

# On the distribution of the product and ratio of products of EGK variates with applications

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**Abstract** We derive novel exact closed-form expressions for the probability density function (PDF) and cumulative distribution function (CDF) of the product and ratio of products of an arbitrary number of independent non-identically distributed (i.n.i.d) extended generalized- $\mathcal{K}$  (EGK) variates. The expressions are given in terms of the Meijer's G-function and can be computed easily using commonly available mathematical software tools. They also subsume those for arbitrary combinations of other well-known variates and can be directly utilized in performance evaluation of wireless communication systems under different scenarios. We present various analytical results that are verified via Monte-Carlo simulations for both the PDF and CDF as well as their application in multiple practical scenarios.

**Keywords** EGK variates · Fading channels · Product of random variables · Ratio of product of random variables

## 1 Introduction

The product and ratio of products of random variables (RVs) arise a lot in the wireless communication literature. For example, they are usually encountered in performance evaluation studies over cascaded fading channels [1], in keyhole channel modeling of multiple-input multiple-output (MIMO) systems [2], in multi-hop communication systems [3] or when modeling the statistics of quantities such as the signal-to-

interference ratio (SIR) in the aforementioned scenarios or simply over composite fading channels [4].

Lots of works in the literature tackled the previously mentioned problems assuming various distributions for the involved RVs. For example, Mekić et al. obtained the distribution of the product of Rayleigh, Weibull and Nakagami- $m$  RVs and applied their results to multi-hop communication systems [5]. Also, Rathie et al. obtained the distribution of the product of generalized shifted gamma RVs and pointed out some applications in wireless communication and multi-carrier systems in [6]. They also obtained the distributions of the product and the ratio of  $\alpha - \mu$  RVs and utilized their expressions to analyze the outage, delay-limited as well as the ergodic capacities over these fading channels in [7]. The statistics of the ratio of products of  $\alpha - \mu$  RVs were also discussed in [8]. The work in [9] investigated the level crossing rate of the SIR, which was represented as the ratio of the product of two  $\kappa - \mu$  RVs and a Nakagami- $m$  one. Also, the SIR statistics where the desired signal experiences gamma long-term fading and  $\kappa - \mu$  short-term fading while the co-channel interference is affected by  $\kappa - \mu$  fading have been studied in [10]. Finally, the work in [11] derived the characteristic function for the product of Gaussian RVs and that for the product of a gamma RV and a zero-mean unity-variance Gaussian one.

Other recent works in the statistics literature that studied the product and ratio of products of RVs in a more abstract form while assuming other less-commonly used distributions include [12–14]. Specifically, in [12], the distribution of the product of two independent generalized trapezoidal RVs has been derived in closed form. The results for the product of two triangular and uniform RVs are then presented as special cases of the main result. In [13], Nagar et al. obtained the distribution of the product and the ratio of independent as well as correlated Macdonald RVs. Finally, Rathie et al. derived

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the exact probability density function (PDF) and cumulative distribution function (CDF) of the product and the quotient of two independent stable Lévy RVs in terms of Fox’s H-function [14].

The extended generalized- $\mathcal{K}$  (EGK) is a general statistical distribution that was recently used to model fading in free-space optical (FSO) environments as well as wireless millimeter wave channels that are candidates for use in 5G technology [15]. This distribution has five parameters, it has some good tail properties and includes most of the well-known fading distributions in the literature as either special or limiting cases ([15, Table I]). In spite of its importance and generality, very few works in the open literature have studied the statistical properties of the product of EGK variates and none has actually studied the ratio of their products. Specifically, the work in [16] has obtained an expression for PDF and CDF only for the product of generalized Nakagami- $m$  variates. These expressions can be used to obtain those for the product of EGK RVs since an EGK variate is, in fact, the product of two generalized Nakagami- $m$  ones. However, these expressions were reported in terms of the Fox’s H-function, which is not very easy to compute. Motivated by this, in this work, we propose to use the Mellin transform to derive novel exact closed-form expressions for both the PDF and the CDF of the product and ratio of products of an arbitrary number of independent non-identically distributed (i.n.i.d.) EGK variates. The obtained expressions are given in terms of the Meijer’s G-function, which is readily available in most mathematical and engineering software packages such as Mathematica, MATLAB and Maple and can be easily computed. They also subsume those for the product and ratio of products of arbitrary combinations of other variates that are special cases of the EGK model. We verify the validity of the obtained expressions via the use of Monte Carlo simulations and we present various selected scenarios where the obtained expressions are applicable. To the best of the authors’ knowledge, the resulting expressions have never been reported before in the literature.

