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NOTION OF BLOWUP OF THE SOLUTION SET OF DIFFERENTIAL EQUATIONS AND AVERAGING OF RANDOM SEMIGROUPS

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We propose a unique approach to studying the violation of the well-posedness of initial boundary-value problems for differential equations. The blowup of the set of solutions of a problem for a differential equation is defined as a discontinuity of a multivalued map associating an initial boundary-value problem with the set of solutions of this problem. We show that such a definition not only describes effects of the solution destruction or its nonuniqueness but also permits prescribing a procedure for extending the solution through the singularity origination instant by using an appropriate random process. Considering the initial boundary-value problems whose solution sets admit singularities of the blowup type and a neighborhood of these problems in the space of problems permits associating the initial problem with the set of limit points of a sequence of solutions of the approximating problems. Endowing the space of problems with the structure of a space with measure, we obtain a random semigroup generated by the initial problem. We study the properties of the mathematical expectations (means) of a random semigroup and their equivalence in the sense of Chernoff to semigroups with averaged generators.

Keywords: boundary-value problem, blowup, dynamical system, Ω-explosion, semigroup, random dynamical system, Chernoff's theorem, averaging

1. Introduction

This paper is devoted to studying the phenomenon of blowup of a solution of an initial boundary-value problem for a differential equation. From the conceptual standpoint, the proposed approach goes back to the classical theory of dynamical systems, where the Ω-explosion phenomenon has been traditionally studied since hyperbolic theory began developing in the 1960s (see, e.g., [1]–[7]). The Ω-explosion in a dynamical system (with continuous or discrete time) is understood as the absence of upper semicontinuity of the map between each point of the considered space of systems and the nonwandering set of the corresponding dynamical system (see [4]).

In the theory of differential equations, the blowup phenomenon is primarily regarded as an unbounded increase in the solution on a finite time interval [8]. The influence of the domain topology on the absence

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of a solution of a nonlinear boundary-value problem of elliptic type is also a phenomenon of the same nature [9]. On the other hand, it is also natural to consider jumplike variations in the cardinality of the solution set of a boundary-value problem as a blowup phenomenon in the theory of differential equations.

We assume that the Cauchy problem for a differential equation exhibits the blowup property if one of the following effects occurs:

1. The interval of existence of the Cauchy problem solution is bounded.

2. The norm of the solution in the Banach space of values of the Cauchy problem solution increases unboundedly on a bounded interval.

3. The integral curve of the Cauchy problem contains points of nonuniqueness of its solutions.

The blowup phenomenon or the peaking regime is traditionally understood as effect 2, but all three effects, as we show below (see Examples 3 and 4 and also [10], [11]), can be exhibited in the same Cauchy problem with different modifications of its statement. We define the phenomenon of blowup of a solution of the Cauchy problem for a differential equation with effects 1–3 as the existence of a discontinuity of the resolving map associating each point z in the topological space Z of Cauchy problems with a point $G(z)$ in the topological space 2^Y of subsets of the Banach space Y of solutions of the Cauchy problems; the point $G(z)$ is the set of solutions of the Cauchy problem z (the rigorous definition of a resolving map is given in Sec. 2.1 below).

We consider several examples showing how the definition introduced above describes peaking regimes and points of the solution nonuniqueness on an integral curve. We classify the blowup phenomena according to the type of the point of discontinuity of the resolving map: there are discontinuity points of the removable type (if the resolving map has a limit), discontinuity points of the pole type (if for any sequence of approximating problems, the sequence of norms of their solutions tends to infinity), and discontinuity points of the essential type (these are all other cases, which necessarily contain different limit points of the sequence of solutions of the approximating problems). The blowup phenomenon and its type depend on the choice of norms in Banach spaces where the Cauchy problem is formulated as an abstract equation (see [10]–[13]). Such a phenomenon is similar to the Ω-explosion phenomena in smooth dynamical systems. For example, the spaces of smooth dynamical systems that admit the C^0-Ω-explosion and do not admit the C^1-Ω-explosion were discussed in [6], [7].

In the framework of the obtained classification, we propose a procedure for extending the solution through the singularity origination instant, i.e., we construct a unique extension in the case of a removable blowup or extend the solution as a random map or the expectation value of a random map in the case of an essential blowup. In the case of a pole-type blowup, it is rather difficult to extend the solution through the blowup origination instant in terms of passing to the limit or averaging over the space of problems. The fact that the blowup is of the pole type shows that there is a discrepancy between the choice of norms and topologies in the statement of the problem and the process or the phenomenon described by this problem. There are examples of how to change the topology in the problem statements such that the discontinuity type of the solution set of the studied problem changes from polar to removable [12] or essential [11], [14].

Studying the problems that are points of essential discontinuity of the resolving map implies the necessity to consider random variables ranging in the space of solutions of the considered problem. Because the solutions of well-posed problems for evolution equations generate semigroups or one-parameter families of maps, we must study random semigroups and random maps.

We consider examples of random semigroups encountered when studying the Cauchy problem for differential equations with singularities violating the problem well-posedness, i.e., violating the existence or uniqueness of the solution. Along with the Cauchy problem, we consider a neighborhood of it in the space of Cauchy problems, which is endowed with the structure of a measurable space with a measure. If the measure is concentrated on a set of Cauchy problems generating a semigroup, then a random semigroup arises. If the measure on the space of Cauchy problems is concentrated in an arbitrary deleted neighborhood of the considered problem and this problem is well posed, then the values of the random semigroup are concentrated in an arbitrary neighborhood of the semigroup generated by the Cauchy problem under study. Otherwise, the initial Cauchy problem is associated not with the semigroup generated by this problem (which does not exist) but with a random semigroup and, in particular, the expectation value of the random semigroup and its iterations.

We note that another reason for studying random semigroups and random operators is the ambiguity in choosing boundary conditions when linear differential equations are considered in a bounded domain [15]. The set of all boundary conditions that can be posed for a given differential equation in a given domain is described in [15], [16]. Endowing this set with the structure of a measurable space with a measure allows determining a random variable ranging in the space of operators representing the boundary-value problems. In particular, this is possible when studying the evolution equations ranging in a set of one-parameter semigroups.

A random semigroup is defined as a random variable ranging in the set of one-parameter semigroups of transformations of Banach spaces. The expectation value $\mathbf{F}(t)$, $t \geq 0$, of a random semigroup is defined by the Pettice integral. The expectation value of a random semigroup is generally not a semigroup. Here, we consider examples where the averaging of a random semigroup preserves the semigroup property.

Although the expectation value is not a semigroup, its iterations (i.e., the sequences $\{(\mathbf{F}(t))^n\}$) as functions of the variable $n \in \{0\} \cup \mathbb{N}$ form a semigroup. The relation between the random semigroups introduced above and random dynamical systems (see [17], [18]) is that the iterations $\{(\mathbf{F}(t))^n\}$, $n \in \mathbb{N}$, are expectation values of a random dynamical system (see [18], [19]). In this case, there is a two-parameter family of operators whose (real and natural) parameters play the role of time, and the operator function $\mathbf{G}(t,n) = (\mathbf{F}(t))^n$, $t \geq 0$, $n \in \mathbb{N}$, has the semigroup property with respect to the integer parameter and generally does not have this property if the parameter is real.

