Theoretical and Mathematical Physics, **185**(2): 1614–1635 (2015)

THE EXTENDED Z_N-TODA HIERARCHY

Chuanzhong Li* **and Jingsong He***

We construct the extended flow equations of a new Z_N-Toda hierarchy taking values in a commutative subalgebra Z_N of $gl(N,\mathbb{C})$. We give the Hirota bilinear equations and tau function of this new extended Z_N-Toda hierarchy. Taking the presence of logarithmic terms into account, we construct some extended vertex operators in generalized Hirota bilinear equations, which might be useful in topological field theory and the Gromov–Witten theory. We present the Darboux transformations and bi-Hamiltonian structure of this hierarchy. Using Hamiltonian tau-symmetry, we obtain another tau function of this hierarchy with some unknown mysterious relation to the tau function derived using the Sato theory.

Keywords: extended Z_N-Toda hierarchy, Hirota quadratic equation, Darboux transformation, bi-Hamiltonian structure

1. Introduction

Being completely integrable systems. the Kadomtsev–Petviashvili (KP) hierarchy and the Toda lattice hierarchy have many important applications in mathematics and physics including the theory of Lie algebra representations, orthogonal polynomials, and random matrix models [1]–[5]. The KP and Toda systems have many kinds of reductions or extensions: the BKP and CKP hierarchies, the extended Toda hierarchy (ETH) [6], [7], the bigraded Toda hierarchy (BTH) [8]–[14], and so on. There are generalizations of another kind, the so-called multicomponent KP system [15], [16] or multicomponent Toda system, which attract more and more attention because they are widely used in many fields such as multiple orthogonal polynomials and nonintersecting Brownian motions.

The multicomponent KP hierarchy was discussed with application to representation theory and the random matrix model in [15], [16]. It was noted in [3] that the tau functions of a $2N$-multicomponent KP generate solutions of the N-multicomponent two-dimensional Toda hierarchy. The multicomponent two-dimensional Toda hierarchy was considered from the standpoint of the Gauss–Borel factorization problem, theory of multiple matrix orthogonal polynomials, nonintersecting Brownian motions, and matrix Riemann–Hilbert problem [17]–[20]. In fact, the multicomponent two-dimensional Toda hierarchy in [18] is a periodic reduction of a bi-infinite matrix-formed two-dimensional Toda hierarchy. The coefficients (or dynamic variables) of the multicomponent two-dimensional Toda hierarchy take values in a complex finite-dimensional

*Department of Mathematics and Ningbo Collaborative Innovation Center of Nonlinear Hazard System of Ocean and Atmosphere, Ningbo University, Ningbo, China, e-mail: lichuanzhong@nbu.edu.cn, hejingsong@nbu.edu.cn.

Chuanzhong Li is supported by the National Natural Science Foundation of China (Grant Nos. 11201251 and 11571192), Zhejiang Provincial Natural Science Foundation of China (Grant No. LY15A010004), and the Natural Science Foundation of Ningbo (Grant Nos. 2015A610157 and 2014A610029).

Jingsong He is supported by the National Natural Science Foundation of China (Grant No. 11271210) and the K. C. Wong Magna Fund in Ningbo University.

Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 185, No. 2, pp. 289–312, November, 2015. Original article submitted January 21, 2015; revised March 5, 2015.

matrix. The multicomponent two-dimensional Toda hierarchy contains the matrix-formed Toda equation as the first flow equation.

Adding additional logarithmic flows to the Toda lattice hierarchy, we obtain the extended Toda hierarchy [6] defined on a Lax operator

$$L = \Lambda + u + e^v \Lambda^{-1}, \quad u, v \in \mathbb{C}, \tag{1.1}$$

which governs the Gromov–Witten invariant of $\mathbb{CP}^1$. The Gromov–Witten potential of $\mathbb{CP}^1$ is actually a tau function of the extended Toda hierarchy, i.e., the Gromov–Witten potential τ of $\mathbb{CP}^1$ generates the expression

$$\frac{d\lambda}{\lambda}(\Gamma^{\delta\#} \otimes \Gamma^{\delta})(\Gamma^{\alpha} \otimes \Gamma^{-\alpha} - \Gamma^{-\alpha} \otimes \Gamma^{\alpha})(\tau \otimes \tau), \tag{1.2}$$

involved in the Hirota quadratic equations [7] of the ETH, regular in λ computed at $q_0 - q_0' = l\epsilon$ for each $l \in \mathbb{Z}$. The extended BTH (EBTH) is an extension of the BTH that includes additional logarithmic flows [8], [10]. Here, we show that Hirota quadratic equations (1.2) can be derived as a reduction on the Lie algebra from the Hirota bilinear equation of the extended Z_N-Toda hierarchy (EZTH). Therefore, the application of our Hirota bilinear equation of the EZTH in the Gromov–Witten theory is an important motivation for our study. The Hirota bilinear equation of the EBTH was equivalently constructed previously in [9] and very recently in [21] because the $t_{1,N}$ and $t_{0,N}$ flows of the EBTH are equivalent. Meanwhile, it was proved that it determines the Gromov–Witten invariant of the total descendent potential of $\mathbb{P}^1$ orbifolds [21]. A natural question concerns the corresponding extended multicomponent Toda hierarchy (as a matrix-formed generalization of the ETH [6]) and the extended multicomponent BTH. There is a class of orbifolds that should be governed by some logarithmic hierarchies. We therefore think this new kind of logarithmic hierarchy might be useful in the theory of Gromov–Witten invariants governed by these two new hierarchies. With this motivation, our paper [22] was devoted to constructing a kind of Hirota quadratic equation taking values in a differential matrix algebra set. This kind of Hirota bilinear equation might be useful in Gromov–Witten theory. In [23], a new hierarchy called the Z_m-KP hierarchy taking values in a maximal commutative subalgebra of $gl(m, \mathbb{C})$ was constructed, and the relation between the Frobenius manifold and the dispersionless reduced Z_m-KP hierarchy was discussed. This inspired us here to consider the Hirota quadratic equation of the commutative version of the extended multicomponent Toda hierarchy, which might be useful in the Frobenius manifold theory.

This paper is structured as follows. In Sec. 2, we recall the factorization problem and construct the logarithmic matrix operators that we use to define the extended flow of the multicomponent Z_N-Toda hierarchy. In Sec. 3, we give the Lax equations of the EZTH and introduce the multicomponent Z_N-Toda equations and the extended equations into this hierarchy. In Sec. 4, we use the Sato equations to prove the Hirota bilinear equations of the EZTH. In Sec. 5, we define the tau function of the EZTH, which leads to the formalism of generalized matrix vertex operators and Hirota quadratic equations in Sec. 6. In Sec. 7, we construct multifold transformations of the EZTH using a determinant technique [24]. In Sec. 8, to prove the integrability of this new hierarchy, we construct the bi-Hamiltonian structure and tau-symmetry of the EZTH. Section 9 is devoted to a brief conclusion and discussion.

2. Factorization and logarithm operators

Let $\widetilde{G}$ be a group containing linear invertible elements of complex $N \times N$ matrices, and let $\tilde{\mathfrak{g}}$ denote its Lie algebra of complex $N \times N$ matrices $M_N(\mathbb{C})$. We now consider the linear space of functions $g\colon \mathbb{R} \to M_N(\mathbb{C})$ with the shift operator Λ acting on these functions as $(\Lambda g)(x) := g(x + \epsilon)$. Left multiplication by $X\colon \mathbb{R} \to M_N(\mathbb{C})$ has the form $X\Lambda^j$, $(X\Lambda^j)(g)(x) := X(x) \circ g(x + j\epsilon)$ defining the product

$$(X(x)\Lambda^i) \circ (Y(x)\Lambda^j) := X(x)Y(x + i\epsilon)\Lambda^{i+j}.$$

The set $\mathfrak{g}$ of Laurent series in Λ as an associative algebra is then a Lie algebra under the standard commutator.

This Lie algebra has the important splitting

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \tag{2.1}$$

where

$$\mathfrak{g}_+ = \left\{ \sum_{j \geq 0} X_j(x)\Lambda^j,\ X_j(x) \in M_N(\mathbb{C}) \right\}, \qquad \mathfrak{g}_- = \left\{ \sum_{j<0} X_j(x)\Lambda^j,\ X_j(x) \in M_N(\mathbb{C}) \right\}.$$

Splitting (2.1) leads us to consider the factorization of $g \in G$

$$g = g_-^{-1} \circ g_+, \quad g_\pm \in G_\pm, \tag{2.2}$$

where $G_\pm$ have $\mathfrak{g}_\pm$ as their Lie algebras, G_+ is the set of invertible linear operators of the form $\sum_{j\geq 0} g_j(x)\Lambda^j$, and G_- is the set of invertible linear operators of the form $1 + \sum_{j<0} g_j(x)\Lambda^j$. This algebra has a maximal commutative subalgebra $Z_N = \mathbb{C}[\Gamma]/(\Gamma^N)$, and $\Gamma = (\delta_{i,j+1})_{ij} \in gl(N,\mathbb{C})$. We set $Z_N(\Lambda) := \mathfrak{g}_c$. We then have the splitting

$$\mathfrak{g}_c = \mathfrak{g}_{c+} \oplus \mathfrak{g}_{c-}, \tag{2.3}$$

where

$$\mathfrak{g}_{c+} = \left\{ \sum_{j \geq 0} X_j(x)\Lambda^j,\ X_j(x) \in Z_N \right\}, \qquad \mathfrak{g}_{c-} = \left\{ \sum_{j<0} X_j(x)\Lambda^j,\ X_j(x) \in Z_N \right\}.$$

Splitting (2.3) leads us to consider the factorization of $g_c \in G_c$

$$g_c = g_{c-}^{-1} \circ g_{c+}, \quad g_{c\pm} \in G_{c\pm}, \tag{2.4}$$

where $G_{c\pm}$ have $\mathfrak{g}_{c\pm}$ as their Lie algebras, G_{c+} is the set of invertible linear operators of the form $\sum_{j\geq 0} g_j(x)\Lambda^j$, and G_{c-} is the set of invertible linear operators of the form $1 + \sum_{j<0} g_j(x)\Lambda^j$.

We now introduce the free operators $W_0, \overline{W}_0 \in G_c$,

$$\begin{aligned} W_0 &:= \exp\left[\sum_{j=0}^{\infty} t_j \frac{\Lambda^j}{\epsilon j!} + s_j \frac{\Lambda^j}{\epsilon j!}(\epsilon\partial - c_j)\right], \qquad && \partial = \frac{\partial}{\partial x}, \\ \overline{W}_0 &:= \exp\left[\sum_{j=0}^{\infty} t_j \frac{\Lambda^{-j}}{\epsilon j!} + s_j \frac{\Lambda^{-j}}{\epsilon j!}(\epsilon\partial - c_j)\right], && c_j = \sum_{i=1}^{j} \frac{1}{i}, \end{aligned} \tag{2.5}$$

where $t_j, s_j \in \mathbb{C}$ plays the role of continuous time.