### 2 The EGK statistical model

Let  $X \geq 0$  be the received signal power in a wireless communication system. Assuming that  $X$  follows the EGK distribution, the PDF  $f_X(x)$  can be expressed as [15]

$$f_X(x) = \frac{\xi \kappa^\xi m x^{\xi m - 1}}{\Gamma(m)\Gamma(m_s)} \Gamma\left(m_s - \frac{m\xi}{\xi_s}, 0, \kappa^\xi x^\xi, \frac{\xi}{\xi_s}\right) \tag{1}$$

where  $\kappa = \frac{\beta\beta_s}{\bar{\gamma}_s}$ ,  $m \geq 0.5$  is the fading figure,  $\xi \geq 0$  is the fading shaping factor,  $m_s \geq 0.5$  represents the shadowing severity,  $\bar{\gamma}_s$  is the average power of the envelope of the received signal and  $\xi_s \geq 0$  represents the shadowing shaping

factor. Furthermore,

$$\beta = \frac{\Gamma\left(m + \frac{1}{\xi}\right)}{\Gamma(m)} \text{ and } \beta_s = \frac{\Gamma\left(m_s + \frac{1}{\xi_s}\right)}{\Gamma(m_s)}.$$

In the previous expressions,  $\Gamma(\cdot)$  is the gamma function and  $\Gamma(\cdot, \cdot, \cdot, \cdot)$  is the extended incomplete gamma function defined as  $\Gamma(\alpha, x, b, \beta) = \int_x^\infty r^{\alpha-1} e^{-r-br^{-\beta}} dr$  where  $\alpha, b, \beta \in \mathbb{C}$  and  $x \in \mathbb{R}^+$ .

### 3 The product of EGK variates

Let  $X_1, X_2, \dots, X_n$  be  $n$  independent EGK RVs with parameters  $(m_i, \xi_i, m_{s_i}, \xi_{s_i}, \bar{\gamma}_{s_i})$  for  $i = 1, 2, \dots, n$  and let  $Y_n$  be their product, i.e.,  $Y_n = \prod_{i=1}^n X_i$ . In this section, we are interested in finding both the PDF and CDF of  $Y_n$ .

#### 3.1 Derivation of the PDF

In order to find the PDF  $f_{Y_n}(y)$ , we use the fact that  $Y_n = Y_{n-1} \times X_n$  in conjunction with the well-known formula for the PDF of the product of two RVs in [17, Example 7.11] to recursively define the required PDF as

$$f_{Y_m}(y) = \int_0^\infty \frac{1}{t} f_{Y_{m-1}}\left(\frac{y}{t}\right) f_{X_m}(t) dt, \tag{2a}$$

$$1 < m \leq n. \tag{2b}$$

$$f_{Y_1}(y) = f_{X_1}(y), \tag{2b}$$

Note that using such a recursive equation to find  $f_{Y_n}(y)$  is tedious in its current form. However, solving it in the Mellin domain is much simpler. The Mellin transform is defined as  $\tilde{f}(s) = \mathcal{M}\{f(x)\} = \int_0^\infty x^{s-1} f(x) dx$  [18, Ch. 8]. Hence, by taking the Mellin transform of the two sides of the recursive relation and using [18, Eq. (8.3.18)], one gets  $\tilde{f}_{Y_m}(s) = \tilde{f}_{Y_{m-1}}(s)\tilde{f}_{X_m}(s)$  and  $\tilde{f}_{Y_1}(s) = \tilde{f}_{X_1}(s)$ . Consequently, it is straightforward to see that  $\tilde{f}_{Y_n}(s) = \prod_{i=1}^n \tilde{f}_{Y_i}(s)$ . The inverse Mellin transform can then be used to find the PDF, which, through direct integration, will lead to the CDF. We start by finding the Mellin transform of  $\Gamma(\alpha, 0, x, \beta)$ , which is given as

$$\mathcal{M}\{\Gamma(\alpha, 0, x, \beta)\} = \int_0^\infty \int_0^\infty r^{\alpha-1} e^{-r-xr^{-\beta}} x^{s-1} dr dx. \tag{3}$$

According to Fubini’s theorem [19], the order of the integration can be switched. Doing that and using the substitution  $u = r^{-\beta}x$  along with the definition of the Gamma function, one directly arrives at

$$\mathcal{M}\{\Gamma(\alpha, 0, x, \beta)\} = \Gamma(s)\Gamma(\beta s + \alpha).$$