Studying the iterations of the expectation value $\{(\mathbf{F}(t/n))^n\}$ allows finding the conditions under which they are equivalent in the Chernoff sense to a semigroup whose generator is the mean of the generator of a random semigroup. The procedure thus introduced for averaging semigroup generators (see [20]) allows introducing a procedure for summing unbounded operators, which is a generalization of the procedure for summing the operators in the sense of quadratic forms. Studying the iterations in the Chernoff sense of the expectation values of random semigroups shows that the two-parameter family of operators $\mathbf{G}(t,n)$ becomes the averaged one-parameter semigroup of operators $\mathbf{U}$ in the special process of passing to the limit as

$$\mathbf{U}(\tau) = \lim_{\substack{n\to\infty,\, t\to 0,\\ nt=\tau}} \mathbf{G}(t,n).$$

Similar processes of passing to the limit along special curves in the space of parameters appear in statistical and quantum statistical mechanics as the thermodynamic limit [21], stochastic limit [22], and the limits of a small density and a weak coupling constant [22], [23]. It was shown in [22] (see Sec. 1.16) that the stochastic limit in the problems of quantum statistical mechanics arises as a generalization of the functional central limit theorem. At the same time, as noted in [24], the central limit theorem is one of the methods for defining the Feynman pseudomeasures.

2. Resolving map of the initial boundary-value problem

We regard the Cauchy problem for a differential equation as the abstract equation (see [11], [13], [25]–[27])

$$\mathbf{A}u = f, \quad f \in X, \quad u \in Y, \quad \mathbf{A} \in B(Y, X), \tag{1}$$

where X and Y are Banach spaces and $B(Y, X)$ is the Banach space of operators acting from the domain $D(\mathbf{A}) \subset Y$ to the space X. Here, $B(Y, X)$ is endowed with the norm $\| \cdot \|_B$ (or $B(Y, X)$ is a topological space of operators acting from the domain $D(\mathbf{A}) \subset Y$ to the space X and endowed with the topology τ_B in this case).

Cauchy problem (1) determines the multivalued map

$$G \colon X \times B(Y, X) \to 2^Y$$

defined on the set $X \times B(Y, X)$, ranging in the set 2^Y of all subsets of Y, and defined by the formula

$$G(f, \mathbf{A}) = M_{f,\mathbf{A}} \equiv \mathbf{A}^{-1}(f).$$

We note that the map G is defined at any point of $X \times B(Y, X)$ and the value of G can be the empty set at some points of this space.

Our goal here is to show that the well-known examples of initial boundary-value problems that admit the blowup phenomena (peaking regime, blowup, violation of uniqueness) are examples of discontinuity points of the multivalued map G regarded as a map of the topological space $Z = X \times B(Y, X)$ to the topological space 2^Y (the topologies introduced in Z and 2^Y are described below), while the well-posed initial boundary-value problems are points of continuity of the map G as a map of one topological space to another topological space. Further, we use the map G to classify the blowup phenomena according to the character of the discontinuity of G.

Supplementing the definition of the map G at discontinuity point $(f_0, \mathbf{A}_0)$ with the value of the upper topological limit $\mathrm{Ls}_{(f,\mathbf{A}) \to (f_0,\mathbf{A}_0)} G(f, \mathbf{A})$ of G as $(f, \mathbf{A}) \to (f_0, \mathbf{A}_0)$ defines an upper semicontinuous multivalued map G (the upper semicontinuity of a map means that for any neighborhood $\mathcal{U}$ of a point $(G(f_0, \mathbf{A}_0))$ in the topological space 2^Y, there is a neighborhood of $(f_0, \mathbf{A}_0)$ in the topological space Z such that the value of $G(f, \mathbf{A})$ is in $\mathcal{U}$ for any $(f, \mathbf{A})$ in this neighborhood) and determines the procedure for extending the dynamical transformation generated by the initial boundary-value problem through the blowup origination instant by the expression $G(f_0, \mathbf{A}_0) = \mathrm{Ls}_{(f,\mathbf{A}) \to (f_0,\mathbf{A}_0)} G(f, \mathbf{A})$.

Introducing a measure μ on the set of initial boundary-value problems Z allows completing the definition of the procedure for the dynamical transformation extension as a random process ξ such that the set of its values is determined by the values of the map G on the deleted neighborhood of the point $(f_0, \mathbf{A}_0)$. The Cauchy problem admitting a blowup phenomenon thus determines not a dynamical system (i.e., not a semigroup of transformations of the space of initial data) but a random semigroup (i.e., a measurable map of a space with a measure into the linear topological space of operator functions whose values are semigroups of operators).

Here, we establish a relation between the extensions of the map G to the point $(f_0, \mathbf{A}_0)$ by using the upper topological limit as $(f, \mathbf{A}) \to (f_0, \mathbf{A}_0)$ and a random process ξ.

Finally, for the problem posed above of extending the dynamical transformations generated by the initial boundary-value problems through the blowup origination instant, it is convenient to represent the dynamical transformations in terms of pseudomeasures on the space of maps of the time interval onto the Banach space corresponding to the problem. In this problem statement, the dynamical transformation is the conditional expectation of a pseudomeasure on the subalgebra generated by a special class of cylinder functionals. The well-posedness of the problem means that the conditional expectation can be represented in the special form of atomic measures.

In conclusion, we note that the theory of partial differential equations studies the continuity of the dependence on the parameter at the highest-order derivative not only on the characteristics of the dynamical system generated by the Cauchy problem (such as the set of solutions) but also on characteristics such as the attractor, nonwandering set, invariant set, and invariant measure. For example, sufficient conditions for the continuity of a multivalued map $\mathcal{A}$ associating each Cauchy problem with its attractor in the space of initial data were obtained in [28].

2.1. Definition of the blowup phenomenon and classification of blowups. If $B(Y,X)$ is a Banach space, then the set $Z = X \times B(Y,X)$ is a Banach space endowed with the norm $\|(f,\mathbf{A})\| = \|f\|_X + \|\mathbf{A}\|_{B(Y,X)}$. If $B(Y,X)$ is a topological space, then the set $Z = X \times B(Y,X)$ is a topological space endowed with the topology τ_Z of the direct product of two topological spaces. The Banach space Y is called the space of solutions of the Cauchy problem, and the topological space Z is called the space of Cauchy problems.

The set $(2^Y,\tau)$ is a topological space with the topology τ generated by the pseudometric given on the set 2^Y by the expressions

$$r_H(A,B) = \max\{\sup_{x\in A}\rho_Y(x,B),\ \sup_{x\in B}\rho_Y(x,A)\} \quad \text{if } A,B \neq \varnothing,$$

$$r_H(A,\varnothing) = r_H(\varnothing,A) = +\infty \quad \text{if } A \neq \varnothing, \qquad r_H(\varnothing,\varnothing) = 0.$$

The function r_H is a pseudometric because it is nonnegative and symmetric and satisfies the triangle inequality on $2^Y \times 2^Y$. But the equality $\rho(A,B) = 0$ does not imply that $A = B$ (for example, in the case where A is an open ball in Y and B is its closure). Therefore, the topology generated by the pseudometric ρ_H is not a Hausdorff topology.

Lemma 1. *The equality $r_H(A,B) = 0$ holds for some $A,B \in 2^Y$ if and only if the closures of the sets A and B in the space Y coincide.*

To prove the lemma, it suffices to show that if a point M belongs to the set A but does not belong to the set B, then it belongs to the closure of B. Indeed, it follows from $r_H(A,B) = 0$ that $\rho_Y(M,B) = 0$. Therefore, if $M \notin B$, then M is in the closure of the set B, which proves the lemma.

We note that the restriction of the function ρ_H to the set 2^Y_c of closed subsets of the set 2^Y is a Hausdorff metric on 2^Y_c.

We now consider the map G as a map of the topological space (Z,τ_Z) to the topological space $(2^Y,\tau)$.

Definition 1. We say that Cauchy problem (1) exhibits the *blowup property* if the point $(f,\mathbf{A}) \in Z$ is a point of discontinuity of the map G.

We draw the reader's attention to the fact that the possibility (or the absence) of the Ω-explosion depends significantly on the space of systems under study (see [6], [7]).

We assume that S is a subset of the topological space Z, the topology on S is induced from the space Z, and the point $(f,\mathbf{A}) \in S$ is a limit point in S. If the map G is considered not on the entire space Z but only on its part S, then the set S is called a topological space of Cauchy problems.

