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# PHASE SPACE OF A GRAVITATING PARTICLE AND DIMENSIONAL REDUCTION AT THE PLANCK SCALE

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*Several approaches to quantizing general relativity suggest that quantum gravity at very short distances behaves effectively as a two-dimensional theory. The mechanism of this dimensional reduction is not yet understood. We attempt to explain it by studying the phase space of a test particle coupled to a gravitational field. The general relativity constraints relate the particle energy–momentum to some curvature invariants taking values in a group manifold. Some directions in the resulting momentum space turn out to be compact, which leads to a kind of "inverse Kaluza–Klein reduction" at short distances.*

**Keywords:** quantum gravity, Wilson loop, magnetic monopole, curved momentum space

## 1. Introduction

Various approaches to quantum gravity indicate that space–time becomes effectively two-dimensional at the Planck scale. These include "causal dynamical triangulations" [1], the "asymptotic safety scenario" [2], and several others. We do not describe them in detail here and suggest [3] for a review. These approaches were applied to (2+1)-, (3+1)-, and (sometimes) higher-dimensional gravity, and the number of microscopic space–time dimensions has always reduced to two.

While evidence for dimensional reduction provided from several independent sources may seem convincing, we still lack an intuitive picture of the mechanism behind this reduction. Approaches to quantum gravity that rely on universality and/or numerical methods leave this mechanism obscured.

Attempting to visualize the disappearance of some space dimensions, we adopt the most intuitive definition of dimension. We place a test particle in space. The dimension of the space is then the number of degrees of freedom of the particle.

If general relativity and a test particle coupled to it are the only ingredients in our theory, how can a change of space dimensionality possibly occur? An external gravitational field can wrap some of the space dimensions in which the particle propagates. This is the well known Kaluza–Klein reduction, but it occurs at large scales. The external gravitational field has no effect at small scales.

The only option that remains is to take the gravitational field created by the test particle itself into account. This field is usually neglected, which is acceptable for most practical applications. It was first noted by Bronstein in 1935 [4] that if we make a measurement at the Planck scale, then we cannot neglect the gravitational field created by this measurement. This back reaction effect limits the scales at which a measurement can possibly be made.

The main difficulty is that the above effect is not seen in the perturbation theory. Not surprisingly, the approaches to quantum gravity listed in [3] that predict dimensional reduction at short distances are all nonperturbative. It is therefore natural to start studying this effect in the framework of an exactly solvable theory.

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In Sec. 2, we summarize the results in (2+1)-dimensional gravity coupled to point particles previously obtained by several authors [5], [6]. The theory is exactly solvable, and the result of the back reaction of the gravitational field on a particle is that such a particle has a curved momentum space. The momentum space curvature is constant, its radius is determined by the Planck energy, and one of the three dimensions in the momentum space is compact.

In Sec. 3, we extend some of these results to (3+1)-dimensional gravity coupled to a single point particle. Although the whole theory is not exactly solvable, it suffices for our purposes to study one particular known solution. The momentum space of the particle is again curved. It is again a space of constant curvature, but its curvature radius now depends on the particle position in coordinate space, namely, on the distance to the origin. For a fixed distance to the origin the curvature radius of the momentum space is the energy scale such that this distance is the Schwarzschild radius. Only two of the four dimensions in this momentum space are noncompact.

## 2. Momentum space of a particle coupled to 2+1 gravity

The 2+1 gravity is a Chern–Simons theory for the $ISO(2,1)$ group. Let $\langle\,\cdot\,\rangle$ be a skew bilinear form that mixes translational and Lorentzian parts of the $iso(2,1)$ algebra. Below, $A$ is the $ISO(2,1)$ connection, $K$ is a fixed $iso(2,1)$ algebra element whose conjugacy class fixes the particle mass and spin, $h$ is an $ISO(2,1)$-valued variable, and $\gamma$ represents the particle worldline. The action of gravity coupled to the particle is

$$S = \kappa\int_M \left\langle A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right\rangle + \int_\gamma \langle h^{-1}\,d_A h\,K\rangle, \tag{1}$$

where $d_A$ is the covariant external derivative with the connectivity $A$. From this, we can derive the constraint equation

$$C = \kappa F(A) - hKh^{-1}\delta^2(x-x_{\mathrm{p}}) = 0.$$

Applying the non-Abelian Stokes theorem [7] to the last equation, we can relate the extrinsic charge $K$ to the value of the Wilson loop $g$ around the particle worldline,

$$he^{K/\kappa}h^{-1} = g.$$

We can also obtain the effective action for the particle taking the gravitational back reaction into account [6], [8]. For this, the solution of the constraints

$$A = \gamma(K\,d\phi + d)\gamma^{-1}, \qquad \gamma(x_0) = \mathrm{Id}, \quad \gamma(x_{\mathrm{p}}) = h,$$

is inserted into the action

$$S = \kappa\int_M \langle A\dot{A}\rangle + \int_\gamma \langle h^{-1}\dot{h}K\rangle + \int_{\neg M}\langle A_0\dot{A}_0\rangle,$$

where the last term is the boundary action in which $A_0$ is the flat connection and $\neg M$ is the completion of $M$ to the whole space. The result is

$$S = \kappa\int \mathrm{Tr}(xu^{-1}\dot{u}),$$

where $u$ is the Lorentzian part of $g$ and $x$ is the translational part of $h$. It can be seen that $u$ plays the role of momentum and is an element of $SO(2,1)$. It satisfies the mass-shell condition

$$\mathrm{Tr}\,u - \cos\frac{m}{\kappa} = 0.$$

The $SO(2,1)$ group manifold has one compact dimension, which is a spatial rotation.