We define the dressing operators W and $\overline{W}$ as

$$W := S \circ W_0, \qquad \overline{W} := \bar{S} \circ \overline{W}_0, \quad S \in G_{c-}, \quad \bar{S} \in G_{c+}. \tag{2.6}$$

Given an element $g \in G_c$, we set $t = (t_j)$ and $s = (s_j)$, $j \in \mathbb{N}$, and consider the factorization problem in G_c similarly to the consideration in [18],

$$W \circ g = \overline{W}, \tag{2.7}$$

i.e., the factorization problem

$$S(t,s) \circ W_0 \circ g = \bar{S}(t,s) \circ \overline{W}_0. \tag{2.8}$$

We note that S and $\bar{S}$ have the expansions

$$\begin{aligned} S &= \mathbb{I}_N + \omega_1(x)\Lambda^{-1} + \omega_2(x)\Lambda^{-2} + \cdots \in G_{c-}, \\ \bar{S} &= \overline{\omega}_0(x) + \overline{\omega}_1(x)\Lambda + \overline{\omega}_2(x)\Lambda^2 + \cdots \in G_{c+}. \end{aligned} \tag{2.9}$$

We also define the symbols of S and $\bar{S}$ as $\mathbb{S}$ and $\bar{\mathbb{S}}$:

$$\begin{aligned} \mathbb{S} &= \mathbb{I}_N + \omega_1(x)\lambda^{-1} + \omega_2(x)\lambda^{-2} + \ldots, \\ \bar{\mathbb{S}} &= \overline{\omega}_0(x) + \overline{\omega}_1(x)\lambda + \overline{\omega}_2(x)\lambda^2 + \ldots. \end{aligned} \tag{2.10}$$

The operators S^{-1} and $\bar{S}^{-1}$ inverse to S and $\bar{S}$ have the expansions

$$\begin{aligned} S^{-1} &= \mathbb{I}_N + \omega_1'(x)\Lambda^{-1} + \omega_2'(x)\Lambda^{-2} + \cdots \in G_{c-}, \\ \bar{S}^{-1} &= \overline{\omega}_0'(x) + \overline{\omega}_1'(x)\Lambda + \overline{\omega}_2'(x)\Lambda^2 + \cdots \in G_{c+}. \end{aligned} \tag{2.11}$$

We also define the symbols of S^{-1} and $\bar{S}^{-1}$ as $\mathbb{S}^{-1}$ and $\bar{\mathbb{S}}^{-1}$

$$\begin{aligned} \mathbb{S}^{-1} &= \mathbb{I}_N + \omega_1'(x)\lambda^{-1} + \omega_2'(x)\lambda^{-2} + \ldots, \\ \bar{\mathbb{S}}^{-1} &= \overline{\omega}_0'(x) + \overline{\omega}_1'(x)\lambda + \overline{\omega}_2'(x)\lambda^2 + \ldots. \end{aligned} \tag{2.12}$$

The Lax operators $\mathcal{L} \in G_c$ are defined by

$$\mathcal{L} := W \circ \Lambda \circ W^{-1} = \overline{W} \circ \Lambda^{-1} \circ \overline{W}^{-1} \tag{2.13}$$

and have the expansions

$$\mathcal{L} = \Lambda + u_1(x) + u_2(x)\Lambda^{-1}. \tag{2.14}$$

In fact, the Lax operators $\mathcal{L} \in G_c$ can also be equivalently defined by

$$\mathcal{L} := S \circ \Lambda \circ S^{-1} = \bar{S} \circ \Lambda^{-1} \circ \bar{S}^{-1}. \tag{2.15}$$

These definitions are a continuous interpolated version of the multicomponent commutative Toda hierarchy, i.e., a continuous spatial parameter x is brought into this hierarchy. With this meaning, the continuous flow $\partial/\partial x$ is missing. To complete these flows, we define the logarithm matrix

$$\begin{aligned} \log_+ \mathcal{L} &= W \circ \epsilon\partial \circ W^{-1} = S \circ \epsilon\partial \circ S^{-1}, \\ \log_- \mathcal{L} &= -\overline{W} \circ \epsilon\partial \circ \overline{W}^{-1} = -\bar{S} \circ \epsilon\partial \circ \bar{S}^{-1}, \end{aligned} \tag{2.16}$$

where ∂ is the derivative with respect to the spatial variable x.

Combining the above logarithm operators together, we can derive the important logarithm matrix

$$\log \mathcal{L} := \frac{1}{2}(\log_+ \mathcal{L} + \log_- \mathcal{L}) = \frac{1}{2}(S \circ \epsilon\partial \circ S^{-1} - \bar{S} \circ \epsilon\partial \circ \bar{S}^{-1}) := \sum_{i=-\infty}^{+\infty} W_i \Lambda^i \in G_c, \tag{2.17}$$

which generates a series of flow equations containing the spatial flow in Lax equations defined below.

3. Lax equations of EZTH

In this section, we use factorization problem (2.7) to derive Lax equations. We first introduce some convenient notation.

Definition 1. The *matrix operators* B_j and D_j are defined as

$$B_j := \frac{\mathcal{L}^{j+1}}{(j+1)!}, \qquad D_j := \frac{2\mathcal{L}^j}{j!}(\log \mathcal{L} - c_j), \quad c_j = \sum_{i=1}^{j} \frac{1}{i}, \quad j \geq 0. \tag{3.1}$$

Definition 2. The EZTH is a hierarchy in which the dressing operators S and $\bar{S}$ satisfy the Sato equations

$$\begin{aligned} &\epsilon\, \partial_{t_j} S = -(B_j)_- S, \qquad \epsilon\, \partial_{t_j} \bar{S} = (B_j)_+ \bar{S}, \\ &\epsilon\, \partial_{s_j} S = -(D_j)_- S, \qquad \epsilon\, \partial_{s_j} \bar{S} = (D_j)_+ \bar{S}. \end{aligned} \tag{3.2}$$

We can then easily obtain the following proposition.

Proposition 1. *The dressing operators W and $\overline{W}$ satisfy the Sato equations*

$$\begin{aligned} &\epsilon\, \partial_{t_j} W = (B_j)_+ W, \qquad \epsilon\, \partial_{t_j} \overline{W} = (B_j)_+ \overline{W}, \\ &\epsilon\, \partial_{s_j} W = \left(\frac{\mathcal{L}^j}{j!}(\log_+ \mathcal{L} - c_j) - (D_j)_- \right) W, \\ &\epsilon\, \partial_{s_j} \overline{W} = \left(-\frac{\mathcal{L}^j}{j!}(\log_- \mathcal{L} - c_j) + (D_j)_+ \right) \overline{W}. \end{aligned} \tag{3.3}$$

From Proposition 1, we derive the following Lax equations for the Lax operators.

Proposition 2. *The Lax equations of the EZTH are*

$$\begin{aligned} &\epsilon\, \partial_{t_j} \mathcal{L} = [(B_j)_+, \mathcal{L}], \qquad \epsilon\, \partial_{s_j} \mathcal{L} = [(D_j)_+, \mathcal{L}], \qquad \epsilon\, \partial_{t_j} \log \mathcal{L} = [(B_j)_+, \log \mathcal{L}], \\ &\epsilon (\log \mathcal{L})_{s_j} = [-(D_j)_-, \log_+ \mathcal{L}] + [(D_j)_+, \log_- \mathcal{L}]. \end{aligned} \tag{3.4}$$

To see this kind of hierarchy more clearly, we give the Z_N-Toda equations as t_0 flow equations.

3.1. The extended Z_N-Toda equations. As a consequence of factorization problem (2.7) and the Sato equations, taking $S \in G_{c-}$ and $\bar{S} \in G_{c+}$ into account, we obtain the t_0 flow of $\mathcal{L}$ in the form $\mathcal{L} = \Lambda + U + V\Lambda^{-1}$:

$$\epsilon\, \partial_{t_0} \mathcal{L} = [\Lambda + U, V\Lambda^{-1}], \tag{3.5}$$

which leads to the matrix Toda equation

$$\begin{aligned} &\epsilon\, \partial_{t_0} U = V(x+\epsilon) - V(x), \\ &\epsilon\, \partial_{t_0} V = U(x)V(x) - V(x)U(x-\epsilon). \end{aligned} \tag{3.6}$$

Of course, we can switch the order of the matrices because Z_N is commutative. We set

$$U = \begin{bmatrix} u_0 & 0 \\ u_1 & u_0 \end{bmatrix}, \qquad V = \begin{bmatrix} v_0 & 0 \\ v_1 & v_0 \end{bmatrix}. \tag{3.7}$$

The specific coupled Toda equation is then

$$\begin{aligned}
\epsilon\,\partial_{t_0}u_0 &= v_0(x+\epsilon) - v_0(x),\\
\epsilon\,\partial_{t_0}u_1 &= v_1(x+\epsilon) - v_1(x),\\
\epsilon\,\partial_{t_0}v_0 &= u_0(x)v_0(x) - v_0(x)v_0(x-\epsilon),\\
\epsilon\,\partial_{t_0}v_1 &= (u_1(x) - u_1(x-\epsilon))v_0(x) - v_1(x)(u_0(x) - u_0(x-\epsilon)).
\end{aligned} \tag{3.8}$$

To obtain the standard matrix Toda equation, we must use the alternative expressions

$$\begin{aligned}
U &:= \omega_1(x) - \omega_1(x+\epsilon) = \epsilon\,\partial_{t_1}\phi(x),\\
V &:= e^{\phi(x)}e^{-\phi(x-\epsilon)} = -\epsilon\,\partial_{t_1}\omega_1(x).
\end{aligned} \tag{3.9}$$

From the Sato equation, we deduce the set of nonlinear partial differential-difference equations

$$\begin{aligned}
&\omega_1(x) - \omega_1(x+\epsilon) = \epsilon\,\partial_{t_1}(e^{\phi(x)})e^{-\phi(x)},\\
&\epsilon\,\partial_{t_1}\omega_1(x) = -e^{\phi(x)}e^{-\phi(x-\epsilon)}.
\end{aligned} \tag{3.10}$$