Definition 2. We say that Cauchy problem (1) exhibits the *blowup property with respect to the topological space of Cauchy problems* S if the point $(f,\mathbf{A})$ is a point of discontinuity of the map $G\colon S \to 2^Y$.

Theorem 1. *If a topological space $B(Y, X)$ is a linearly connected subset in a normed linear space, then a point $(f, \mathbf{A}) \in Z = X \times B(Y, X)$ is a discontinuity point of the map $G\colon Z \to 2^Y$ if and only if there is a curve $\Gamma \subset Z$ such that $(f, \mathbf{A}) \in \Gamma$ and $(f, \mathbf{A})$ is a discontinuity point of the map $G|_\Gamma$.*

Proof. If a point $(f_0, \mathbf{A}_0) \in Z$ is a continuity point of the map $G\colon Z \to 2^Y$ (this means that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|f - f_0\|_X + \|\mathbf{A} - \mathbf{A}_0\|_B < \delta$ implies $\rho_H(G(f, \mathbf{A}), G(f_0, \mathbf{A}_0)) < \varepsilon$), then it is necessarily a continuity point of the restriction of G to the curve Γ containing the point $(f_0, \mathbf{A}_0)$.

But if the point $(f_0, \mathbf{A}_0) \in Z$ is a discontinuity point of the map $G\colon Z \to 2^Y$, then there is a $\varepsilon > 0$ such that for any $\delta > 0$, there exists a $(f, \mathbf{A}) \in O_\delta(f_0, \mathbf{A}_0)$ such that $\rho_H(G(f, \mathbf{A}), G(f_0, \mathbf{A}_0)) > \varepsilon$. Therefore, there is a sequence of points $\{M_k\} = \{(f_k, \mathbf{A}_k)\}$ converging to the point $M_0 = (f_0, \mathbf{A}_0)$ in the space Z and satisfying the condition $\rho_H(G(f_k, \mathbf{A}_k), G(f_0, \mathbf{A}_0)) > \varepsilon$ for all $k \in \mathbb{N}$. It then follows from the linear connectivity of the set Z that there is a curve $\Gamma \subset Z$ containing the points $\{M_1, M_2, \dots, M_k, \dots, M_0\}$ and such that the restriction of G to this curve is discontinuous at the point M_0. The theorem is proved.

The definition of the blowup of the solution set allows proposing the following classification of the blowup phenomena for defining the procedure of completing the definition of the solution of the problem in the cases where the absence of a solution or its nonuniqueness follow from the initial statement of the problem. We note that the type of the equation under study has no importance in the proposed approach.

Definition 3. 1. A discontinuity point $(f_0, \mathbf{A}_0) \in Z$ of a map G is called a *point of removable blowup* (*with respect to the set* S) *of a solution of problem* (1) if the limit $\lim_{(f,\mathbf{A})\to(f_0,\mathbf{A}_0)} G(f, \mathbf{A})$ exists. Otherwise, the discontinuity point $(f_0, \mathbf{A}_0) \in Z$ of G is called a *point of nonremovable blowup* (*with respect to* S) *of a solution of problem* (1).

2. A point $(f_0, \mathbf{A}_0) \in Z$ of nonremovable blowup (with respect to S) of a solution of problem (1) is called a *point of pole-type blowup* if the limit $\lim_{(f,\mathbf{A})\to(f_0,\mathbf{A}_0)} \inf_{u\in G(f,\mathbf{A})} \|u\|_Y = +\infty$ exists.

3. A point $(f_0, \mathbf{A}_0) \in Z$ of nonremovable blowup (with respect to S) of a solution of problem (1) is called a *point of essential blowup* if it is not a point of pole-type blowup.

Remark 1. If the upper limit $\overline{\lim}_{(f,\mathbf{A})\to(f_0,\mathbf{A}_0)} \sup_{u\in G(f,\mathbf{A})} \|u\|_Y < +\infty$ is finite and the upper topological limit $\mathrm{Ls}_{(f,\mathbf{A})\to(f_0,\mathbf{A}_0)} G(f, \mathbf{A})$ (i.e., the set $M(f_0, \mathbf{A}_0) = \{y \in Y : \exists (f_k, \mathbf{A}_k) : (f_k, \mathbf{A}_k) \to (f_0, \mathbf{A})0),\ \exists y_k \in G(f_k, \mathbf{A}_k) : \lim_{k\to\infty} y_k = y\}$) consists of more than one point, then the point $(f_0, \mathbf{A}_0)$ is a point of essential blowup.

We now define a multivalued map that associates each problem $(f_0, \mathbf{A}_0) \in Z$ with the set of limit points in the space of solutions:

$$M(f_0, \mathbf{A}_0) = \mathrm{Ls}_{(f,\mathbf{A})\to(f_0,\mathbf{A}_0)} G(f, \mathbf{A}).$$

3. Examples of blowup phenomena for ordinary and partial differential equations

3.1. Examples and classification of solution blowups. We illustrate Definitions 1–3 with examples of ill-posed Cauchy problems for differential equations all of whose solutions do not admit violation of their existence or uniqueness on a finite interval.

Example 1 (Absence of blowup of the solution set)**.** We consider the Cauchy problem for the ordinary differential equation

$$y' - f(y) = g(t), \quad t \in (0, +\infty), \qquad y(0) = y_0, \quad y_0 \in \mathbb{R}, \qquad g \in C([0, T]), \tag{2}$$

whose solution on the interval $[0,T]$, $T>0$, is a function $y\in C^1([0,T])$ satisfying the differential equation and the initial condition. We assume that $f_0\in C^1(\mathbb{R})$ (where $C^1(\mathbb{R})$ is the Banach space of continuously differentiable functions on the real line $\mathbb{R}$ endowed with the standard norm), $y_0=y_0^*\in\mathbb{R}$, and $g\in C([0,T])$. Then there exists a number $\sigma>0$ such that Cauchy problem (2) has a unique solution in the linear space $C^1([0,T])$ for any $y_0\in O_\sigma(y_0^*)$, for any $g\in[0,T]$ and for any $f\in O_\sigma(f_0)$. In this case, if the sequence (g_n,y_{0n},f_n) of elements of the Banach space $C([0,T])\times\mathbb{R}\times C^1(\mathbb{R})$ converges to an element (g_n,y_n^*,f_n) as $n\to\infty$ and if the function y_n is a solution of problem (2) with the initial condition y_{0n}, the right-hand side g_n, and the function f_n for each value $n\in\mathbb{N}$, then the sequence $\{y_n\}$ uniformly converges as $n\to\infty$ on the interval $[0,T]$ to the solution y of problem (2) with the initial condition y_0^*, the right-hand side g, and the function f_0.

Hence, if we set $X=C([0,T])\times\mathbb{R}$ and $Y=C^1([0,T])$, then Cauchy problem (2) takes form (1) for $(g,y_0)\in X$ and $\mathbf{A}(u)=\mathbf{D}(u)-\mathbf{F}_f(u)$, where $\mathbf{D}(u)=u'$ and $\mathbf{F}_f(u)(t)=f(u(t))$, $t\in(0,T)$, $f\in C^1(\mathbb{R})$. We then take the set $S=\{((g,y_0),\mathbf{A})=((g,y_0),\mathbf{D}-\mathbf{F}_f),\ (g,y_0,f)\in C([0,T])\times\mathbb{R}\times C^1(\mathbb{R})\}$, introduce the topology on S generated by the norm of the Banach space $C([0,T])\times\mathbb{R}\times C^1(\mathbb{R})$, and consider the map $G\colon S\to 2^{C^1([0,T])}$ such that $G(g,y_0,f)=M_{g,y_0,f}$, where $M_{g,y_0,f}$ is the set of solutions of Cauchy problem (2). Then according to the classical theorem on the existence, uniqueness, and continuous dependence of the Cauchy problem solution on the problem data, $M_{g,y_0,f}$ is a one-point set, and G is a continuous map of the topological space S to the topological space 2^Y (to the Banach space Y because this is a one-point map).