Finally, using the above result along with the homogeneity property of the Mellin transform followed by the identities in [18, Eqs. (8.31), (8.32), (8.33)], we get

$$\tilde{f}_{X_i}(s) = \frac{k_i}{\lambda_i^s} \Gamma\left(\frac{s}{\xi_i} + m_i - \frac{1}{\xi_i}\right) \Gamma\left(\frac{s}{\xi_{s_i}} + m_{s_i} - \frac{1}{\xi_{s_i}}\right), \quad (4)$$

where  $k_i = \frac{\lambda_i}{\Gamma(m_i)\Gamma(m_{s_i})}$  and  $\lambda_i = \frac{\beta_{l_i}\beta_{s_{l_i}}}{\gamma_{s_i}}$ . Hence,

$$\tilde{f}_{Y_n}(s) = \frac{k}{\lambda^s} \prod_{i=1}^n \Gamma\left(\frac{s-1}{\xi_i} + m_i\right) \Gamma\left(\frac{s-1}{\xi_{s_i}} + m_{s_i}\right), \quad (5)$$

where  $k = \prod_{i=1}^n k_i$  and  $\lambda = \prod_{i=1}^n \lambda_i$ . Now, taking the inverse Mellin transform and using the change of variable  $s = w - 1$ , the required PDF is obtained as

$$\begin{aligned} f_{Y_n}(y) &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{f}_{Y_n}(w) y^{-w} dw \\ &= \frac{k}{2\pi i \lambda y} \int_{\sigma-1-i\infty}^{\sigma-1+i\infty} (\lambda y)^{-s} \prod_{i=1}^n \Gamma\left(\frac{s}{\xi_i} + m_i\right) \\ &\quad \Gamma\left(\frac{s}{\xi_{s_i}} + m_{s_i}\right) ds. \end{aligned} \quad (6)$$

To find  $f_{Y_n}(y)$  in a more tractable form, let  $\xi_i$  and  $\xi_{s_i}$  be rational numbers for  $1 \leq i \leq n$ . Note that such condition will not affect the generality of the results since any real number could be approximated by rational numbers with an arbitrarily small error. As a result, we can write  $\xi_i = \frac{a_i}{b_i}$  and  $\xi_{s_i} = \frac{a_{s_i}}{b_{s_i}}$  where  $a_i, b_i, a_{s_i}, b_{s_i}$  are positive integers such that  $\text{GCD}(a_i, b_i) = \text{GCD}(a_{s_i}, b_{s_i}) = 1, \forall i$ , where GCD is the greatest common divisor. Now, letting  $z = \text{LCM}(a_1, a_2, \dots, a_n, a_{s_1}, a_{s_2}, \dots, a_{s_n})$ , where LCM is the least common multiple and by using the change of variable  $s = zu$  and setting  $\gamma = z(\sigma - 1)$ , we get

$$\begin{aligned} f_{Y_n}(y) &= \frac{kz}{2\pi i \lambda y} \times \\ &\int_{\gamma-i\infty}^{\gamma+i\infty} (\lambda y)^{-zu} \prod_{i=1}^n \Gamma\left(\frac{z}{\xi_i} u + m_i\right) \Gamma\left(\frac{z}{\xi_{s_i}} u + m_{s_i}\right) du. \end{aligned} \quad (7)$$

For a more compact notation, we also let  $A_i = \frac{z}{\xi_i}, A_{s_i} = \frac{z}{\xi_{s_i}}, B_i = \frac{\xi_i m_i}{z}$  and  $B_{s_i} = \frac{\xi_{s_i} m_{s_i}}{z}$ . Noting that  $A_i$  and  $A_{s_i}$  are positive integers, Gauss's multiplication formula [20, Eq. (6.1.20)] can be applied to the integrand in (7) to arrive at

$$f_{Y_n}(y) = \frac{\eta}{2\pi i y} \int_{\gamma-i\infty}^{\gamma+i\infty} (\lambda h y)^{-zu} \times$$

$$\prod_{i=1}^n \left[ \prod_{j=0}^{A_i-1} \Gamma\left(u + B_i + \frac{j}{A_i}\right) \prod_{j=0}^{A_{s_i}-1} \Gamma\left(u + B_{s_i} + \frac{j}{A_{s_i}}\right) \right] du \quad (8)$$

where

$$\begin{aligned} \eta &= \frac{kzH}{\lambda A^*} (2\pi)^{n-A}, \quad A = \frac{1}{2} \sum_{i=1}^n (A_i + A_{s_i}), \\ A^* &= \sqrt{\prod_{i=1}^n A_i A_{s_i}}, \quad h_i = A_i^{A_i}, \quad h_{s_i} = A_{s_i}^{A_{s_i}}, \\ H &= \prod_{i=1}^n h_i^{B_i} h_{s_i}^{B_{s_i}} \quad \text{and} \quad h^{-z} = \prod_{i=1}^n h_i h_{s_i}. \end{aligned}$$