We note that if we combine the two first equations, then we obtain

$$\epsilon^2\,\partial_{t_1}^2\phi(x) = e^{\phi(x+\epsilon)}e^{-\phi(x)} - e^{\phi(x)}e^{-\phi(x-\epsilon)},$$

which is an $N\times N$ matrix-valued extension of the Toda equation, whence the original Toda equation follows for $N = 1$. For $N = 2$, the equation for $\phi = \begin{bmatrix} \phi_0 & 0 \\ \phi_1 & \phi_0 \end{bmatrix}$ is the coupled Toda system

$$\begin{aligned}
&\epsilon^2\,\partial_{t_1}^2\phi_0(x) = e^{\phi_0(x+\epsilon)-\phi_0(x)} - e^{\phi_0(x)-\phi_0(x-\epsilon)},\\
&\epsilon^2\,\partial_{t_1}^2\phi_1(x) = (\phi_1(x+\epsilon) - \phi_1(x))e^{\phi_0(x+\epsilon)-\phi_0(x)} - (\phi_1(x) - \phi_1(x-\epsilon))e^{\phi_0(x)-\phi_0(x-\epsilon)}.
\end{aligned}$$

In the calculation, we use the identity

$$\exp\begin{pmatrix} \phi_0 & 0 \\ \phi_1 & \phi_0 \end{pmatrix} = \begin{bmatrix} e^{\phi_0} & 0 \\ \phi_1 e^{\phi_0} & e^{\phi_0} \end{bmatrix}.$$

In addition to the Z_N-Toda equations above, together with logarithmic flows, the EZTH also contains some extended flow equations (see Sec. 4 below). Here, we consider the extended flow equations in the simplest case, i.e., the s_0 flow for $\mathcal{L} = \Lambda + u_0 + u_1\Lambda^{-1}$,

$$\epsilon\,\partial_{s_0}\mathcal{L} = [(S\epsilon\,\partial_x S^{-1})_+, \mathcal{L}] = [\epsilon\,\partial_x S\,S^{-1}, \mathcal{L}] = \epsilon\mathcal{L}_x, \tag{3.11}$$

which leads to the specific equation

$$\partial_{s_0}U = U_x, \qquad \partial_{s_0}V = V_x. \tag{3.12}$$

To see the extended equations clearly, we must rewrite the extended flows in the Lax equations of the EZTH as in the following lemma.

Lemma 1. *The extended flows in the Lax formulation of the EZTH can be given equivalently by*

$$\epsilon\frac{\partial\mathcal{L}}{\partial s_j}=[D_j,\mathcal{L}],$$
$$D_j=\left(\frac{\mathcal{L}^j}{j!}(\log_+\mathcal{L}-c_j)\right)_+-\left(\frac{\mathcal{L}^j}{j!}(\log_-\mathcal{L}-c_j)\right)_-, \tag{3.13}$$

which can also be rewritten in the form

$$\epsilon\frac{\partial\mathcal{L}}{\partial s_n}=[\overline{D}_n,\mathcal{L}],$$
$$\overline{D}_j=\frac{\mathcal{L}^j}{j!}\epsilon\partial+\left[\frac{\mathcal{L}^j}{j!}\left(\sum_{k<0}W_k(x)\Lambda^k-c_j\right)\right]_+-\left[\frac{\mathcal{L}^j}{j!}\left(\sum_{k\geq 0}W_k(x)\Lambda^k-c_j\right)\right]_-. \tag{3.14}$$

We can then derive the s_1 flow equation of the EZTH as

$$\epsilon U_{s_1}=(1-\Lambda)(V(\Lambda^{-1}-1)^{-1}\epsilon(\log V)_x)-2(\Lambda-1)V+\frac{\epsilon}{2}U_x^2+\epsilon V_x,$$
$$\epsilon V_{s_1}=((\Lambda^{-1}-1)^{-1}\epsilon V_xV^{-1}+2)(U(x-\epsilon)-U(x))V+$$
$$+\epsilon V_xU(x-\epsilon)+\epsilon(U_x(x-\epsilon)+U_x(x))V,$$

where $U=U(x)$ and $V=V(x)$. To give a linear description of the EZTH, we introduce the matrix wave functions ψ and $\overline{\psi}$ defined by

$$\psi=W\cdot\chi,\qquad\bar{\psi}=\overline{W}\cdot\bar{\chi}, \tag{3.15}$$

where

$$\chi(z):=z^{x/\epsilon}\mathbb{I}_N,\qquad\bar{\chi}(z):=z^{-x/\epsilon}\mathbb{I}_N, \tag{3.16}$$

and the dot denotes the action of an operator on a function. We note that $\Lambda\cdot\chi=z\chi$, and we can define the asymptotic expansions

$$\psi=z^{x/\epsilon}(\mathbb{I}_N+\omega_1(x)z^{-1}+\cdots)\psi_0(z),$$
$$\psi_0:=\exp\left[\sum_{j=1}^{\infty}t_j\frac{z^j}{\epsilon j!}+s_j\frac{z^j}{\epsilon j!}(\log z-c_j)\right],\quad z\to\infty,$$
$$\bar{\psi}=z^{-x/\epsilon}(\overline{\omega}_0(x)+\overline{\omega}_1(x)z+\cdots)\bar{\psi}_0(z),$$
$$\bar{\psi}_0:=\exp\left[\sum_{j=0}^{\infty}t_j\frac{z^{-j}}{\epsilon j!}+s_j\frac{z^{-j}}{\epsilon j!}(\log z-c_j)\right],\quad z\to 0. \tag{3.17}$$

We obtain linear equations in the following proposition.

Proposition 3. *The matrix wave functions ψ and $\bar{\psi}$ satisfy the Sato equations*

$$\mathcal{L}\cdot\psi=z\psi,\qquad\mathcal{L}\cdot\bar{\psi}=z\bar{\psi},$$
$$\epsilon\,\partial_{t_j}\psi=(B_j)_+\cdot\psi,\qquad\epsilon\,\partial_{t_j}\bar{\psi}=(B_j)_+\cdot\bar{\psi},$$
$$\epsilon\,\partial_{s_j}\psi=\left(\frac{\mathcal{L}^j}{\epsilon j!}(\log_+\mathcal{L}-c_j)-(D_j)_-\right)\cdot\psi,$$
$$\epsilon\,\partial_{s_j}\bar{\psi}=\left(-\frac{\mathcal{L}^j}{\epsilon j!}(\log_-\mathcal{L}-c_j)+(D_j)_+\right)\cdot\bar{\psi}. \tag{3.18}$$

4. Hirota bilinear equations

From Lax equations, we can find that the s_0 flow is equivalent to the spatial flow ∂_x. Based on this fact, we derive Hirota bilinear equations, which are equivalent to the Lax equations of the EZTH, in the following proposition.

Proposition 4. *The operators W and $\overline{W}$ are matrix-valued wave operators of the EZTH if and only if the Hirota bilinear equations*

$$W\Lambda^r W^{-1} = \overline{W}\Lambda^{-r}\overline{W}^{-1}, \quad r \in \mathbb{N}, \tag{4.1}$$

are satisfied.

Proof. Let

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots), \qquad \beta = (\beta_1, \beta_2, \dots) \tag{4.2}$$

be a multi-index and

$$\partial^\alpha := \partial_{t_0}^{\alpha_0}\partial_{t_0}^{\alpha_1}\partial_{t_2}^{\alpha_2}\dots, \qquad \partial^\beta := \partial_{s_1}^{\beta_1}\partial_{s_2}^{\beta_2}\dots. \tag{4.3}$$

Also let $\partial^\theta = \partial^\alpha\partial^\beta$. We first prove that the formulated statement leads to the expression

$$W(x,t,\Lambda)\Lambda^r W^{-1}(x,t',\Lambda) = \overline{W}(x,t,\Lambda)\Lambda^{-r}\overline{W}^{-1}(x,t',\Lambda) \tag{4.4}$$

for all integers $r \geq 0$. Using the same method as in [7], [9], by induction on α, we prove that

$$W(x,t,\Lambda)\Lambda^r(\partial^\theta W^{-1}(x,t,\Lambda)) = \overline{W}(x,t,\Lambda)\Lambda^{-r}(\partial^\theta\overline{W}^{-1}(x,t,\Lambda)). \tag{4.5}$$

This is obviously true for $\theta = 0$ by the definition of matrix-valued wave operators. Let Eq. (4.5) hold for some $\theta \neq 0$. We note that

$$\epsilon\,\partial_{p_j}W := \begin{cases} [(\partial_{t_j}S)S^{-1} + S\Lambda^j S^{-1}]W, & p_j = t_j, \\ [(\partial_{s_j}S)S^{-1} + S\Lambda^j\partial_x S^{-1}]W, & p_j = s_j, \end{cases}$$

and

$$\epsilon\,\partial_{p_j}\overline{W} := \begin{cases} (\partial_{t_j}\bar{S})\bar{S}^{-1}\overline{W}, & p_j = t_j, \\ [(\partial_{\bar{S}_j}\bar{S})\bar{S}^{-1} + \bar{S}\Lambda^{-j}\partial_x\bar{S}^{-1}]\overline{W}, & p_j = s_j, \end{cases}$$

which further leads to

$$\epsilon\,\partial_{p_j}W := \begin{cases} (B_j)_+W, & p_j = t_j, \\ [-(D_j)_- + \dfrac{\mathcal{L}^j}{\epsilon j!}(\log_+\mathcal{L} - c_j)]W, & p_j = s_j, \end{cases}$$

and

$$\epsilon\,\partial_{p_j}\overline{W} := \begin{cases} (B_j)_+\overline{W}, & p_j = t_j, \\ [(D_j)_+ - \dfrac{\mathcal{L}^j}{\epsilon j!}(\log_-\mathcal{L} - c_j)]\overline{W}, & p_j = s_j. \end{cases}$$

This further implies

$$(\partial_{p_j}W)\Lambda^r(\partial^\theta W^{-1}) = (\partial_{p_j}\overline{W})\Lambda^{-r}(\partial^\theta\overline{W}^{-1}).$$

Taking (4.5) into account, we obtain

$$W\Lambda^r(\partial_{p_j}\partial^\theta W^{-1}) = \overline{W}\Lambda^{-r}(\partial_{p_j}\partial^\theta \overline{W}^{-1}).$$

Therefore, if we increase the power of ∂_{p_j} by 1, Eq. (4.5) is still satisfied. The proof by induction is completed. Expanding both sides of Eq. (4.4) in Taylor series in $t = t'$ and $s = s'$, we can finish the proof of Eq. (4.4).