Example 2 (Blowup of the removable type)**.** Let $f_0(y)=(y^2)^{1/3}\in C(\mathbb{R})$, $g\equiv 0$, and $y_0\le 0$. Then Cauchy problem (2) has a unique solution on the interval $[0,T)$ for $T\le T_*=3|y_0|^{1/3}$, but for each $T>T_*$, this problem has a one-parameter continuous family of solutions, and the set of branch points of the integral curves of the solutions is the interval $\{(t,y)=(t,0)\colon t\in[T_*,T)\}$. The time T_* is the instant of violation of the uniqueness of the Cauchy problem solution. For Cauchy problem (2) on the half-line $\mathbb{R}_+$, the set of solutions is continuous and unbounded in the space $C^1(\mathbb{R})$. But this property of the Cauchy problem is unstable under small perturbations (in the space $C^1(\mathbb{R})$) of the function f_0 in a neighborhood of the function $f_0\in C(\mathbb{R})$. For example, we consider a perturbation of f_0 along the curve $\gamma=\{f_\varepsilon=f_0+\varepsilon^{2/3},\ \varepsilon\in(-1,1)\}$ in the space $C(\mathbb{R})$. For any function $f_\varepsilon\in\gamma$, $\varepsilon\ne 0$, Eq. (1) has a unique solution $y_\varepsilon\in C([0,+\infty))\cap C^1(0,+\infty)$, and the function $y_\varepsilon(x)$, $x\in\mathbb{R}$, is uniquely defined by the expression $y_\varepsilon^{1/3}-\varepsilon\arctan(y_\varepsilon^{1/3}/\varepsilon)=x/3+C(y_0,\varepsilon)$. Therefore, the family y_ε uniformly converges on any interval $[0,T]$ to the function $u=(x/3+y_0^{1/3})^3$ defined on the interval $[0,+\infty)$.

We set $X=C(\mathbb{R})\times\mathbb{R}$ and $Y=C([0,T))\cap C^1((0,T))$, and Cauchy problem (2) then takes form (1) for $(g,y_0)=(0,0)\in X$ and $\mathbf{A}(u)=\mathbf{D}(u)-\mathbf{F}_f(u)$, where $\mathbf{D}(u)=u'$ and $\mathbf{F}_f(u)(t)=f(u(t))$, $t\in(0,T)$, $f\in C(\mathbb{R})$. We consider the curve $\Gamma=\{((g,y_0),\mathbf{A}_\varepsilon)=((0,0),\mathbf{D}-\mathbf{F}_{f_\varepsilon}),\ \varepsilon\in(-1,1)\}$ in the space of Cauchy problems. The map $G\colon\Gamma\to 2^Y$ associating the element $f_\varepsilon\in\gamma$ with the set M_f of solutions of Cauchy problem (2) then has a point of removable discontinuity f_0 because the limit $\lim_{\varepsilon\to 0}M_{f_\varepsilon}=\{u\}$, which does not coincide with the continual set M_{f_0}, exists.

Example 3 (Pole-type blowup in one statement of the problem and a removable blowup in another statement)**.** The Cauchy problem for the Hopf equation has the form

$$u'_t-uu'_x=f,\quad (t,x)\in(0,T)\times\mathbb{R},\qquad u|_{t=+0}=u_0. \tag{3}$$

Here, $T\in(0,+\infty)$, $(f,u_0)\in C((0,T)\times\mathbb{R})\times C^1(\mathbb{R})\equiv X$, and the space $C(G)$ is the Banach space of continuous numerical functions defined on the domain of G in a finite-dimensional Euclidean space where the norm of its elements is defined as $\|u\|_{C(G)}=\sup_{x\in G}|u(x)|$. We set $f=\theta\equiv 0$ and fix an operator

$\mathbf{A}u = (u_t' - uu_x', \lim_{t\to+0} u)$ acting from a Banach space Y into a Banach space X. The properties of the map G depend on the choice of the space Y.

A. If $Y_1 = C^1((0,T)\times\mathbb{R}) \cap C([0,T], C^1(\mathbb{R}))$, then the set $M_{(0,u_0),\mathbf{A}}$ of solutions of Cauchy problem (3) can be empty for some $u_0 \in C^1(\mathbb{R})$ (see [10], [12]).

B. If $Y_2 = L_{1,\mathrm{loc}}((0,T)\times\mathbb{R})$, then $M_{(0,u_0),\mathbf{A}}$ is a continuous set for any $u_0 \in C^1(\mathbb{R})$ (see O. A. Oleinik's examples in [10]).

In terms of the proposed classification, any problem in Example 1 does not exhibit the blowup property, the problem considered in Example 2 exhibits the property of a blowup removable along the curve (considered in the example) in the space of initial data, the Cauchy problem in Example 3 with the solution space Y_1 exhibits the property of a nonremovable pole-type blowup, and the Cauchy problem in Example 3 with the solution space Y_2 exhibits the property of a blowup removable along the set of regularizations by uniformly elliptic linear second-order differential operators.

To consider Example 3 in more detail, we fix a certain finite function $u_0 \in C^1(\mathbb{R})$. In the space of Cauchy problems, we choose a curve $\Gamma = \{((0,u_0), \mathbf{A}_\epsilon),\ \epsilon \in [0,1)\}$, where $\mathbf{A}_\epsilon u = u_t' - uu_x' - \epsilon u_{xx}''$. For each $\varepsilon \in (0,1)$, the set $M_{(0,u_0),\mathbf{A}_\varepsilon}$ consists of one point $u_\varepsilon = \mathbf{A}_\varepsilon^{-1}((0,u_0)) \in Y_1 \cap Y_2$. Then we have the following assertions:

1. *The point* $((0,u_0),\mathbf{A}_0)$ *is a point of removable discontinuity for the map* $G\colon \Gamma \to 2^{Y_2}$ *because the sequence* $\{u_\varepsilon\}$, $\varepsilon \in (0,1)$, $\varepsilon \to +0$, *has a limit in the space* Y_2 *as* $\varepsilon \to +0$. Indeed, according to the results obtained in [12], the sequence converges in the topological vector space $L_{1,\mathrm{loc}}((0,T)\times\mathbb{R})$.

2. *The point* $((0,u_0),\mathbf{A}_0)$ *is a point of nonremovable pole-type discontinuity for the map* $G\colon \Gamma \to 2^{Y_1}$ (because the sequence $\{u_\varepsilon\}$ diverges in the space Y_1 and $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{Y_1} = +\infty$; see [10], [12]).

The effects of a nonremovable pole-type blowup of solutions of nonlinear partial differential equations, similar to the effect considered in Example 3 in the space Y_1, were studied in [12], [29]–[32] and in several other works.

There are several examples of Cauchy problems for differential equations where solution destruction (similar to the cases considered in Example 3A; see [11], [29], [32]) or a solution uniqueness violation (similar to the cases considered in Examples 2 and 3B; see [11], [33], [34]) can occur, and these examples illustrate the blowup phenomenon in the sense of Definition 1.

An efficient method for studying such problems is the vanishing viscosity method, as well as similar approximation methods (see [11], [28], [32], [34]–[36]) based on the removability of a discontinuity point of the restriction of G to a special set S (depending on the regularization method) in Definition 2.

3.2. Applications of the blowup phenomena classification to determining the solution extension procedure. The classification of the blowup phenomena for the solution set of differential equations permits determining the following three situations that arise in studying initial boundary-value problems whose solutions admit a blowup.

1. If a problem $z_0 \in Z$ is a point of removable discontinuity of the map G, then G is defined at z_0 by continuity. In several problems, such a redefinition of G allows extending the solution uniquely through the blowup origination instant (see Examples 1 and 3B), and this is the goal in the regularization method (see [10], [12], [28], [34], [35]).