Finally, using the definition of the Meijer's G-function in [21, Eq. (07.34.02.0001.01)] and the identity in [21, Eq. (07.34.17.0011.01)], we get the following result.

$$\begin{aligned} f_{Y_n}(y) &= \eta \lambda h G_{0,2A}^{2A,0} \left[ (\lambda h y)^z \left| \Delta\left(A_i, m_i - \frac{1}{\xi_i}\right)_{i=1:n}, \Delta\left(A_{s_i}, m_{s_i} - \frac{1}{\xi_{s_i}}\right)_{i=1:n} \right. \right], \end{aligned} \quad (9)$$

where

$$\Delta(a, b) = \frac{b}{a}, \frac{b+1}{a}, \dots, \frac{b+a-1}{a}.$$

To the best of the authors' knowledge, this obtained PDF expression has never been reported before in the literature and it is easy to evaluate as it is given in terms of the Meijer's G-function.

### 3.2 Derivation of the CDF

The CDF can be obtained by direct integration of the expression in (9). Towards that end, we first use [21, Eq. (07.34.17.0011.01)] to put the Meijer's G-function in a more suitable form for integration as follows

$$\begin{aligned} f_{Y_n}(y) &= \eta (\lambda h)^z y^{z-1} \times \\ &G_{0,2A}^{2A,0} \left[ (\lambda h y)^z \left| \Delta(A_i, m_i - A_i)_{i=1:n}, \Delta(A_{s_i}, m_{s_i} - A_{s_i})_{i=1:n} \right. \right]. \end{aligned} \quad (10)$$

By letting  $u = (\lambda h y)^z$ , integrating (10) and using [21, Eq. (07.34.21.0001.01)], the following expression for the CDF immediately follows:

$$F_{Y_n}(y) = \frac{\eta}{z} \times$$

$$G_{1,2A+1}^{2A,1} \left[ (\lambda h y)^z \left| \Delta(A_i, m_i)_{i=1:n}, \Delta(A_{s_i}, m_{s_i})_{i=1:n}, 0 \right. \right]. \tag{11}$$

$$\begin{aligned} \tilde{f}_{Y_n}(s) &= \frac{\lambda h}{2\pi i} \int_0^\infty y^{s-1} \int_{\gamma-i\infty}^{\gamma+i\infty} \eta(\lambda h)^{-zs} \prod_{i=1}^n \\ &\left[ \prod_{j=0}^{A_i-1} \Gamma\left(s + B_i - \frac{1}{z}\right) \right. \\ &\left. + \frac{j}{A_i} \right) \prod_{j=0}^{A_{s_i}-1} \Gamma\left(s + B_{s_i} - \frac{1}{z} + \frac{j}{A_{s_i}}\right) \right] y^{-zs} ds dy. \end{aligned} \tag{14}$$

### 4 The ratio of products of EGK variates

Now, we switch our attention to find the distribution of  $Q = \frac{Y_n}{W_m}$ , where  $Y_n = \prod_{i=1}^n X_i$  and  $W_m = \prod_{j=1}^m R_j$ ,  $X_1, X_2, \dots, X_n$  and  $R_1, R_2, \dots, R_m$  are  $n$  and  $m$  independent EGK RVs with parameters  $(m_i, \xi_i, m_{s_i}, \xi_{s_i}, \bar{\gamma}_{s_i})$  and  $(m'_j, \xi'_j, m'_{s_j}, \xi'_{s_j}, \bar{\gamma}'_{s_j})$ , respectively, for  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m$ .

#### 4.1 Derivation of the PDF

Using standard analysis, the PDF of the ratio can be obtained as

$$f_Q(q) = \int_0^\infty x f_{Y_n}(qx) f_{W_m}(x) dx. \tag{12}$$

Clearly, given the nature of the integrand, the integration in (12) is not easy to deal with. Hence, we again propose to use Mellin transform to find  $f_Q(q)$ . Taking the Mellin transform for both sides of (12) and using [18, Eqs. (8.3.2), (8.3.19)], we get

$$\tilde{f}_Q(s) = \tilde{f}_{Y_n}(s) \tilde{f}_{W_m}(2-s). \tag{13}$$

Although the Mellin transform of  $f_{Y_n}(y)$  (and consequently of  $f_{W_m}(w)$ ) is readily available from (5), we opt for an alternative expression that will lead to a simpler result as we will show in the sequel. Before finding these new expressions, we first note that the result in (9) still holds after replacing  $z$  with any of its multiples. Hence, for convenience, throughout the following derivation,  $z$  will be redefined so that it applies for both the numerator ( $Y_n$ ) and the denominator ( $W_m$ ), i.e.,