Conversely, by separating the negative and the positive parts of the equation, we can prove that S and $\bar{S}$ are a pair of matrix-valued wave operators. ■

For a description in terms of matrix-valued wave functions, we need the following symbolic definitions. If the series have the forms

$$W(x,t,s,\Lambda) = \sum_{i\in\mathbb{Z}} a_i(x,t,s,\partial_x)\Lambda^i, \qquad \overline{W}(x,t,s,\Lambda) = \sum_{i\in\mathbb{Z}} b_i(x,t,s,\partial_x)\Lambda^i,$$

$$W^{-1}(x,t,s,\Lambda) = \sum_{i\in\mathbb{Z}} \Lambda^i a'_i(x,t,s,\partial_x), \qquad \overline{W}^{-1}(x,t,s,\Lambda) = \sum_{j\in\mathbb{Z}} \Lambda^j b'_j(x,t,s,\partial_x),$$

then we define their corresponding left symbols $\mathcal{W}$ and $\overline{\mathcal{W}}$ and right symbols $\mathcal{W}^{-1}$ and $\overline{\mathcal{W}}^{-1}$ as

$$\mathcal{W}(x,t,s,\lambda) = \sum_{i\in\mathbb{Z}} a_i(x,t,s,\partial_x)\lambda^i, \qquad \mathcal{W}^{-1}(x,t,s,\lambda) = \sum_{i\in\mathbb{Z}} a'_i(x,t,s,\partial_x)\lambda^i,$$

$$\overline{\mathcal{W}}(x,t,s,\lambda) = \sum_{i\in\mathbb{Z}} b_i(x,t,s,\partial_x)\lambda^i, \qquad \overline{\mathcal{W}}^{-1}(x,t,s,\bar{t},\lambda) = \sum_{j\in\mathbb{Z}} b'_j(x,t,s,\partial_x)\lambda^j.$$

We can now write the Hirota bilinear equation in another form (see Proposition 5 below), defining the residue as

$$\mathrm{Res}_\lambda \sum_{n\in\mathbb{Z}} \alpha_n\lambda^n = \alpha_{-1}$$

using a proof similar to the proof in [3], [7], [9].

Proposition 5. *Let $s_0 = s'_0$. The operators S and $\bar{S}$ are matrix-valued wave operators of the Z_N-Toda hierarchy if and only if the Hirota bilinear identity*

$$\mathrm{Res}_\lambda\{\lambda^{r+m-1}\mathcal{W}(x,t,s,\epsilon\partial_x,\lambda)\mathcal{W}^{-1}(x-m\epsilon,t',s',\epsilon\partial_x,\lambda)\} =$$
$$= \mathrm{Res}_\lambda\{\lambda^{-r+m-1}\overline{\mathcal{W}}(x,t,s,\epsilon\partial_x,\lambda)\overline{\mathcal{W}}^{-1}(x-m\epsilon,t',s',\epsilon\partial_x,\lambda)\} \tag{4.6}$$

is satisfied for all $m \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Proof. Let $m \in \mathbb{Z}$, $r \in \mathbb{N}$, and $s_0 = s'_0$. We set

$$W(x,t,s,\Lambda) = \sum_{i\in\mathbb{Z}} a_i(x,t,s,\partial_x)\Lambda^i, \qquad \overline{W}(x,t,s,\Lambda) = \sum_{i\in\mathbb{Z}} b_i(x,t,s,\partial_x)\Lambda^i,$$

$$W^{-1}(x,t,s,\Lambda) = \sum_{i\in\mathbb{Z}} \Lambda^i a'_i(x,t,s,\partial_x), \qquad \overline{W}^{-1}(x,t,\Lambda) = \sum_{j\in\mathbb{Z}} \Lambda^j b'_j(x,t,s,\partial_x)$$

and compare the coefficients of Λ^{-m} in Eq. (4.4):

$$\sum_{i+j=-m-r} a_i(x,t,s,\partial_x)a'_j(x-m\epsilon,t',s',\partial_x) = \sum_{i+j=-m+r} b_i(x,t,s,\partial_x)b'_j(x-m\epsilon,t',s',\partial_x).$$

This equality can be written also as Eq. (4.6). ■

To express the Hirota quadratic function in terms of tau functions, we must first define the tau function of the EZTH and prove that it exists, which we do in the next section.

5. Tau functions of EZTH

We introduce the sequences

$$t - [\lambda] := (t_j - \epsilon(j-1)!\lambda^j,\ 0 \le j \le \infty). \tag{5.1}$$

A matrix-valued function $\tau \in Z_N$ depending on only the dynamical variables t and ϵ is called a *matrix tau-function of the EZTH* if it provides symbols related to matrix-valued wave operators as

$$\begin{aligned}
\mathbb{S} &:= \frac{\tau(s_0 + x - \epsilon/2, t_j - \epsilon(j-1)!/\lambda^j, s; \epsilon)}{\tau(s_0 + x - \epsilon/2, t, s; \epsilon)},\\
\mathbb{S}^{-1} &:= \frac{\tau(s_0 + x + \epsilon/2, t_j + \epsilon(j-1)!/\lambda^j, s; \epsilon)}{\tau(s_0 + x + \epsilon/2, t, s; \epsilon)},\\
\bar{S} &:= \frac{\tau(s_0 + x + \epsilon/2, t_j + \epsilon(j-1)!\lambda^j, s; \epsilon)}{\tau(s_0 + x - \epsilon/2, t, s; \epsilon)},\\
\bar{S}^{-1} &:= \frac{\tau(s_0 + x - \epsilon/2, t_j - \epsilon(j-1)!\lambda^j, s; \epsilon)}{\tau(s_0 + x + \epsilon/2, t, s; \epsilon)}.
\end{aligned} \tag{5.2}$$

Here, division means multiplication of the numerator matrix by the inverse of the denominator matrix. We can obtain the solution U, V in terms of tau functions as

$$U = (\log \tau)_{xx}, \qquad V = \log \frac{\tau(x+\epsilon)\tau(x-\epsilon)}{\tau^2(x)}. \tag{5.3}$$

For $N = 2$, we obtain

$$\begin{bmatrix} u_0 & 0 \\ u_1 & u_0 \end{bmatrix} = \begin{bmatrix} (\log \tau_0)_{xx} & 0 \\ \left(\dfrac{\tau_1}{\tau_0}\right)_{xx} & (\log \tau_0)_{xx} \end{bmatrix},$$

$$\exp\left(\begin{bmatrix} v_0 & 0 \\ v_1 & v_0 \end{bmatrix}\right) = \begin{bmatrix} e^{v_0} & 0 \\ v_1 e^{v_0} & e^{v_0} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{\tau_0(x+\epsilon)\tau_0(x-\epsilon)}{\tau_0} & 0 \\ \dfrac{\tau_1(x+\epsilon)\tau_0(x-\epsilon) + \tau_0(x+\epsilon)\tau_1(x-\epsilon)}{\tau_0} - \dfrac{\tau_0(x+\epsilon)\tau_0(x-\epsilon)\tau_1}{\tau_0^2} & \dfrac{\tau_0(x+\epsilon)\tau_0(x-\epsilon)}{\tau_0} \end{bmatrix}.$$

This implies

$$\begin{aligned}
&u_0 = (\log \tau_0)_{xx}, \qquad u_1 = \left(\frac{\tau_1}{\tau_0}\right)_{xx},\\
&v_0 = \log \frac{\tau_0(x+\epsilon)\tau_0(x-\epsilon)}{\tau_0}, \qquad v_1 = (\Lambda - 1 + \Lambda^{-1})\frac{\tau_1(x)}{\tau_0(x)}.
\end{aligned} \tag{5.4}$$

Using Proposition 5, we can prove the following lemma.

Lemma 2. *The equations*

$$\sum_{k=1}^{N}\mathbb{S}(x,t,\lambda_1)_{ik}\mathbb{S}^{-1}(x+\epsilon,t+[\lambda_2],\lambda_1)_{kj}=\sum_{k=1}^{N}\bar{S}(x,t,\lambda_2)_{ik}\bar{S}^{-1}(x,t-[\lambda_1^{-1}],\lambda_2)_{kj},$$
$$\sum_{k=1}^{N}\mathbb{S}(x,t,\lambda_1)_{ik}\mathbb{S}^{-1}(x,t-[\lambda_2^{-1}],\lambda_1)_{kj}=\sum_{k=1}^{N}\mathbb{S}(x,t,\lambda_2)_{ik}\mathbb{S}^{-1}(x,t-[\lambda_1^{-1}],\lambda_2)_{kj}, \tag{5.5}$$
$$\sum_{k=1}^{N}\bar{S}(x,t,\lambda_1)_{ik}\bar{S}^{-1}(x+\epsilon,t+[\lambda_2],\lambda_1)_{kj}=\sum_{k=1}^{N}\bar{S}(x,t,\lambda_2)_{ik}\bar{S}^{-1}(x+\epsilon,t+[\lambda_1],\lambda_2)_{kj}$$

hold.

Using Lemma 2, we can prove the following important proposition, which asserts the existence of the matrix-valued tau functions.

Proposition 6. *For a given pair of wave operators $\mathbb{S}$ and $\bar{S}$ of the EZTH, there exists a corresponding matrix-valued invertible tau function $\tau \in Z_N$, and it is unique up to multiplication by a nonvanishing function independent of t_j, $j \geq 1$.*

Proof. Here, we note that the Z_N-valued tau function $\tau(x,\mathbf{t})$ corresponding to the wave operators $\mathbb{S}$ and $\bar{S}$ is in fact $\tau(x-\epsilon/2,\mathbf{t})$.

System (5.5) is equivalent to

$$\log\mathbb{S}=\left(\exp\left(-\epsilon\sum_{j=0}^{\infty}j!\lambda^{-(j+1)}\,\partial_{t_j}\right)-1\right)\log\tau,$$
$$\log\bar{S}=\left(\exp\left(\epsilon\partial_x+\epsilon\sum_{j=0}^{\infty}j!\lambda^{j+1}\,\partial_{t_j}\right)-1\right)\log\tau,$$
$$\partial_{s_0}\log\tau(x,\mathbf{t})=\partial_x\log\tau(x,\mathbf{t}).$$

Using Lemma 2, we can now prove the existence of the tau function of this hierarchy. ■

Given the tau functions of the EZTH, the natural question arises about the expression for the Hirota bilinear equation in terms of tau functions, and we answer this question in the next section using generalized vertex operators.