2. If a problem $z_0 \in Z$ is a point of essential discontinuity of the map G, then the extension of the solution through the blowup origination instant is defined as a random process whose values are in the set of limit values of G as $z \to z_0$. In this case, the dynamical transformation of the space of the initial

data of the problem looses its uniqueness and invertibility properties and becomes a random map. Such an approach demonstrates a relation between the phenomena of randomness, irreversibility, and nonuniqueness.

3. If a problem $z_0 \in Z$ is a point of pole-type discontinuity of the map G, then G in some cases has an infinitely remote point as a unique limit and hence can be extended uniquely through the blowup origination instant as a function whose values can be infinite (see [37]). But the values of G in the general case tend to infinity as $z \to z_0$ along different curves in the infinite-dimensional space Y; it is then also difficult to define the dynamical transformation as a random process because its values do not belong to the spaces related to the problem statement. This indicates that it is expedient to enlarge the functional spaces for extending the solutions through the blowup origination instant. In this case, to extend the solution through the blowup origination instant, a method is proposed (see [10], [12]) in which the Banach spaces X and Y and the topological space $B(Y, X)$ are chosen such that the values of G remain bounded in a neighborhood of z_0. In the case of such a choice of topologies and norms, the pole-type point z_0 can become a point of removable blowup (see [10], [12], [35]) or a point of essential blowup (see [11], [14], [34]).

We further continue to study random dynamical transformations arising in the case of an essential blowup of the set of solutions of initial boundary-value problems. The case of a removable blowup is a rather regular case and is related to studying well-posed initial boundary-value problems. But the case of pole-type blowup requires certain changes in the problem statement such that the character of the blowup singularity becomes either removable or essential.

The following example shows how the blowup phenomenon in the Cauchy problems posed for the same equation can exhibit any of the three types of blowup depending on the choice of the space Y of solutions.

Example 4 (Degenerate maximal symmetric operator)**.** We assume that in the Hilbert space $H = L_2(\mathbb{R})$, a degenerate second-order differential operator $\mathbf{L}$ is given by the differential expression

$$\mathbf{L}v(x) = \frac{\partial}{\partial x}\left(g(x)\frac{\partial}{\partial x}v(x)\right) + \frac{i}{2}\left[a(x)\frac{\partial}{\partial x}v(x) + \frac{\partial}{\partial x}(a(x)v(x))\right], \quad x \in \mathbb{R},$$

where the coefficients $g = \chi_{(-\infty,0)}$ and $a\chi_{(0,+\infty}$ are indicator functions of half-lines. In this case, the operator $\mathbf{L}$ defined by the differential expression on the domain

$$D(\mathbf{L}) = \left\{u \in W_2^1(\mathbb{R})\colon u|_{(-\infty,0)} \in W_2^2(-\infty,0),\ u'(-0) = \frac{i}{2}u(0)\right\}$$

is closed and maximal symmetric, while the adjoint operator has a wider domain

$$D(\mathbf{L}^*) = \Big\{u \in L_2(\mathbb{R})\colon u_{(0,+\infty)} \in W_2^1(0,+\infty),$$
$$u|_{(-\infty,0)} \in W_2^2(-\infty,0),\ u'(-0) = \frac{i}{2}u(0)\Big\}.$$

The maximal symmetric operator $\mathbf{L}$ was considered in [11] as an example of the Schrödinger operator degenerate on a half-line. Such an operator is not a generator of the semigroup $e^{-it\mathbf{L}}$, $t \geq 0$, in the space H, but its adjoint is a generator of the contracting semigroup $e^{-it\mathbf{L}^*}$, $t \geq 0$, in H (see [11]).

A regularization of such an operator $\mathbf{L}$ is a one-parameter family of self-adjoint operators given by the differential expression

$$\mathbf{L}_\varepsilon v(x) = \frac{\partial}{\partial x}\left(g_\varepsilon(x)\frac{\partial}{\partial x}v(x)\right) + \frac{i}{2}\left[a(x)\frac{\partial}{\partial x}v(x) + \frac{\partial}{\partial x}(a(x)v(x))\right],$$

where $g_\varepsilon(x) = g(x) + \varepsilon$, $x \in \mathbb{R}$, $\varepsilon \in E = (0, 1]$, on the domain

$$D(\mathbf{L}) = \Big\{ u \in W_2^1(\mathbb{R})\colon u|_{(-\infty,0)} \in W_2^2(-\infty, 0),$$

$$u|_{(0,+\infty)} \in W_2^2(0, +\infty),\ (1+\varepsilon)u'(-0) = \varepsilon u'(+0) + \frac{i}{2}u(0) \Big\}.$$

The self-adjoint operators $\mathbf{L}_\varepsilon$ are generators of the unitary groups $e^{-it\mathbf{L}_\varepsilon}$, $t \geq 0$, for $\varepsilon \in (0, 1]$. Then, as shown in [11], [32], the Cauchy problem

$$i\frac{d}{dt}u(t) = \mathbf{L}u(t), \quad t > 0, \qquad u(+0) = u_0, \tag{4}$$

admits a blowup phenomenon along the one-parameter family of Cauchy problems

$$i\frac{d}{dt}u(t) = \mathbf{L}_\varepsilon u(t), \quad t > 0, \quad \varepsilon \in [0, 1], \qquad u(+0) = u_0,$$

for all u_0 in the infinite-dimensional subspace $H_1 = \bigcup_{t \geq 0} \operatorname{Ker}(e^{-it\mathbf{L}^*}) \subset L_2(\mathbb{R})$.

We note that if as the solution space Y, we choose the space $C_w([0, +\infty), H)$ of weakly continuous maps of the half-line $[0, +\infty)$ to the space H endowed with the topology of weak convergence in H and if this convergence is uniform on each interval $[0, T]$, $T > 0$, then the Cauchy problem admits a removable blowup phenomenon, and the generalized sequence $\{e^{-it\mathbf{L}_\varepsilon}u_0,\ \varepsilon \to +0\}$ of solutions of the regularized problems converges in $C_w([0, +\infty), H)$ to the function $e^{-it\mathbf{L}^*}u_0$, $t \geq 0$.

If as the solution space Y, we choose the space $C([0, +\infty), H)$ of continuous maps of the half-line $[0, +\infty)$ to the space H endowed with the topology of convergence in the space H and if this convergence is uniform on each interval $[0, T]$, $T > 0$, then the Cauchy problem admits an essential blowup phenomenon, and the generalized sequence $\{e^{-it\mathbf{L}_\varepsilon}u_0, \varepsilon \to +0\}$ of solutions of the regularized problems does not contain subsequences converging in $C([0, +\infty), H)$ but remains bounded in this Banach space.

If the condition $u_0 \in W_2^1(\mathbb{R})$ is satisfied and if as the solution space Y, we choose the space $C([0, +\infty), W_2^1(\mathbb{R}))$ of continuous maps of the half-line $[0, +\infty)$ to the space $W_2^1(\mathbb{R})$ endowed with the topology of convergence in the space $W_2^1(\mathbb{R})$ and if this convergence is uniform on each interval $[0, T]$, $T > 0$, then the Cauchy problem admits a pole-type blowup phenomenon, and the generalized sequence $\{e^{-it\mathbf{L}_\varepsilon}u_0, \varepsilon \to +0\}$ of solutions of the regularized problems satisfies the condition

$$\lim_{\varepsilon \to +0} \|e^{-it\mathbf{L}_\varepsilon}u_0\|_{C([0,+\infty), W_2^1(\mathbb{R}))} = +\infty.$$

The Cauchy problems for differential equation (4) can hence exhibit all possible types of blowup of the solutions depending on the choice of the space where the problem for (4) is posed.

