemeny's method and Slater's rule, *Social Choice and Welfare*, forthcoming.

Klamler, Ch. (2003b), On the closeness aspect of three voting rules: Borda-Copeland-maximin, *Group Decision and Negotiation*, forthcoming.

Le Breton, M. and Truchon M. (1997), A Borda measure for social choice functions, *Mathematical Social Sciences* 34, 249–272.

Litvak, B.G. (1982), *Information Given by the Experts. Methods of Acquisition and Analysis* (Radio and Communication, Moscow, Russian).

Meskanen, T. (2004), private communication.

Nanson, E.J. (1882), Methods of election, *Transactions and Proceedings of the Royal Society of Victoria XIX*, 197–240.

Nitzan, S. 1981, Some measures of closeness to unanimity and their implications, *Theory and Decision* 13, 129–138.

Nurmi, H. 1988, Discrepancies in the outcomes resulting from different voting schemes, *Theory and Decision* 25, 193–208.

Nurmi, H. and Suojanen M. (2003), Assessing contestability of electoral outcomes: An illustration of Saari's geometry of elections in the light of the 2001 British parliamentary elections, *Quality and Quantity*, forthcoming.

Ratliff, Th.C. (2001), A comparison of Dodgson's method and Kemeny's rule, *Social Choice and Welfare 18*, 79–89.

Ratliff, Th.C. (2002), A comparison of Dodgson's method and the Borda count, *Economic Theory 20*, 357–372.

Saari, D.G. (1990), The Borda dictionary, *Social Choice and Welfare* 7, 279-317.

Saari, D.G. (1995), *Basic Geometry of Voting* Berlin, Heidelberg, New York, Springer-Verlag.

Saari, D.G. (1999), Explaining all three-alternative voting outcomes, *Journal of Economic Theory* 87, 313–355.

Saari, D.G. (2000), Mathematical structure of voting paradoxes I: Pairwise vote, *Economic Theory* 15, 1–53.

Saari, D.G. (2001a), *Chaotic Elections. A Mathematician Looks at Voting* Providence, RI: American Mathematical Society.

Saari, D.G. (2001b), Analyzing a nail-biting election, *Social Choice and Welfare* 18, 415–430.

Saari, D.G. (2002), Adopting a plurality perspective, *Mathematics of Operations Research* 27, 45–64.

Saari, D.G. and Barney S. (2003), Consequences of reversing preferences, *Mathematical Intelligencer* 25, 17–31.

Saari, D.G. and Merlin V. (2000), A geometric examination of Kemeny's rule, *Social Choice and Welfare* 17, 403–438.

Tabarrok, A. (2001), President Perot or fundamentals of voting theory illustrated with the 1992 election, *Public Choice 106*, 275–297.

Young, H.P. (1974), An axiomatization of Borda's rule, *Journal of Economic Theory 9*, 43–52.

Address for correspondence: Hannu Nurmi, Academy of Finland, Department of Political Science, University of Turku, 20014 TURKU, Finland.