6. Generalized matrix vertex operators and Hirota quadratic equations

In this section, we continue to discuss the fundamental properties of the EZTH tau function, i.e., the Hirota quadratic equations of the EZTH. We therefore introduce the vertex operators

$$\Gamma^{\pm a}:=\exp\left(\pm\frac{1}{\epsilon}\left(\sum_{j=0}^{\infty}t_j\frac{\lambda^{j+1}}{(j+1)!}+s_j\frac{\lambda^j}{j!}(\log\lambda-c_j)\right)\right)\times\exp\left(\mp\frac{\epsilon}{2}\partial_{s_0}\mp[\lambda^{-1}]_\partial\right),$$
$$\Gamma^{\pm b}:=\exp\left(\pm\frac{1}{\epsilon}\left(\sum_{j=0}^{\infty}t_j\frac{\lambda^{-j-1}}{(j+1)!}-s_j\frac{\lambda^{-j}}{j!}(\log\lambda-c_j)\right)\right)\times\exp\left(\mp\frac{\epsilon}{2}\partial_{s_0}\mp[\lambda]_\partial\right),$$

where

$$[\lambda]_\partial := \epsilon \sum_{j=0}^{\infty} j!\lambda^{j+1}\, \partial_{t_j}.$$

Because of the presence of the logarithm $\log\lambda$, the vertex operators $\Gamma^{\pm a}\otimes\Gamma^{\mp a}$ and $\Gamma^{\pm b}\otimes\Gamma^{\mp b}$ are multivalued functions. In passing between different branches around $\lambda=\infty$, monodromy factors appear:

$$\begin{aligned} M^a &= \exp\biggl\{\pm\frac{2\pi i}{\epsilon}\sum_{j\geq 0}\frac{\lambda^j}{j!}(s_j\otimes 1 - 1\otimes s_j)\biggr\}, \\ M^b &= \exp\biggl\{\pm\frac{2\pi i}{\epsilon}\sum_{j\geq 0}\frac{\lambda^{-j}}{j!}(s_j\otimes 1 - 1\otimes s_j)\biggr\}. \end{aligned} \tag{6.1}$$

To compensate this difficulty, we generalize the concept of a vertex operator, which is already not scalar-valued but takes values in a differential operator algebra in Z_N. We therefore introduce the vertex operators

$$\begin{aligned} \Gamma_a &= \exp\biggl(-\sum_{j>0}\frac{j!\lambda^{j+1}}{\epsilon}(\epsilon\partial_x)s_j\biggr)e^{x\partial_{s_0}}, \\ \Gamma_b &= \exp\biggl(-\sum_{j>0}\frac{j!\lambda^{-(j+1)}}{\epsilon}(\epsilon\partial_x)s_j\biggr)e^{x\partial_{s_0}}, \\ \Gamma_a^{\#} &= e^{x\partial_{s_0}}\exp\biggl(\sum_{j>0}\frac{j!\lambda^{j+1}}{\epsilon}(\epsilon\partial_x)s_j\biggr), \\ \Gamma_b^{\#} &= e^{x\partial_{s_0}}\exp\biggl(\sum_{j>0}\frac{j!\lambda^{-(j+1)}}{\epsilon}(\epsilon\partial_x)s_j\biggr). \end{aligned} \tag{6.2}$$

Then

$$\begin{aligned} \Gamma_a^{\#}\otimes\Gamma_a &= e^{x\partial_{s_0}}\exp\biggl(\sum_{j>0}\frac{j!\lambda^{j+1}}{\epsilon}(\epsilon\partial_x)(s_j-s'_j)\biggr)e^{x\partial_{s'_0}}, \\ \Gamma_b^{\#}\otimes\Gamma_b &= e^{x\partial_{s_0}}\exp\biggl(\sum_{j>0}\frac{j!\lambda^{-(j+1)}}{\epsilon}(\epsilon\partial_x)(s_j-s'_j)\biggr)e^{x\partial_{s'_0}}. \end{aligned} \tag{6.3}$$

After some calculations, we obtain

$$\begin{aligned} (\Gamma_a^{\#}\otimes\Gamma_a)M^a =\ & \exp\biggl\{\pm\frac{2\pi i}{\epsilon}\sum_{j>0}\frac{\lambda^j}{j!}(s_j-s'_j)\biggr\}\times \\ & \times\exp\biggl(\pm\frac{2\pi i}{\epsilon}\biggl((s_0+x)-\biggl(s'_0+x+\sum_{j>0}\frac{\lambda^j}{j!}(s_j-s'_j)\biggr)\biggr)\biggr)(\Gamma_a^{\#}\otimes\Gamma_a) = \\ =\ & \exp\biggl(\pm\frac{2\pi i}{\epsilon}(s_0-s'_0)\biggr)(\Gamma_a^{\#}\otimes\Gamma_a), \\ (\Gamma_b^{\#}\otimes\Gamma_b)M^b =\ & \exp\biggl\{\pm\frac{2\pi i}{\epsilon}\sum_{j>0}\frac{\lambda^{-j}}{j!}(s_j-s'_j)\biggr\}\times \\ & \times\exp\biggl(\pm\frac{2\pi i}{\epsilon}\biggl((s_0+x)-\biggl(s'_0+x+\sum_{j>0}\frac{\lambda^{-j}}{j!}(s_j-s'_j)\biggr)\biggr)\biggr)(\Gamma_b^{\#}\otimes\Gamma_b) = \\ & =\exp\biggl(\pm\frac{2\pi i}{\epsilon}(s_0-s'_0)\biggr)(\Gamma_b^{\#}\otimes\Gamma_b). \end{aligned}$$

Hence, if $s_0 - s_0' \in \mathbb{Z}\epsilon$, then all $(\Gamma_a^{\#} \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a})$ and $(\Gamma_b^{\#} \otimes \Gamma_b)(\Gamma^{-b} \otimes \Gamma^b)$ are single-valued near $\lambda = \infty$.

We note that the vertex operators introduced above take values in a Z_N-valued differential operator algebra,

$$\mathbb{C}[\partial, x, t, s, \epsilon] := \left\{ f(x,t,\epsilon) \;\middle|\; f(x,t,s,\epsilon) = \sum_{i\geq 0} \sum_{k\geq 0}^{N} c_{ik}(x,t,s,\epsilon)\Gamma^k \partial^i \right\}.$$

Theorem 1. *The invertible Z_N-valued matrix $\tau(t,s,\epsilon)$ is a tau function of the EZTH if and only if it satisfies the Hirota quadratic equation of the EZTH*

$$\operatorname{Res}_\lambda \lambda^{r-1}(\Gamma_a^{\#} \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a})(\tau \otimes \tau) = \operatorname{Res}_\lambda \lambda^{-r-1}(\Gamma_b^{\#} \otimes \Gamma_b)(\Gamma^{-b} \otimes \Gamma^b)(\tau \otimes \tau), \tag{6.4}$$

computed at $s_0 - s_0' = l\epsilon$ for all $l \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Proof. We need only prove that the Hirota bilinear equations are equivalent to the hypothesis of Proposition 5. By direct calculation, we obtain the four identities

$$\begin{aligned}
&\Gamma_a^{\#}\Gamma^a\tau = \tau\left(s_0 + x - \frac{\epsilon}{2}, t, s\right)\lambda^{s_0/\epsilon}\mathcal{W}(x,t,s,\epsilon\partial_x,\lambda)\lambda^{\mathbb{I}_N x/\epsilon},\\
&\Gamma_a\Gamma^{-a}\tau = \lambda^{-(s_0+x)/\epsilon}\mathcal{W}^{-1}(x,t,s,\epsilon\partial_x,\lambda)\tau\left(x + s_0 + \frac{\epsilon}{2}, t, s\right),\\
&\Gamma_b^{\#}\Gamma^{-b}\tau = \tau\left(x + s_0 - \frac{\epsilon}{2}, t, s\right)\lambda^{s_0/\epsilon}\overline{\mathcal{W}}(x,t,s,\epsilon\partial_x,\lambda)\lambda^{x\mathbb{I}_N/\epsilon},\\
&\Gamma_b\Gamma^b\tau = \lambda^{-s_0/\epsilon}\lambda^{-x\mathbb{I}_N/\epsilon}\overline{\mathcal{W}}^{-1}(x,t,s,\epsilon\partial_x,\lambda)\tau\left(x + s_0 + \frac{\epsilon}{2}, t, s\right).
\end{aligned} \tag{6.5}$$

Equations (6.5) can be proved by a method similar to the method given in [7], [9]. Substituting Eqs. (6.5) in Hirota bilinear equations (6.4), we obtain Eq. (4.6). ■

Performing a transformation of the form $\lambda \to \lambda^{-1}$ on Eq. (6.4), we obtain

$$\operatorname{Res}_\lambda \lambda^{r-1}((\Gamma_a^{\#} \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a} - \Gamma^{-a} \otimes \Gamma^a))(\tau \otimes \tau) = 0, \tag{6.6}$$

evaluated at $s_0 - s_0' = l\epsilon$ for all $l \in \mathbb{Z}$ and $r \in \mathbb{N}$. This means that the expression

$$\frac{d\lambda}{\lambda}((\Gamma_a^{\#} \otimes \Gamma_a)(\Gamma^a \otimes \Gamma^{-a} - \Gamma^{-a} \otimes \Gamma^a))(\tau \otimes \tau) \tag{6.7}$$

is regular in λ at $s_0 - s_0' = l\epsilon$ for all $l \in \mathbb{Z}$. For $N = 1$, expression (6.7) is exactly Hirota quadratic equation (1.2) of the ETH obtained in [7]. As is known, the vertex operator in fact gives one special Bäcklund transformation of the EZTH. To obtain more information about the relations between different solutions of the EZTH, in the next section, we construct the Darboux transformation for the EZTH using the determinant technique as in [24], [25].

7. Darboux transformation of the EZTH

In this section, we consider the action of the Darboux transformation for the EZTH on the Lax operator

$$\mathcal{L} = \Lambda + U + V\Lambda^{-1}, \tag{7.1}$$

i.e.,

$$\mathcal{L}^{[1]} = \Lambda + U^{[1]} + V^{[1]}\Lambda^{-1} = W\mathcal{L}W^{-1}, \tag{7.2}$$

where W is the Darboux transformation operator. That means that after the Darboux transformation, the spectral problem for the $N\times N$ matrix-valued function ϕ

$$\mathcal{L}\phi = \Lambda\phi + U\phi + V\Lambda^{-1}\phi = \lambda\phi \tag{7.3}$$

becomes

$$\mathcal{L}^{[1]}\phi^{[1]} = \lambda\phi^{[1]}. \tag{7.4}$$

To preserve the invariance of the Lax pair for the EZTH in Proposition 2, i.e., to satisfy the equalities

$$\begin{aligned}
&\partial_{t_j}\mathcal{L}^{[1]} = [(B_j^{[1]})_+, \mathcal{L}^{[1]}], \qquad \partial_{s_j}\mathcal{L}^{[1]} = [(D_j^{[1]})_+, \mathcal{L}^{[1]}],\\
&B_j^{[1]} := B_j(\mathcal{L}^{[1]}), \qquad D_j^{[1]} := D_j(\mathcal{L}^{[1]}),\\
&\partial_{t_j}\log\mathcal{L}^{[1]} = [(B_j^{[1]})_+, \log\mathcal{L}^{[1]}],\\
&(\log\mathcal{L}^{[1]})_{s_j} = [-(D_j^{[1]})_-, \log_+\mathcal{L}^{[1]}] + [(D_j^{[1]})_+, \log_-\mathcal{L}^{[1]}],
\end{aligned} \tag{7.5}$$

the dressing operator W must satisfy the dressing equation

$$W_{t_j} = -W(B_j)_+ + (WB_jW^{-1})_+W, \quad j \geq 0, \tag{7.6}$$

where W_{t_j} denotes the derivative of W with respect to t_j.