Example 5 (Essential blowup and a multivalued map)**.** In several cases, the Cauchy problem $(f, \mathbf{A})$ as a point in the domain of the resolving map G is an essential singular point of G with a set of partial limits $Ls_{(f,A)\to(f_0,A_0)}G(f, A)$ at this point (see [11], [34], [35]). In this case, the Cauchy problem $(f, \mathbf{A})$ is associated not with a single solution but with either the set $\mathrm{Ls}_{(f,A)\to(f_0,A_0)}G(f, A)$ or (if the set of partial limits is endowed with the structure of a measurable space with a measure) a random variable whose range coincides with the set $M(f_0, \mathbf{A}_0) = \mathrm{Ls}_{(f,A)\to(f_0,A_0)}G(f, A)$.

We now consider an example of a problem admitting an essential blowup phenomenon. We recall that a set $\Gamma \subset H \times H$ is called a strong (or weak) graph limit of a sequence $\{\mathbf{L}_n\}$ ranging in the set $\mathrm{Cl}(H)$ of closed linear operators in a Hilbert space H, $(u, v) \in \Gamma$, if and only if there is a sequence $\{u_n\}$ of elements of H such that $u_n \in D(\mathbf{L}_n)$ for each $n \in \mathbb{N}$ and $\|u_n - u\| \to 0$, and $\|\mathbf{L}_n u_n - v\| \to 0$ as $n \to \infty$ (or the sequence $\{\mathbf{L}_n u_n\}$ weakly converges in H to a vector v as $n \to \infty$).

For each closed operator $\mathbf{A}$ in the Hilbert space H, we define the function on the set $\mathrm{Cl}(H)$ of closed operators

$$\delta_{\mathbf{A}}(\mathbf{L}) = \sup_{u \in \Gamma_{\mathbf{A}},\, \|u\|=1} \mathrm{dist}(u, \Gamma_{\mathbf{L}}),$$

where $\Gamma_{\mathbf{A}}$ and $\Gamma_{\mathbf{L}}$ are closed subspaces of $\mathcal{H} = H \times H$ that are graphs of the operators $\mathbf{A}$ and $\mathbf{L}$ (see [38]). For each $\varepsilon > 0$, we call the set of operators $\mathbf{L} \in \mathrm{Cl}(H)$ such that $\delta_{\mathbf{A}}(\mathbf{L}) < \varepsilon$ an ε-neighborhood of $\mathbf{A} \in \mathrm{Cl}(H)$. We let τ_Γ denote the least topology in $\mathrm{Cl}(H)$ that contains all such neighborhoods.

The sequence $\{\mathbf{L}_n\}$ of self-adjoint operators in a Hilbert space H is called a self-adjoint regularization of a symmetric operator $\mathbf{L}$ if the strong graph limit of this sequence contains the graph of the operator $\mathbf{L}$ (each of such sequences converges to $\mathbf{L}$ in the topology τ_Γ).

Theorem 2. *If an operator* $\mathbf{L}$ *is a symmetric operator in a Hilbert space* H *with finite indices* (n_-, n_+): $n_- = n_+ = n \in \mathbb{N}$, *then for any self-adjoint regularization* $\{\mathbf{L}_n\}$ *of* $\mathbf{L}$ *in the topology* τ_Γ, *the condition*

$$G(u_0, \mathbf{L}) = \mathrm{Ls}_{\mathbf{L}_\epsilon \to \mathbf{L}}(e^{-i\mathbf{L}_\epsilon t} u_0) = \bigcup_{L_\sigma \in \Sigma(\mathbf{L})} e^{-i\mathbf{L}_\sigma t} u_0$$

is satisfied, where $\Sigma(\mathbf{L})$ *is the set of self-adjoint extensions of* $\mathbf{L}$.

The assertion of this theorem follows from Theorem 9.5 (also see Corollary 9.5) in [11]. Theorem 2 provides a sufficient condition for Cauchy problem (4) to be a discontinuity point of essential type for the resolving map.

4. Averaging of approximating regularizations and extension of the solution

The notion of a statistical solution was proposed (see [39], [40]) for problems for differential equations such that the uniqueness of their solution was either absent or not established. In this case, instead of the Cauchy problem for a function defined on an interval and ranging in a Banach space H of the problem, it was proposed to consider the Cauchy problem for a function defined on an interval and ranging in the space of measures on H or to consider a problem for the measure on the space of maps of an interval to H. We use a different approach where the Cauchy problem itself is considered as a random variable ranging in a space $B(Y, X)$. In some cases, we can average random solutions and define the solution extension in terms of their mean value. The method for averaging random variables that range a set of one-parameter semigroups was developed in [20]. To apply this method to differential equations with an essential blowup of their solutions, we first repeat necessary definitions and assertions given in that paper.

4.1. Random solutions and sets of solutions. Let E be a topological space and 2^E be the algebra of all subsets of E. For each point $\varepsilon_0 \in E$, we let $W(E, \varepsilon_0)$ denote the set of all nonnegative normalized finite-additive measures on a measurable space $(E, 2^E)$ that are concentrated in an arbitrary neighborhood of ε_0 in the sense that for any $\mu \in W(E, \varepsilon_0)$, the equality $\mu(A) = 0$ holds for any set $A \subset E$ for which ε_0 is not a limit point and in addition $\mu(\{\varepsilon_0\}) = 0$.

We generalize the definition of a random variable as follows. A random variable is a measurable map of a space with a finite-additive nonnegative normalized measure $(\Omega, \mathcal{F}, \mu)$ into a measurable space $(T, \mathcal{A})$

(i.e., the preimage of any element of the algebra $\mathcal{A}$ is an element of the algebra $\mathcal{F}$, in which case $\mathcal{A}$ is not a sigma-algebra and the measure μ is not sigma-additive). Speaking about a random variable ranging in a topological space (T, τ), we assume that the space T is equipped with an algebra of subsets $\mathcal{A}_\tau$ generated by the topology τ and that this is the least algebra with the topology τ (i.e., contains all open sets of (T, τ), in which case it is possible that not all Borel sets enter $\mathcal{A}_\tau$ because it is not a sigma-algebra).

Let 2^Z be the sigma-algebra of all subsets of the topological space of problems Z. Let $W(Z)$ be the set of nonnegative normalized finite-additive measures on the measurable space $(Z, 2^Z)$. For each $z_0 \in Z$, we let $W(Z, z_0)$ denote the set of measures $\mu \in W(Z)$ such that $\mu(\{z_0\}) = 0$ and $\mu(A) = 0$ for any set $A \in 2^Z$ from which the point z_0 is isolated in the topology τ_Z.

In the topological space $(2^Y, \tau_H)$, we define the algebra $\mathcal{A}_Y$ as the least algebra containing all ε-neighborhoods of all points of the space 2^Y in the pseudometric ρ_H.

Definition 4. The *random set of solutions* of a problem $z_0 \in Z$ is a random variable G defined on the measurable space with a measure $(Z, 2^Z, \mu)$ such that $\mu \in W(Z, z_0)$ and ranging in the measurable space $(2^Y, \mathcal{A}_Y)$.

This definition is a generalization of the traditional definition of the solution of problem (1) (as a partial preimage of the vector f under the map $\mathbf{A}$) in the following sense.

Theorem 3. *If the map G is continuous at a point $z_0 \in Z$ and $G(z_0) = y_0 \in Y$, then*

$$\mu(\{z \in Z \colon \rho_H(G(z),\, \{y_0\}) > \varepsilon\}) = 0$$

for any $\mu \in W(Z, z_0)$ and any $\varepsilon > 0$.