By using the change of variable  $t = zs$  and the inverse Mellin transform formula [18, Eq. (8.2.6)], we directly get

$$\begin{aligned} \tilde{f}_{Y_n}(s) &= \frac{\eta}{z} (\lambda h)^{1-s} \prod_{i=1}^n \left[ \prod_{j=0}^{A_i-1} \Gamma\left(\frac{s}{z} + B_i - \frac{1}{z} + \frac{j}{A_i}\right) \times \right. \\ &\left. \prod_{j=0}^{A_{s_i}-1} \Gamma\left(\frac{s}{z} + B_{s_i} - \frac{1}{z} + \frac{j}{A_{s_i}}\right) \right]. \end{aligned} \tag{15}$$

Using similar steps while adopting the previous notations, the Mellin transform of the PDF of  $W_m$  can be obtained as

$$\begin{aligned} \tilde{f}_{W_m}(s) &= \frac{\eta'}{z} (\lambda' h')^{1-s} \prod_{i=1}^m \left[ \prod_{j=0}^{A'_i-1} \Gamma\left(\frac{s}{z} + B'_i - \frac{1}{z} + \frac{j}{A'_i}\right) \times \right. \\ &\left. \prod_{j=0}^{A'_{s_i}-1} \Gamma\left(\frac{s}{z} + B'_{s_i} - \frac{1}{z} + \frac{j}{A'_{s_i}}\right) \right]. \end{aligned} \tag{16}$$

Substituting (15) and (16) into (13), rearranging the terms, taking the inverse Mellin transform and using the change of variable  $t = s/z$  along with [21, Eq. (07.34.17.0011.01)], we finally arrive at the result in (17) shown below. In this equation,  $\eta_Q = \eta\eta'/z$ ,  $\lambda_Q = \lambda/\lambda'$  and  $h_Q = h/h'$ . Like (9), this PDF expression is also novel and has never been reported before in the literature.

$$f_Q(q) = \frac{\eta_Q}{q} (\lambda_Q h_Q q)^z G_{2A',2A}^{2A,2A'} \left[ (\lambda_Q h_Q q)^z \left| \Delta(-A'_i, m'_i)_{i=1:m}, \Delta(-A'_{s_i}, m'_{s_i})_{i=1:m} \right. \right] \tag{17}$$

$$F_Q(q) = \frac{\eta_Q}{z} G_{2A'+1,2A+1}^{2A,2A'+1} \left[ (\lambda_Q h_Q q)^z \left| 1, \Delta(-A'_i, m'_i - A'_i)_{i=1:m}, \Delta(-A'_{s_i}, m'_{s_i} - A'_{s_i})_{i=1:m} \right. \right], \tag{18}$$

$z = \text{LCM}(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$  where  $\Omega_1 = a_1, \dots, a_n$ ,  $\Omega_2 = a_{s_1}, \dots, a_{s_n}$ ,  $\Omega_3 = a'_1, \dots, a'_m$  and  $\Omega_4 = a'_{s_1}, \dots, a'_{s_m}$ . Now, using the definition of the Meijer's G-function and (9), one gets the following result.

#### 4.2 Derivation of the CDF

Integrating (17) and using [21, Eq. (07.34.21.0001.01)], one directly arrives at the CDF expression given in (18) shown

above. In the next section, we present various numerical results for the PDF and CDF of both the product and ratio of products along with a selected application.

## 5 Numerical and simulation results

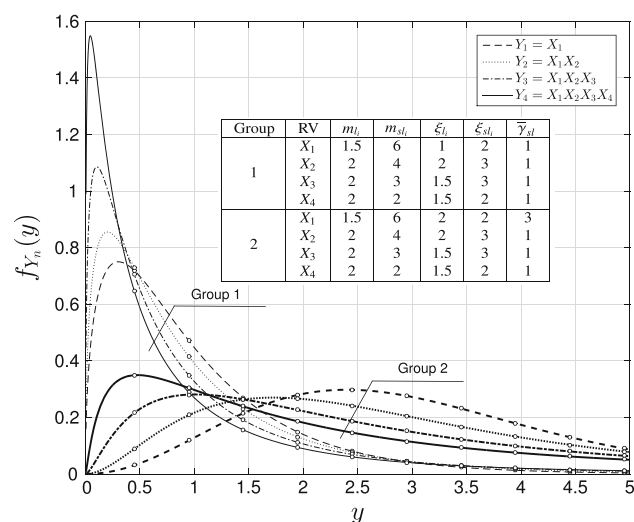
### 5.1 Verification of the PDFs and CDFs