We now formulate an important theorem, which we use to obtain new solutions.

Theorem 2. *If ϕ is the first wave function of the EZTH, then the Darboux transformation operator for the EZTH*

$$W(\lambda) = (1 - \phi(\phi(x-\epsilon))^{-1}\Lambda^{-1}) = \phi\circ(1-\Lambda^{-1})\circ\phi^{-1}, \tag{7.7}$$

generates new solutions $U^{[1]}$ and $V^{[1]}$ from seed solutions U and V:

$$\begin{aligned}
&U^{[1]} = U + (\Lambda - 1)\phi(\phi(x-\epsilon))^{-1},\\
&V^{[1]} = \Lambda^{-1}V\frac{\phi\Lambda^{-2}\phi}{\Lambda^{-1}\phi^2}.
\end{aligned} \tag{7.8}$$

We introduce the notation $\phi_i = \phi_i^{[0]} := \phi|_{\lambda=\lambda_i}$. We can then choose a specific one-time Darboux transformation for the EZTH in the form

$$W_1(\lambda_1) = \mathbb{I}_N - \phi_1(\phi_1(x-\epsilon))^{-1}\Lambda^{-1}. \tag{7.9}$$

As the same time, we can obtain the action of the Darboux transformation on a wave function ϕ as

$$\phi^{[1]} = (\mathbb{I}_N - \phi_1(x)(\phi_1(x-\epsilon))^{-1}\Lambda^{-1})\phi. \tag{7.10}$$

Iterating the Darboux transformation, we then write the action of the jth Darboux transformation on the $(j-1)$th solution as

$$\begin{aligned}
\phi^{[j]} &= \left(\mathbb{I}_N - \frac{\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}}\Lambda^{-1}\right)\phi^{[j-1]}, \\
U^{[j]} &= U^{[j-1]} + (\Lambda - 1)\frac{\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}}, \\
V^{[j]} &= (\Lambda^{-1}V^{[j-1]})\frac{\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}}\frac{\Lambda^{-2}\phi_j^{[j-1]}}{\Lambda^{-1}\phi_j^{[j-1]}},
\end{aligned} \tag{7.11}$$

where $\phi_i^{[j-1]} := \phi^{[j-1]}|_{\lambda=\lambda_i}$ are the wave functions corresponding to different spectral parameter values and the $(j-1)$th solutions $U^{[j-1]}$ and $V^{[j-1]}$. It can be verified that $\phi_i^{[j-1]} = 0$, $i = 1, 2, \ldots, j-1$.

After iterating the Darboux transformation, we can generalize it to an n-fold transformation.

Theorem 3. *The n-fold Darboux transformation for the EZTH has the form*

$$W_n = \mathbb{I}_N + t_1^{[n]}\Lambda^{-1} + t_2^{[n]}\Lambda^{-2} + \cdots + t_n^{[n]}\Lambda^{-n}, \tag{7.12}$$

where

$$W_n \cdot \phi_i|_{i\le n} = 0. \tag{7.13}$$

The Darboux transformation constructs a new solution from a seed solution:

$$\begin{aligned}
U^{[n]} &= U + (\Lambda - 1)t_1^{[n]}, \\
V^{[n]} &= t_n^{[n]}(x)(\Lambda^{-n}V)t_n^{[n]-1}(x-\epsilon),
\end{aligned} \tag{7.14}$$

where

$$(W_n)_{ij} = \frac{1}{\Delta_n}\left|\begin{matrix}
\delta_{ij} & 0 & \cdots & \Lambda^{-1} & \cdots & 0 \\
0 & \phi_{1,11}(x-\epsilon) & \cdots & \phi_{1,j1}(x-\epsilon) & \cdots & \phi_{1,N1}(x-\epsilon) \\
0 & \phi_{1,12}(x-\epsilon) & \cdots & \phi_{1,j2}(x-\epsilon) & \cdots & \phi_{1,N2}(x-\epsilon) \\
-\phi_{1,ii}(x) & \phi_{1,1i}(x-\epsilon) & \cdots & \phi_{1,ji}(x-\epsilon) & \cdots & \phi_{1,Ni}(x-\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \phi_{1,1N}(x-\epsilon) & \cdots & \phi_{1,jN}(x-\epsilon) & \cdots & \phi_{1,NN}(x-\epsilon) \\
0 & \phi_{2,11}(x-\epsilon) & \cdots & \phi_{2,j1}(x-\epsilon) & \cdots & \phi_{2,N1}(x-\epsilon) \\
0 & \phi_{2,12}(x-\epsilon) & \cdots & \phi_{2,j2}(x-\epsilon) & \cdots & \phi_{2,N2}(x-\epsilon) \\
-\phi_{2,ii}(x) & \phi_{2,1i}(x-\epsilon) & \cdots & \phi_{2,ji}(x-\epsilon) & \cdots & \phi_{2,Ni}(x-\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \phi_{2,1N}(x-\epsilon) & \cdots & \phi_{2,jN}(x-\epsilon) & \cdots & \phi_{2,NN}(x-\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \phi_{n,11}(x-\epsilon) & \cdots & \phi_{n,j1}(x-\epsilon) & \cdots & \phi_{n,N1}(x-\epsilon) \\
0 & \phi_{n,12}(x-\epsilon) & \cdots & \phi_{n,j2}(x-\epsilon) & \cdots & \phi_{n,N2}(x-\epsilon) \\
-\phi_{n,ii}(x) & \phi_{n,1i}(x-\epsilon) & \cdots & \phi_{n,ji}(x-\epsilon) & \cdots & \phi_{n,Ni}(x-\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \phi_{n,1N}(x-\epsilon) & \cdots & \phi_{n,jN}(x-\epsilon) & \cdots & \phi_{n,NN}(x-\epsilon)
\end{matrix}\right. \cdots$$

$$
\cdots\left.\begin{matrix}
0 & \cdots & 0 & \cdots & \Lambda^{-n} & \cdots & 0 \\
\phi_{1,11}(x-2\epsilon) & \cdots & \phi_{1,N1}(x-2\epsilon) & \cdots & \phi_{1,j1}(x-n\epsilon) & \cdots & \phi_{1,N1}(x-n\epsilon) \\
\phi_{1,12}(x-2\epsilon) & \cdots & \phi_{1,N2}(x-2\epsilon) & \cdots & \phi_{1,j2}(x-n\epsilon) & \cdots & \phi_{1,N2}(x-n\epsilon) \\
\phi_{1,1i}(x-2\epsilon) & \cdots & \phi_{1,Ni}(x-2\epsilon) & \cdots & \phi_{1,ji}(x-n\epsilon) & \cdots & \phi_{1,Ni}(x-n\epsilon) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\phi_{1,1N}(x-2\epsilon) & \cdots & \phi_{1,NN}(x-2\epsilon) & \cdots & \phi_{1,jN}(x-n\epsilon) & \cdots & \phi_{1,NN}(x-n\epsilon) \\
\phi_{2,21}(x-2\epsilon) & \cdots & \phi_{2,N1}(x-2\epsilon) & \cdots & \phi_{2,j1}(x-n\epsilon) & \cdots & \phi_{2,N1}(x-n\epsilon) \\
\phi_{2,12}(x-2\epsilon) & \cdots & \phi_{2,N2}(x-2\epsilon) & \cdots & \phi_{2,j2}(x-n\epsilon) & \cdots & \phi_{2,N2}(x-n\epsilon) \\
\phi_{2,1i}(x-2\epsilon) & \cdots & \phi_{2,Ni}(x-2\epsilon) & \cdots & \phi_{2,ji}(x-n\epsilon) & \cdots & \phi_{2,Ni}(x-n\epsilon) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\phi_{2,1N}(x-2\epsilon) & \cdots & \phi_{2,NN}(x-2\epsilon) & \cdots & \phi_{2,jN}(x-n\epsilon) & \cdots & \phi_{2,NN}(x-n\epsilon) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\phi_{n,21}(x-2\epsilon) & \cdots & \phi_{n,N1}(x-2\epsilon) & \cdots & \phi_{n,j1}(x-n\epsilon) & \cdots & \phi_{n,N1}(x-n\epsilon) \\
\phi_{n,12}(x-2\epsilon) & \cdots & \phi_{n,N2}(x-2\epsilon) & \cdots & \phi_{n,j2}(x-n\epsilon) & \cdots & \phi_{n,N2}(x-n\epsilon) \\
\phi_{n,1i}(x-2\epsilon) & \cdots & \phi_{n,Ni}(x-2\epsilon) & \cdots & \phi_{n,ji}(x-n\epsilon) & \cdots & \phi_{n,Ni}(x-n\epsilon) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\phi_{n,1N}(x-2\epsilon) & \cdots & \phi_{n,NN}(x-2\epsilon) & \cdots & \phi_{n,jN}(x-n\epsilon) & \cdots & \phi_{n,NN}(x-n\epsilon)
\end{matrix}\right|,
$$

$$
\Delta_n = \begin{vmatrix}
\phi_1(x-\epsilon) & \phi_1(x-2\epsilon) & \cdots & \phi_1(x-n\epsilon) \\
\phi_2(x-\epsilon) & \phi_2(x-2\epsilon) & \cdots & \phi_2(x-n\epsilon) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_n(x-\epsilon) & \phi_n(x-2\epsilon) & \cdots & \phi_n(x-n\epsilon)
\end{vmatrix}.
$$

It can be easily verified that $W_n\phi_i = 0$, $i = 1, 2, \ldots, n$.

Taking a seed solution $U = (0)_{N\times N}$, $V = \mathbb{I}_N$ and then using Theorem 3, we can obtain the nth new solution of the EZTH as

$$
\begin{aligned}
U^{[n]} &= (1-\Lambda^{-1})\,\partial_{t_0}\log \overline{W}_r(\phi_1,\phi_2,\ldots,\phi_n),\\
V^{[n]} &= e^{(1-\Lambda^{-1})(1-\Lambda^{-1})\log \overline{W}_r(\phi_1,\phi_2,\ldots,\phi_n)},
\end{aligned}
\tag{7.15}
$$

where $\overline{W}_r(\phi_1,\phi_2,\ldots,\phi_n)$ is the Hankel function expressed in terms of Γ,

$$
\overline{W}_r(\phi_1,\phi_2,\ldots,\phi_n) = \det(\Lambda^{-j+1}\phi_{n+1-i})_{1\le i,j\le n}. \tag{7.16}
$$

In the definition of the Hankel function in terms of Γ, it is understood that in the calculation process, each element ϕ_{n+1-i} is treated not in the matrix form but as a scalar polynomial in Γ. After obtaining the values of $U^{[n]}$ and $V^{[n]}$ in terms of Γ, we rewrite the result in matrix form.