Proof. Because the map G is continuous at z_0, the preimage of any ε-neighborhood of the point y_0 in the space 2^Y is in the topology τ_Z and contains z_0 and a limit point in the topology τ_Z of the set $G^{-1}(O_\varepsilon(\{y_0\}))$. In this case, the preimage of any neighborhood $O_\varepsilon(A)$, $\varepsilon > 0$, $A \in 2^Y$, in the space 2^Y is measurable in the space $(Z, 2^Z)$, and the measure of the preimage of any ε-neighborhood of the point $A \in 2^Y$ is equal to unity if this neighborhood contains the point $\{y_0\}$ or to zero if this neighborhood is isolated from y_0. Therefore, by the definition of the set $W(Z, z_0)$, the measure of any such preimages of the ε-neighborhood of $\{y_0\} \in 2^Y$ is equal to unity. The theorem is proved.

Remark 2. Thus, a random variable G defined on a measurable space with a measure $(Z, 2^Z, \mu)$ and ranging in the measurable space $(2^Y, \mathcal{A}_Y)$ takes values with probability one in an arbitrary ε-neighborhood of the point y_0. In this sense, the definition introduced above is a generalization of the notion of a solution of the problem $z_0 \in Z$.

But if the initial problem $z_0 \in Z$ has no solution (see Examples 3 and 4), then the role of the solution is played by a random set of solutions with a measure chosen in the class $W(Z, z_0)$.

In what follows, we study random sets of solutions and their averaging in the following situation in detail. Let $z_0 \in Z$ be a problem of the evolution equation and $E \subset Z$ be a set of problems such that $z_0 \in E$ and the problem z determines a strongly continuous semigroup of transformations of the Hilbert space H for each $z \in E \setminus z_0$ (see Example 4). In this case, if the support of the measure μ is in the set E, then the random set of solutions G becomes a random solution, and it is meaningful to consider a random variable $\xi \colon E \to C(\mathbb{R}_+, B(H))$ ranging in the set of semigroups of bounded transformations of the space H (see Example 4). The random semigroups and their expectation values are studied in Sec. 4.2.

4.2. Random semigroups, their expectation values, and iterations. Let Y be a Banach space and $\Lambda_s = C_s(\mathbb{R}_+, B(Y))$ be a topological vector space of strongly continuous operator-valued maps of the half-line $\mathbb{R}_+ = [0, +\infty)$ to the Banach space $B(Y)$ of bounded linear operators defined everywhere in Y. On the space Λ_s, we define a family of functionals acting on an arbitrary element $z \in \Lambda$ by the rule $\Phi_{t,A,g}(z) = \sup_{s\in[0,t]} \|z(t)A\|$, $t \in \mathbb{R}_+$, $A \in Y$. On the space Λ_s, we also consider the topology τ_s generated by the family of functionals $\Phi_{t,A}$, $t \in \mathbb{R}_+$, $A \in Y$.

Definition 5. A *random semigroup* is a measurable map ξ of a space with a measure to a linear topological space $C_s(\mathbb{R}_+, B(Y))$ whose values are one-parameter semigroups.

Definition 6. The *expectation value of a random semigroup* ξ is an operator-valued function of a parameter of the semigroup whose value at each point t is equal to the Pettice integral over the measure μ of the map $E \to B(Y)$ acting by the rule $\varepsilon \to \xi_\varepsilon(t)$, $\varepsilon \in E$:

$$\mathbf{F}_\mu(t) = \mathbf{M}[\xi](t) \equiv \int_E \xi_\varepsilon(t)\, d\mu. \tag{5}$$

In the study of the Cauchy problem z_0 for an evolution differential equation that can be represented in form (1) and admits an essential blowup phenomenon, the role of the set E is played by a deleted neighborhood in the topological space S of Cauchy problems for which the problem z_0 is a point of the closure and the Cauchy problem z has a unique solution Y for each $z \in S \setminus z_0$. If the measure μ on the measurable space $(S, 2^S)$ belongs to the set $W(S, z_0)$, then the random variable $\xi\colon E \to Y$ is an extension of the solution of the problem z_0 through the blowup origination instant in the sense of Definition 4.

As the set E, we take the set $G(X)$ of all generators of strongly continuous semigroups acting in the space X; this set is endowed with the topology τ_s on the set of strongly continuous semigroups generated by these generators (see Sec. 9 in [11]). We consider a random variable defined on the measurable space $(E, 2^E, \mu)$ with a measure $\mu \in W(E)$ and ranging in the space $\Lambda_w = C_w(\mathbb{R}_+, B(X))$ of weakly continuous maps of the half-line $\mathbb{R}_+$ to the Banach space $B(X)$ of bounded linear transformations of the Banach space X dual to the X_*. On the space Λ_w, we introduce a family of functionals acting on an arbitrary element $z \in \Lambda_w$ by the rule $\varphi_{t,A,g}(z) = \langle z(t)A, g\rangle$, $t \in \mathbb{R}_+$, $A \in X$, $g \in X_*$, and define the topology τ_w generated by the family of functionals $\varphi_{t,A,g}$, $t \in \mathbb{R}_+$, $A \in X$, $g \in X_*$. Then the space Λ_w endowed with the structure of the minimal algebra $\mathcal{A}_w$ of subsets, which contains the topology τ_w, is a measurable space, and the map $\xi\colon E \to \Lambda_w$ is a random variable. The expectation value of the random variable $\xi\colon E \to \Lambda_w$ is called the Pettice integral, which is given by (5), i.e., an element $M\xi \in \Lambda_w$ such that

$$\langle M\xi(t)A, g\rangle = \int_E \langle \xi_\varepsilon(t)A, g\rangle\, d\mu(\varepsilon) \tag{6}$$

for any $t \in \mathbb{R}_+$, $A \in X$, $g \in X_*$. If the values of the map ξ are uniformly bounded in (ε, t) on the set $E \times \mathbb{R}_+$, then the integral is the Radon integral of a bounded numerical function over a finite-additive measure. In this case, it is well defined, and relation (6) defines a bounded linear transformation $M\xi(t) \in B(X)$ for each $t \in \mathbb{R}_+$.

Theorem 4. *If there is a linear subspace D dense in X and such that the family of maps $\xi_\varepsilon(t)A \in C(\mathbb{R}_+, X)$, $\varepsilon \in E$, is weakly (or strongly) uniformly Lipschitzian for each $A \in D$ and the family of maps ξ is uniformly bounded, then $M\xi(t) \in C_w(\mathbb{R}_+, B(X))$ (or $M\xi(t) \in C_s(\mathbb{R}_+, B(X))$).*

The proof of Theorem 4 was published in [20].

Remark 3. The expectation value of a random semigroup can be not a semigroup. A trivial example is given by averaging of the set of two unitary semigroups e^{-it}, $t \geq 0$, and e^{it}, $t \geq 0$, acting in a one-dimensional Hilbert space $\mathbf{C}$. The half-sum of these two semigroups is a one-parameter family $F(t)$, $t \geq 0$, of transformations of the space $\mathbf{C}$ acting as the operator of multiplication by the number $\cos t$. But because $\cos(t+s) \neq (\cos t)(\cos s)$, the family of averaged transformations $F(t)$, $t \geq 0$, is not a semigroup.

4.3. Averaging of random semigroups by Chernoff's iterations. Following [41], we use the equivalence relation for operator-valued functions on the space Π of strongly continuous operator-valued functions $\mathbf{F}\colon [0,+\infty) \to B(X)$ such that $\mathbf{F}(0) = \mathbf{I}$.

Definition 7 [20]**.** We say that operator-valued functions $\mathbf{F}$ and $\mathbf{G}$ acting from a closed right half-neighborhood of zero on the number axis into the Banach space $B(X)$ of bounded linear operators acting in a Banach space X are *equivalent in the Chernoff sense* if the condition

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| \left(\left(\mathbf{G}\left(\frac{t}{n}\right)\right)^n - \left(\mathbf{F}\left(\frac{t}{n}\right)\right)^n \right) u \right\| = 0$$

is satisfied for each $T > 0$ and each $u \in X$.