In Fig. 1, we first present numerical results for the PDF of  $Y_n = \prod_{i=1}^n X_i$  for two arbitrarily-selected groups of EGK-distributed RVs whose parameters are summarized in the table in the figure. We also present results obtained via Monte-Carlo simulations represented as circles in the same figure. Clearly, there is a perfect agreement between the two sets of results thus confirming the validity of the derived PDF expression. Moreover, the obtained expressions are versatile enough to give results for any number of EGK variates that might be of interest.

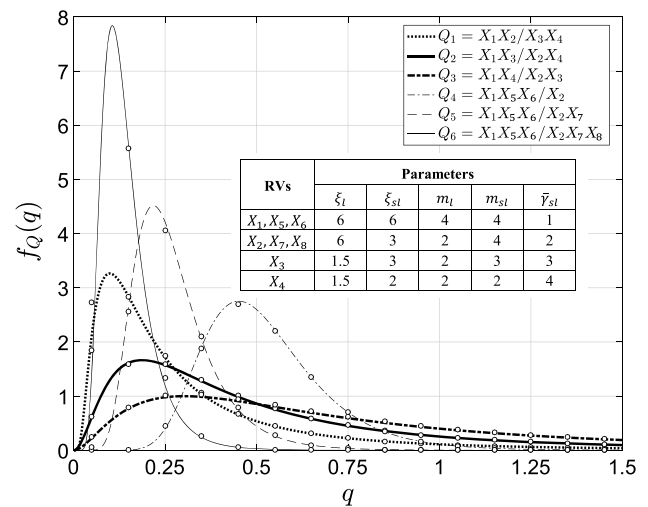
Figure 2 shows the obtained PDF expression in (17) for six different ratios of products of EGK RVs whose parameters are summarized as shown in the figure. The figure also shows results obtained via Monte Carlo simulations. Again, perfect agreement is observed between the two sets of results. Results pertaining to the CDF will be discussed in the following subsection within the context of an application.

### 5.2 Applications

The results introduced in this work can find use in a number of practical scenarios. In this subsection, we enumerate some



**Fig. 1** Analytical (lines) and simulated (markers) PDFs of the product of two groups of  $n$  EGK variates with their parameters summarized in the table shown



**Fig. 2** Analytical (lines) and simulated (markers) PDFs for six different ratios of products of EGK RVs whose parameters are summarized in the shown table

of these examples and validate the obtained analytical results through simulations.

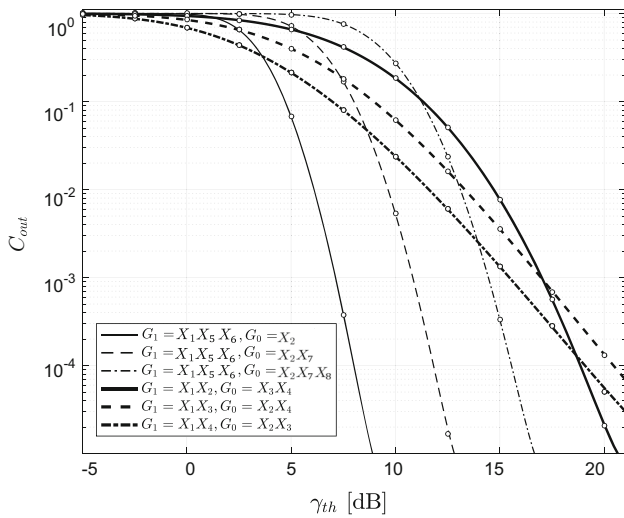
#### 5.2.1 Outage capacity calculation in multi-hop cognitive networks

In this scenario, we are interested in calculating the outage capacity in an amplify-and-forward relay network in a spectrum sharing scenario like the one described in [22]. The outage capacity in this case can be calculated as  $C_{out} = P[G_1/G_0 < N_0(2^{R_0} - 1)/\gamma_{th}]$ , where  $G_0$  and  $G_1$  are the secondary-to-primary and primary-to-secondary multi-hop channel powers, respectively,  $N_0/2$  is the additive white Gaussian noise power spectral density,  $R_0$  is the transmission rate and  $\gamma_{th}$  is the peak interference power. Since  $G_0$  and  $G_1$  represent multihop channels, then these RVs effectively consist of a product of other RVs. Assuming that these RVs are EGK-distributed, the above expression can be directly given in terms of the CDF of the ratio of products as  $C_{out} = F_Q(N_0(2^{R_0} - 1)/\gamma_{th})$ . Assuming  $N_0 = 1$  W/Hz and  $R_0 = 1$  bits/Hz, Fig. 3 shows both the analytical and simulated outage capacity for different combinations of the EGK RVs used earlier in Fig. 2. Perfect agreement can still be noticed between the two sets of results thus confirming the validity of the CDF expression of  $Q$ .