7.1. Soliton solutions. We investigated the first Darboux transformation for the EZTH above; we now use it to obtain new solutions from trivial seed solutions. In particular, we find some matrix-valued soliton solutions using the first Darboux transformation.

For $N = 2$, we take a seed solution of the form $U = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$, $V = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. The initial wave function ϕ_i then satisfies

$$
\begin{aligned}
&\Lambda\phi + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Lambda^{-1}\phi = \begin{bmatrix}\lambda_1 & 0\\ \lambda_2 & \lambda_1\end{bmatrix}\phi, \quad 1\le i\le n.\\
&\phi = \exp\!\left(\frac{x}{\epsilon}\log\begin{bmatrix}z_1 & 0\\ z_2 & z_1\end{bmatrix}\right), \quad z_1 \ne 0,
\end{aligned}
\tag{7.17}
$$

and moreover

$$z_1 + z_1^{-1} = \lambda_1, \qquad z_2 + z_1^{-1} - \frac{z_2}{z_1^2} = \lambda_2.$$
$$S = E + \omega_1 \Lambda^{-1} + v\frac{x}{\epsilon}\Lambda^{-1} - \frac{x}{\omega_1}\epsilon\Lambda^{-2} + \ldots, \qquad \omega_1 = \text{const.} \tag{7.18}$$

Under such an initial condition, the operator A_1 in Lemma 1 becomes

$$A_1 = \left(\Lambda + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1}\right)\epsilon\partial - \left(\Lambda - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1}\right),$$
$$\frac{\partial \phi}{\partial s_1} = \left[\left(\Lambda + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1}\right)\epsilon\partial - \left(\Lambda - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1}\right)\right]\phi. \tag{7.19}$$

The solution ϕ expressed in terms of x and s_1 can then be chosen in the form

$$\phi = \exp\left(\frac{x + \lambda s_1}{\epsilon}\log Z + \frac{s_1}{\epsilon}(-Z + Z^{-1})\right), \qquad Z = \begin{bmatrix} z_1 & 0 \\ z_2 & z_1 \end{bmatrix}, \qquad \lambda = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \lambda_1 \end{bmatrix}, \tag{7.20}$$

where

$$Z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z^{-1} = \begin{bmatrix} z_1 & 0 \\ z_2 & z_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1^{-1} & 0 \\ -\dfrac{z_2}{z_1^2} & z_1^{-1} \end{bmatrix} = \lambda = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \lambda_1 \end{bmatrix},$$

and

$$U^{[1]} = (\Lambda - 1)\frac{\cosh(((x + \lambda s_1)/\epsilon)\log Z + (s_1/\epsilon)(-Z + Z^{-1}))}{\cosh(((x + \lambda s_1)/\epsilon)\log Z + (s_1/\epsilon)(-Z + Z^{-1}) - \log Z)},$$
$$V^{[1]} = (1 - \Lambda^{-1})^2 \log\left[2\cosh\left(\frac{x + \lambda s_1}{\epsilon}\log Z + \frac{s_1}{\epsilon}(-Z + Z^{-1})\right)\right]. \tag{7.21}$$

Using

$$\log Z = \begin{bmatrix} \log z_1 & 0 \\ \dfrac{z_2}{z_1} & \log z_1 \end{bmatrix}, \qquad \cosh\left(\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}\right) = \begin{bmatrix} \cosh a & 0 \\ b\cosh a & \cosh a \end{bmatrix},$$

we can obtain the concrete elements of the new solutions $U^{[1]}$, $V^{[1]}$ in the form

$$\begin{aligned}
u_0^{[1]} &= (\Lambda - 1)\frac{\cosh((x/\epsilon)\log z_1 + (s_1/\epsilon)(\lambda_1 - z_1 + z_1^{-1}))}{\cosh(((x - \epsilon)/\epsilon)\log z_1 + (s_1/\epsilon)(\lambda_1 - z_1 + z_1^{-1}))}, \\
u_1^{[1]} &= \left(\frac{x}{\epsilon}\frac{z_2}{z_1} + \frac{s_1}{\epsilon}\left(\lambda_2 - z_2 - \frac{z_2}{z_1^2}\right)\right) \times \\
&\quad \times \frac{\cosh((x/\epsilon)\log z_1 + (s_1/\epsilon)(\lambda_1 - z_1 + z_1^{-1}))}{\cosh(((x - \epsilon)/\epsilon)\log z_1 + (s_1/\epsilon)(\lambda_1 - z_1 + z_1^{-1}))} - \\
&\quad - \left(\frac{x - \epsilon}{\epsilon}\frac{z_2}{z_1} + \frac{s_1}{\epsilon}\left(\lambda_2 - z_2 - \frac{z_2}{z_1^2}\right)\right) \times \\
&\quad \times \frac{\cosh((x/\epsilon)\log z_1 + (s_1/\epsilon)(\lambda_1 - z_1 + z_1^{-1}))}{\cosh^2(((x - \epsilon)/\epsilon)\log z_1 + (s_1/\epsilon)(\lambda_1 - z_1 + z_1^{-1}))}, \\
v_0^{[1]} &= (1 - \Lambda^{-1})^2 \log\left[2\cosh\left(\frac{x}{\epsilon}\log z_1 + \frac{s_1}{\epsilon}(\lambda_1 - z_1 + z_1^{-1})\right)\right], \\
v_1^{[1]} &= (1 - \Lambda^{-1})^2 \log\left(\frac{x}{\epsilon}\frac{z_2}{z_1} + \frac{s_1}{\epsilon}\left(\lambda_2 - z_2 - \frac{z_2}{z_1^2}\right)\right).
\end{aligned} \tag{7.22}$$

Taking $z_2 = \lambda_2 = u_1^{[1]} = v_1^{[1]} = 0$, we reduce these soliton solutions to soliton solutions of the scalar-valued extended Toda chain [26].

8. Bi-Hamiltonian structure and tau symmetry

In this section, to describe the integrability of the EZTH, we construct the bi-Hamiltonian structure and tau symmetry of the EZTH as in [23]. For a matrix $A = (a_{ij}) = \sum_{i=0}^{N-1} a_i \Gamma^i$, the vector field ∂_A over EZTH is defined by

$$\partial_A = \sum_{i=0}^{N-1} \sum_{k\geq 0} a_i^{(k)} \left(\frac{\partial}{\partial u_i^{(k)}} + \frac{\partial}{\partial v_i^{(k)}} \right). \tag{8.1}$$

For a function $\bar{f} = \int f\, dx$, we have

$$\partial_A \bar{f} = \int \sum_{i=0}^{N-1} \sum_{k\geq 0} a_i^{(k)} \left(\frac{\partial f}{\partial u_i^{(k)}} + \frac{\partial f}{\partial v_i^{(k)}} \right) dx = \int \mathrm{Tr}_N \sum_{k\geq 0} A^{(k)} \left(\frac{\delta f}{\delta u^{(k)}} + \frac{\delta f}{\delta v^{(k)}} \right) dx, \tag{8.2}$$

where

$$\left(\frac{\delta}{\delta u}\right)_{ij} = \frac{\delta}{\delta u_{ji}}, \qquad \left(\frac{\delta}{\delta v}\right)_{ij} = \frac{\delta}{\delta v_{ji}}, \tag{8.3}$$

and

$$\mathrm{Tr}_N\, A = \mathrm{Tr} \begin{bmatrix} \frac{1}{N} & \frac{1}{N-1} & \cdots & 1 \\ 0 & \frac{1}{N} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{N} \end{bmatrix} A. \tag{8.4}$$

In this section, we consider the EZTH using the Lax operator

$$\mathcal{L} = \Lambda + u + e^v \Lambda^{-1}, \quad u, v \in Z_N. \tag{8.5}$$

We can then introduce the Hamiltonian brackets as

$$\begin{gathered} \{\bar{f}, \bar{g}\} = \int \mathrm{Tr}_N \sum_{w,w'} \frac{\delta f}{\delta w} \{w, w'\} \frac{\delta g}{\delta w'}\, dx, \\ w, w' = u_i \text{ or } v_j, \quad 0 \leq i, j \leq N-1. \end{gathered} \tag{8.6}$$

For $u(x) = \sum_{i=0}^{N-1} u_i(x)\Gamma^i$ and $v(x) = \sum_{i=0}^{N-1} v_i(x)\Gamma^i$, the bi-Hamiltonian structure for the EZTH can be defined by two compatible Poisson brackets, which is a matrix generalization of the ETH in [6]:

$$\begin{aligned} &\{v_i(x), v_j(y)\}_1 = \{u_i(x), u_j(y)\}_1 = 0, \\ &\{u_i(x), v_j(y)\}_1 = \frac{1}{\epsilon}\delta_{ij}[e^{\epsilon\partial_x} - 1]\delta(x-y), \\ &\{u_i(x), u_j(y)\}_2 = \frac{1}{\epsilon}\delta_{ij}[e^{\epsilon\partial_x} e^{v(x)} - e^{v(x)} e^{-\epsilon\partial_x}]_i \delta(x-y), \\ &\{u_i(x), v_j(y)\}_2 = \frac{1}{\epsilon}\delta_{ij} u_i(x)[e^{\epsilon\partial_x} - 1]\delta(x-y), \\ &\{v_i(x), v_j(y)\}_2 = \frac{1}{\epsilon}\delta_{ij}[e^{\epsilon\partial_x} - e^{-\epsilon\partial_x}]\delta(x-y). \end{aligned} \tag{8.7}$$

For any difference operator $A = \sum_k A_k \Lambda^k$, we define the residue $\operatorname{Res} A = A_0$. In the following theorem, we prove that the Poisson structure above can be regarded as the Hamiltonian structure for the EZTH.

Theorem 4. *The flows of the EZTH are Hamiltonian systems of the form*

$$\frac{\partial u_i}{\partial t_{k,j}} = \{u_i, H_{k,j}\}_1, \qquad \frac{\partial v_i}{\partial t_{k,j}} = \{v_i, H_{k,j}\}_1, \quad k = 0, 1, \quad j \geq 0, \tag{8.8}$$

with $t_{0,j} = t_j$ and $t_{1,j} = s_j$. They satisfy the bi-Hamiltonian recurrence relations

$$\{\,\cdot\,, H_{1,n-1}\}_2 = n\{\,\cdot\,, H_{1,n}\}_1 + 2\{\,\cdot\,, H_{0,n-1}\}_1, \qquad \{\,\cdot\,, H_{0,n-1}\}_2 = (n+1)\{\,\cdot\,, H_{0,n}\}_1.$$

The Hamiltonians here have the form

$$H_{k,j} = \int h_{k,j}(u, v; u_x, v_x; \ldots; \epsilon)\, dx, \quad k = 0, 1, \quad j \geq 0, \tag{8.9}$$

where

$$h_{0,j} = \frac{1}{(j+1)!} \operatorname{Tr}_N \operatorname{Res} \mathcal{L}^{j+1}, \qquad h_{1,j} = \frac{2}{j!} \operatorname{Tr}_N \operatorname{Res}[\mathcal{L}^j(\log \mathcal{L} - c_j)]. \tag{8.10}$$

Proof. For $\beta = 0$, which corresponds to the original Toda hierarchy, the proof is the same as in [6].