Theorem 5 [20]**.** *Let $\{\mathbf{L}_n\}$ be a sequence of generators of strongly continuous contracting semigroups in a Banach space X. Let $\{\mu_n\}$ be a sequence of nonnegative numbers such that the sum of the series composed of these numbers is equal to unity. We assume that there is a linear subspace $D \subset H$ that is an essential domain of each of the operators $\mathbf{L}_n$, $n \in \mathbb{N}$, and satisfies the condition that the number series $\sum_{n=1}^{\infty} \mu_n \|\mathbf{L}_n x\|$ converges for any $x \in D$. Then, if the operator $\mathbf{S}$ defined on the space D by the expression $\mathbf{S}x = \sum_{k=1}^{\infty} \mu_k \mathbf{L}_k x$ can be closed and if its closure is a generator of the strongly continuous semigroup $e^{t\mathbf{S}}$, $t \geq 0$, then the expectation value of the random semigroup $\mathbf{F}_\mu(t) = \sum_{n\in\mathbb{N}} e^{t\mathbf{L}_n}\mu_n$, $t \geq 0$, is equivalent in the Chernoff sense to the semigroup $e^{t\mathbf{S}}$, $t \in \mathbb{R}_+$.*

Corollary 1 [20]**.** *If $\{\mathbf{L}_j\}$ is a uniformly bounded sequence of operators in the space $B(X)$, then Theorem 5 holds.*

Therefore, by Corollary 1, the operation of averaging of generators of semigroups, defined by Theorem 5, is a generalization of the operation of averaging of bounded linear operators in the Banach space $B(X)$.

Definition 8. We say that the generator of a strongly continuous semigroup $\mathbf{L}$ is the integral of the function $\mathbf{L}(\varepsilon)$, $\varepsilon \in E$, ranging in the set of generators of strongly continuous semigroups, over a nonnegative normalized measure μ on the algebra 2^E in the exponential sense if the operator-valued function $\mathbf{G}(t) = \int_E e^{t\mathbf{L}(\varepsilon)}\, d\mu(\varepsilon)$ (where the integral is understood in the Pettice sense), $t \geq 0$, is equivalent in the Chernoff sense to the semigroup $e^{t\mathbf{L}}$.

We say that the generator of a strongly continuous semigroup $\mathbf{L}$ is the sum of generators of strongly continuous semigroups $p_1\mathbf{L}_1$ and $p_2\mathbf{L}_2$ for arbitrary $p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 = 1$, in the exponential sense if the operator-valued function $\mathbf{G}(t) = p_1 e^{t\mathbf{L}_1} + p_2 e^{t\mathbf{L}_2}$, $t \geq 0$, is equivalent in the Chernoff sense to the semigroup $e^{t\mathbf{L}}$.

The sum of operators is commutative, and the zero operator is the zero element. The associativity of the operation of summation must be studied; Corollary 1 gives sufficient conditions for the associativity $p_1\mathbf{L}_1 + (p_2\mathbf{L}_2 + p_3\mathbf{L}_3) = (p_1\mathbf{L}_1 + p_2\mathbf{L}_2) + p_3\mathbf{L}_3$.

Corollary 2 [20]**.** *If the sequence of iterations* $\mathbf{G}_n(t) = [\mathbf{F}_\mu(t/n)]^n$*,* $n \in \mathbb{N}$*, converges to the expectation value* $\mathbf{F}_\mu$ *of a random semigroup* $\mathbf{U}_\varepsilon$*,* $\varepsilon \in E$*, uniformly on each interval of the half-line* $\mathbb{R}_+$*, then the expectation value of the random semigroup* $\mathbf{F}_\mu(\,\cdot\,)$ *is equivalent in the Chernoff sense to the semigroup* $\mathbf{V}(\,\cdot\,)$ *whose generator is a generalized exponential average of the generators* $\mathbf{L}_\varepsilon$*,* $\varepsilon \in E$*, of the random semigroup.*

4.4. Examples of applying random semigroup averaging to the approximation of initial boundary-value problems admitting an essential blowup. Regarding a Cauchy problem as a point z_0 in the topological vector space Z and choosing a measure on the topological space Z allows associating a Cauchy problem admitting an essential blowup of the solution set with the following objects:

1. a random transformation of the space of initial data,
2. an averaged transformation of the space of initial data,
3. the Feynman–Chernoff iterations of an averaged transformation, and
4. the limit (or set of limit points) of the Feynman–Chernoff iterations of an averaged transformation.

For example, the choice of a measure on the space of approximating operators in the class $W(E, 0)$ (see Sec. 4.1) associates the Cauchy problem for the Schrödinger equation with degeneration on one of the half-lines (see Example 4) with a random solution such that its expectation value is a solution of the Cauchy problem with the adjoint operator. The choice of a measure on the space of approximating operators in the class $W(E)$ associates the Cauchy problem for the Schrödinger equation on two half-lines with a random solution whose expectation value is a solution of the Cauchy problem for the Schrödinger equation with a generating operator coinciding with one of the self-adjoint extensions of the operator $\mathbf{L}$.

A Cauchy problem admitting an essential blowup of solutions thus generates a random transformation of the space of initial data that can be averaged according to Definition 6 and whose generating operator can be averaged according to Definition 8, and the result of averaging can be illustrated by Example 6 given below.

We consider a generalized sequence of Hamiltonians $\mathbf{L}_\varepsilon$, $\varepsilon \to +0$, converging in the topology of strong graph-convergence to the limit Hamiltonian $\mathbf{L}$ (such a sequence was considered in Example 4; also see [11], [32]).

Example 6. Let $\mathbf{L}$ be the maximal symmetric operator in a Hilbert space H. We assume that a nonnegative normalized pure finite-additive measure μ is chosen on the set $\mathbb{N}$ and a sequence of self-adjoint operators $\{\mathbf{L}_n\}$ satisfies the condition that its strong graph limit Γ contains the graph $\Gamma_{\mathbf{L}}$ of the operator $\mathbf{L}$ (as in Example 4). In this case, if the indices (n_-, n_+) of the operator $\mathbf{L}$ satisfy the condition $n_+ = 0$, then the sequence of semigroups $e^{-it\mathbf{L}_n}$, $t \geq 0$, converges in the strong operator topology to an isometric semigroup $e^{-it\mathbf{L}}$, $t \geq 0$, uniformly on each interval. We then have

$$e^{-it\mathbf{L}} = \int_{\mathbb{N}} e^{-it\mathbf{L}_n}\, d\mu(n), \quad t \geq 0.$$

And if $n_- = 0$ (as in Example 4), then the sequence of semigroups $e^{-it\mathbf{L}_n}$, $t \geq 0$, converges in the weak operator topology to a contracting semigroup $e^{-it\mathbf{L}^*}$, $t \geq 0$, uniformly on each interval. We then have

$$e^{-it\mathbf{L}^*} = \int_{\mathbb{N}} e^{-it\mathbf{L}_n}\, d\mu(n), \quad t \geq 0.$$

Therefore, for any choice of a nonnegative normalized pure finite-additive measure μ on the set $\mathbb{N}$, the equalities $\mathbf{L} = \int_{\mathbb{N}} \mathbf{L}_n\, d\mu(n)$ if $n_+ = 0$ and $\mathbf{L}^* = \int_{\mathbb{N}} \mathbf{L}_n\, d\mu(n)$ if $n_+ > 0$ hold in the sense of Definition 8.

Example 6 also shows that the type of the blowup of solutions of the Cauchy problem for Eq. (4) depends on the spectral properties of the generating operator of the Cauchy problem $\mathbf{L}$.

Remark 4. If a sequence of self-adjoint operators $\{\mathbf{L}_n\}$ satisfies the condition that its strong graph limit Γ contains the graph $\Gamma_{\mathbf{L}}$ of the maximal symmetric operator $\mathbf{L}$ or coincides with it, then this does not mean that the common domain $D = \bigcap_{n\in\mathbb{N}} D(\mathbf{L}_n)$ is dense in the space H and the manifold D can even be trivial.

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