#### 5.2.2 Amount of fading of cascaded EGK channels

The amount of fading  $A_f$  is defined in [23, Eq. (1.27)] as the ratio of the variance to the mean square of the instantaneous SNR, i.e.,  $A_f := \frac{\mathbb{E}[\gamma^2]}{\mathbb{E}[\gamma]^2} - 1$ , where  $\mathbb{E}(\cdot)$  is the expectation operator. In other words,  $A_f$  is the square of the coefficient of variation of the SNR. In this application, we are interested





**Fig. 3** Analytical (lines) and simulated (markers) outage capacity for the scenario described in [22] with different combinations of the EGK RVs

in calculating  $A_f$  for a series of  $N$  cascaded EGK fading channels in which the total SNR  $\gamma$  is the product of the fading of the individual channels [16, Eq. (18)]. Using the fact that the Mellin transform of  $\gamma$  is related to its moments via  $\hat{f}(s) = \mathbb{E}[\gamma^{s-1}]$  as well as the result obtained in (5) for the Mellin transform of the PDF of the product of EGK variates, one can directly arrive at the following result for  $A_f$

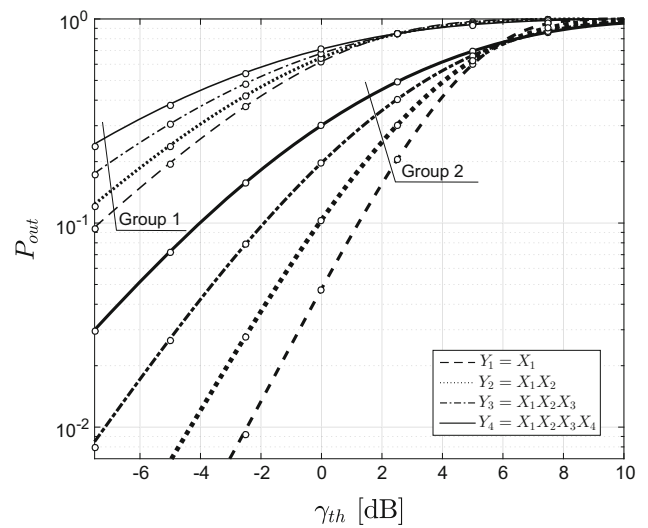
$$A_f = \prod_{i=1}^N \frac{\Gamma(m_i) \Gamma(m_{s_i}) \Gamma(m_i + \frac{2}{\xi_i}) \Gamma(m_{s_i} + \frac{2}{\xi_{s_i}})}{\Gamma(m_i + \frac{1}{\xi_i})^2 \Gamma(m_{s_i} + \frac{1}{\xi_{s_i}})^2} - 1. \tag{19}$$

5.2.3 Outage probability over cascaded EGK channels

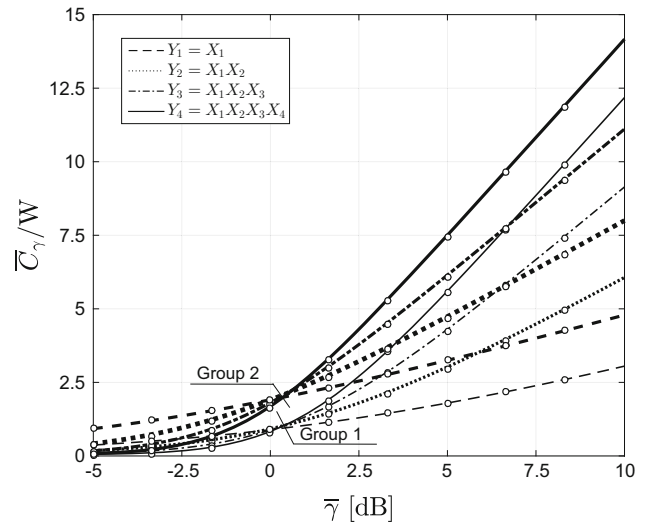
The outage probability  $P_{out}$  is defined as the probability that the instantaneous error rate exceeds a specified threshold value or equivalently that the instantaneous SNR  $\gamma$  falls below a certain threshold  $\gamma_{th}$ . Hence,  $P_{out} := \int_0^{\gamma_{th}} f_\gamma(y)dy = F_\gamma(\gamma_{th})$ , where  $f_\gamma(\cdot)$  is the PDF of the SNR  $\gamma$  and  $F_\gamma(\cdot)$  is its CDF. As in the previous subsection, we assume communication over a group of cascaded EGK fading channels. In this case, the OP immediately follows by using the CDF result in (11) for the product of EGK variates. Figure 4 shows the analytical as well as the simulation results for the OP of two groups of cascaded EGK fading channels whose parameters are summarized in the Table in Fig. 1.