Here, we prove that the flows $\partial/\partial t_{1,n}$ are also Hamiltonian systems with respect to the first Poisson bracket. The identity

$$\operatorname{Tr}_N \operatorname{Res}[\mathcal{L}^n d(S \epsilon \partial_x S^{-1})] \sim \operatorname{Tr}_N \operatorname{Res} \mathcal{L}^{n-1} d\mathcal{L} \tag{8.11}$$

was proved in [6]. It shows the validity of the equivalence relation

$$\operatorname{Tr}_N \operatorname{Res}(\mathcal{L}^n d \log_+ \mathcal{L}) \sim \operatorname{Tr}_N \operatorname{Res}(\mathcal{L}^{n-1} d\mathcal{L}). \tag{8.12}$$

Here, the equivalence relation $\sim$ is up to an x-derivative of another 1-form.

By analogy with (8.11), we obtain the equivalence relation

$$\operatorname{Tr}_N \operatorname{Res}[\mathcal{L}^n d(\bar{S} \epsilon \partial_x \bar{S}^{-1})] \sim -\operatorname{Tr}_N \operatorname{Res} \mathcal{L}^{n-1} d\mathcal{L}, \tag{8.13}$$

i.e.,

$$\operatorname{Tr}_N \operatorname{Res}(\mathcal{L}^n d \log_- \mathcal{L}) \sim \operatorname{Tr}_N \operatorname{Res}(\mathcal{L}^{n-1} d\mathcal{L}). \tag{8.14}$$

Combining (8.12) with (8.14), we obtain

$$\operatorname{Tr}_N \operatorname{Res}(\mathcal{L}^n d \log \mathcal{L}) \sim \operatorname{Tr}_N \operatorname{Res}(\mathcal{L}^{n-1} d\mathcal{L}). \tag{8.15}$$

Let

$$A_{\alpha,n} = \sum_k a_{\alpha,n+1;k} \Lambda^k. \tag{8.16}$$

From

$$\frac{\partial \mathcal{L}}{\partial t_{k,n}} = [(B_{k,n})_+, \mathcal{L}] = [-(B_{k,n})_-, \mathcal{L}], \qquad B_{0,n} = B_n, B_{1,n} = D_n, \tag{8.17}$$

we then derive the equation

$$\begin{aligned} \epsilon \frac{\partial u}{\partial t_{\beta,n}} &= a_{\beta,n;1}(x+\epsilon) - a_{\beta,n;1}(x) \in Z_N, \\ \epsilon \frac{\partial v}{\partial t_{\beta,n}} &= a_{\beta,n;0}(x-\epsilon) e^{v(x)} - a_{\beta,n;0}(x) e^{v(x+\epsilon)} \in Z_N, \quad \beta = 0, 1. \end{aligned} \tag{8.18}$$

Equivalence relation (8.12) now easily follows from these two equations. Using (8.12), we obtain

$$
\begin{aligned}
d\tilde{h}_n &= \frac{2}{n!} d \operatorname{Tr}_N \operatorname{Res}[\mathcal{L}^n(\log_+ \mathcal{L} - c_n)] \sim \\
&\sim \frac{2}{(n-1)!} \operatorname{Tr}_N \operatorname{Res}[\mathcal{L}^{n-1}(\log_+ \mathcal{L} - c_n) d\mathcal{L}] + \frac{2}{n!} \operatorname{Tr}_N \operatorname{Res}[\mathcal{L}^{n-1} d\mathcal{L}] = \\
&= \frac{2}{(n-1)!} \operatorname{Tr}_N \operatorname{Res}[\mathcal{L}^{n-1}(\log_+ \mathcal{L} - c_{n-1}) d\mathcal{L}] = \\
&= \operatorname{Tr}_N \operatorname{Res}[a_{1,n;0}(x)\, du + a_{1,n;1}(x-\epsilon) e^{v(x)}\, dv]. \qquad (8.19)
\end{aligned}
$$

This yields the identities

$$
\frac{\delta H_{1,n}}{\delta u} = a_{1,n;0}(x), \qquad \frac{\delta H_{1,n}}{\delta v} = a_{1,n;1}(x-\epsilon) e^{v(x)}. \qquad (8.20)
$$

These identities agree with the Lax equation

$$
\begin{aligned}
\frac{\partial u_i}{\partial t_{1,n}} &= \{u_i, H_{1,n}\}_1 = \frac{1}{\epsilon}[e^{\epsilon\, \partial_x} - 1] \frac{\delta H_{1,n}}{\delta v_i} = \frac{1}{\epsilon}(a_{1,n;1}(x+\epsilon) - a_{1,n;1}(x))_i, \\
\frac{\partial v_i}{\partial t_{1,n}} &= \{v_i, H_{1,n}\}_1 = \frac{1}{\epsilon}[1 - e^{\epsilon\, \partial_x}] \frac{\delta H_{1,n}}{\delta u_i} = \qquad (8.21) \\
&= \frac{1}{\epsilon}[a_{1,n;0}(x-\epsilon) e^{v(x)} - a_{1,n;0}(x) e^{v(x+\epsilon)}]_i.
\end{aligned}
$$

From the identities obtained above, we can see that the flows $\partial/\partial t_{1,n}$ are Hamiltonian systems of form (8.8). In the case $\beta = 1$, the recurrence relation follows from the trivial identities

$$
\begin{aligned}
n\frac{2}{n!}\mathcal{L}^n(\log_\pm \mathcal{L} - c_n) &= \mathcal{L}\frac{2}{(n-1)!}\mathcal{L}^{n-1}(\log_\pm \mathcal{L} - c_{n-1}) - 2\frac{1}{n!}\mathcal{L}^n = \\
&= \frac{2}{(n-1)!}\mathcal{L}^{n-1}(\log_\pm \mathcal{L} - c_{n-1})\mathcal{L} - 2\frac{1}{n!}\mathcal{L}^n.
\end{aligned}
$$

For $\beta = 1$, we then obtain

$$
\begin{aligned}
n a_{1,n+1;1}(x) &= a_{1,n;0}(x+\epsilon) + u a_{1,n;1}(x) + e^v a_{1,n;2}(x-\epsilon) - 2a_{0,n+1;1}(x) = \\
&= a_{1,n;0}(x) + u(x+\epsilon) a_{1,n;1}(x) + e^{v(x+2\epsilon)} a_{1,n;2}(x) - 2a_{0,n+1;1}(x).
\end{aligned}
$$

This further leads to

$$
\begin{aligned}
\{u_i, H_{1,n-1}\}_2 &= \{[\Lambda e^{v(x)} - e^{v(x)}\Lambda^{-1}] a_{1,n;0}(x) + u(x)[\Lambda - 1] a_{1,n;1}(x-\epsilon) e^{v(x)}\}_i = \\
&= n[a_{1,n+1;1}(x) e^{v(x+\epsilon)} - a_{1,n+1;1}(x-\epsilon) e^{v(x)}]_i + \\
&\quad + 2[a_{0,n+1;0}(x) e^{v(x+\epsilon)} - a_{0,n+1;0}(x-\epsilon) e^{v(x)}]_i.
\end{aligned}
$$

This is exactly the recurrence relation for flows in terms of u. The analogous recurrence relation for flows in terms of v can be derived similarly. ■

As in [6], the existence of the tau symmetry for the EZTH is proved in the following theorem.

Theorem 5. *The EZTH has the tau-symmetry property*

$$\frac{\partial h_{\alpha,m}}{\partial t_{\beta,n}} = \frac{\partial h_{\beta,n}}{\partial t_{\alpha,m}}, \quad \alpha,\beta = 0,1, \quad m,n \geq 0. \tag{8.22}$$

Proof. We prove the theorem for the case where $\alpha = 1$ and $\beta = 0$:

$$\begin{aligned} \frac{\partial h_{1,m}}{\partial t_{0,n}} &= \frac{2}{m!(n+1)!}\operatorname{Tr}_N \operatorname{Res}[-(\mathcal{L}^{n+1})_-, \mathcal{L}^m(\log_+\mathcal{L} - c_m)] = \\ &= \frac{2}{m!(n+1)!}\operatorname{Tr}_N \operatorname{Res}[(\mathcal{L}^m(\log_+\mathcal{L} - c_m))_+, (\mathcal{L}^{n+1})_-] = \\ &= \frac{2}{m!(n+1)!}\operatorname{Tr}_N \operatorname{Res}[(\mathcal{L}^m(\log_+\mathcal{L} - c_m))_+, \mathcal{L}^{n+1}] = \frac{\partial h_{0,n}}{\partial t_{1,m}}. \end{aligned} \tag{8.23}$$

The theorem is proved similarly for the other cases. ■

This property justifies the following alternative definition of another kind of tau function for the EZTH.

Definition 3. Another tau function in Z_N for the EZTH can be defined using the expressions in terms of the Hamiltonian densities

$$h_{\beta,n} = \epsilon(\Lambda - 1)\frac{\partial \log \bar{\tau}}{\partial t_{\beta,n}}, \quad \beta = 0,1, \quad n \geq 0, \tag{8.24}$$

where $t_{0,j} = t_j$ and $t_{1,j} = s_j$.

Having two different definitions of tau functions for this hierarchy, we pose the question of some mysterious relations between these two kinds of tau functions. One function arises from the Sato theory, and the other arises from the tau symmetry of the Hamiltonian.

9. Conclusions and discussion

We have constructed a new hierarchy called the EZTH and extended the Sato theory to this hierarchy including the Sato equations, matrix wave operators, Hirota quadratic equations, and existence of the tau function. Similarly to the ETH and EBTH in the respective Gromov–Witten theory of $\mathbb{CP}^1$ and orbifolds, this hierarchy deserves further study in view of its potential applications in topological quantum fields and the Gromov–Witten theory. Based on two different definitions of the tau functions for this hierarchy, it would be interesting to discover mysterious deep relations between these two kinds of tau functions, of which one is defined using the Sato theory and the other arises from the tau symmetry of the Hamiltonian. This is not easy, and we plan to study this question in our future work.

Acknowledgments. The authors thank Todor E. Milanov and Dafeng Zuo for the valuable discussion.

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