5.2.4 Average capacity of cascaded EGK channels

The last application we present here is related to the calculation of the normalized average capacity  $\bar{C}_\gamma/W$  (where  $W$



**Fig. 4** Analytical (lines) and simulated (markers) outage probability for two groups of cascaded EGK fading channels whose parameters are summarized in the Table in Fig. 1



**Fig. 5** Analytical (lines) and simulated (markers) normalized average capacity for two groups of cascaded EGK fading channels whose parameters are summarized in the Table in Fig. 1

is the transmission bandwidth) of a series of cascaded EGK fading channels. In order to arrive at such result, we start by re-writing the Meijer’s G-function in the PDF of the product of EGK variates in (9) in terms of Fox’s H-function using [18, Eq. (8.3.3)] to get the expression below.

$$f_{Y_n}(y) = \frac{\eta\lambda h}{z} H_{0,2A}^{2A,0} \left[ \lambda h y \left| \Delta \left( A_i, m_i - \frac{1}{\xi_i}, \frac{1}{z} \right)_{i=1:n}, \Delta \left( A_{s_i}, m_{s_i} - \frac{1}{\xi_{s_i}}, \frac{1}{z} \right)_{i=1:n} \right. \right], \tag{20}$$

In (20),  $\Delta(a, b, c) = (b/a, c), (b+1/a, c), \dots, (b+a-1/a, c)$ . Now, using the capacity result for a single EGK channel as a

special case from [24, Eq. (13)], one arrives at the following expression.

$$\bar{C}_\gamma/W = \frac{\eta}{z \ln(2)} H_{2,2A+2}^{2A+2,1} \left[ \lambda h \left| \begin{matrix} (0, 1), (1, 1) \\ (0, 1), (0, 1), \Delta(A_i, m_i, \frac{1}{z})_{i=1:n}, \Delta(A_{s_i}, m_{s_i}, \frac{1}{z})_{i=1:n} \end{matrix} \right. \right], \tag{21}$$

This last result can be further expressed in terms of the Meijer’s G function like all the other expressions in this paper by first going to the Mellin domain, then using Gauss’s multiplication theorem and the relation  $\frac{\Gamma(s)}{\Gamma(s+1)} = \frac{\Gamma(\frac{1}{z}s)}{z\Gamma(\frac{1}{z}s+1)}$  followed by the inverse Mellin transform in addition to [18, Eq. (8.3.3)] to yield the following final result.

$$C_\gamma/W = \frac{(2\pi)^{1-z}\eta}{z \ln(2)} G_{z+1,2A+z+1}^{-2A+z+1,z} \left[ (\lambda h)^z \left| \begin{matrix} \Delta(z, 0), 1 \\ 0, \Delta(z, 0), \Delta(A_i, m_i)_{i=1:n}, \Delta(A_{s_i}, m_{s_i})_{i=1:n} \end{matrix} \right. \right], \tag{22}$$

Figure 5 shows the analytical results of the normalized average capacity obtained from (22) for two groups of cascaded EGK fading channels whose parameters are summarized in the Table in Fig. 1 as well as results obtained via Monte Carlo simulations. In this figure, for Group 1, all the channels have the same average SNR  $\bar{\gamma}$  while for Group 2,  $\bar{\gamma}_1 = 3\bar{\gamma}$ ,  $\bar{\gamma}_2 = \bar{\gamma}_3 = \bar{\gamma}_4 = \bar{\gamma}$ . As noticed in all other applications, perfect agreement between the two sets of results is observed.

### 6 Conclusion

We derived novel closed-form exact versatile expressions for the PDF and CDF of the product and ratio of products of any number of i.n.i.d EGK variates. We verified the validity of the obtained expressions through the use of Monte Carlo simulations. We also presented a practical application where the derived expressions can be used thus showing their applicability in a multitude of wireless communications scenarios. The Mellin transform approach used in this paper can actually be used to describe the statistical properties of other more generalized RVs, e.g., the Málaga and Fox’s H-function distributions.

#### Compliance with ethical standards

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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