The constructions considered in this section have been studied in detail and generalized in the framework called "doubly special relativity" [9].

## 3. The (3+1)-gravity results

What obstacles could we expect in extending the above results to 3+1 gravity? First, 3+1 gravity has infinitely many degrees of freedom, gravitational waves. But gravitational waves do not appear if we consider a situation with one freely moving particle. In other words, we can consider a "minisuperspace" model obtained from gravity by freezing all but a finite number of degrees of freedom. The degrees of freedom remaining unfrozen are the particle position with respect to a chosen origin.

The second problem is that unlike in two spatial dimensions, a point particle in three-dimensional space does not seem to select a topologically distinct Wilson loop that would contain information about the particle momentum. Such a loop in fact exists, but we must manipulate the solution in order to see this explicitly. This can be done most naturally in the framework of the pure gauge formulation of 3+1 gravity. This is MacDowell–Mansouri formulation [10], which is somewhat analogous to the Chern–Simons formulation of 2+1 gravity, but it works only for a nonzero cosmological constant (for definiteness, we choose it positive here).

Let $A^{IJ}$ be an $SO(4,1)$ connection, where capital indices $I, J, \ldots$ take the values $0,1,2,3,4$, and $v^I$ be a $SO(4,1)$ normalized 0-form vector. The particle is introduced into the theory analogously to 2+1 gravity in (1), but $h$ and $K$ are now $SO(4,1)$, and we have the ordinary trace instead of the skew bilinear form $\langle \cdot \rangle$.

The action for gravity coupled to the particle is then

$$S = \frac{l^2}{8\pi G} \int \epsilon_{IJKLM} F^{IJ}(A) v^K \wedge F^{LM}(A) + \lambda(v^I v_I - 1) + \int_\gamma \mathrm{Tr}(h^{-1}\, d_A h K), \tag{2}$$

where $F^{IJ}(A)$ is the curvature 2-form of the connection $A$. The second term in (2) is the normalization condition for $v^I$, introduced with a Lagrangian multiplier. The last term is the particle action. In (2), $l = 1/\sqrt{\Lambda}$ is the cosmological length.

The first term in (2) reduces to the Cartan–Weil action for gravity with a cosmological constant plus the Euler term by change of notation. We introduce a connection $\omega^{IJ}$ with respect to which $v^I$ is covariantly constant, $d_\omega v^I = 0$. Its curvature $R^{IJ}(\omega)$ is an $SO(3,1)$ curvature in the subgroup that leaves $v^I$ stable because $R^{IJ}(\omega) v_I = 0$. We then introduce the tetrad

$$e^I = l\, d_A v^I.$$

This is indeed a tetrad because $e^I v_I = 0$. The $SO(4,1)$ connection can then be decomposed as

$$A^{IJ} = \omega^{IJ} - \frac{1}{l}(v^I e^J - v^J e^I),$$

which leads to the curvature decomposition

$$F^{IJ} = R^{IJ} + \frac{1}{l^2} e^I \wedge e^J + \frac{1}{l}(v^J\, d_\omega e^I - v^I\, d_\omega e^J). \tag{3}$$

Substituting this in (2), we see that the last term in (3), which contains the torsion, does not enter the action. The first two terms produce the gravity action. In particular, the cross term is the Cartan–Weil term.

Varying this action with respect to the time component of $A$, we obtain the set of constraint equations

$$d_A(\epsilon_{IJKLM}v^K F^{LM}) = (hKh^{-1})_{IJ}\delta^3(x - x_{\mathrm{p}}). \tag{4}$$

Using these constraints, we can try to express the particle energy–momentum, which is the translational part of the charge in the right-hand side of (4) in terms of some integral characteristics of the geometry.

In what follows, we sometimes use an index-free notation, for example, $(\epsilon vF)_{IJ} \equiv \epsilon_{IJKLM}v^K F^{LM}$.

We can take an arbitrary ball $B$ containing our particle $x_{\mathrm{p}} \in B$, where the reference point is on its boundary. We take $\gamma(x) \in SO(4,1)$ such that $\gamma(x_{\mathrm{p}}) = h$. The charge $K$ can then be expressed as an integral over the ball:

$$K = \int_B \gamma^{-1}\, d_A(\epsilon vF)\gamma. \tag{5}$$

To convert it to a boundary integral in the general case, we would need something like a "non-Abelian Gauss theorem," which unfortunately does not yet exist. But if $F$ is a solution of constraints (4) with a $\delta$-function in the right-hand side, then there exists a $\gamma$ such that

$$K = \int_B \gamma^{-1}\, d_A(\epsilon vF)\gamma = \int_{\partial B} \gamma^{-1}(\epsilon vF)\gamma \tag{6}$$

simply because the $\delta$-function is the ordinary divergence of some vector field.

The last expression is reminiscent of the magnetic charge in the Yang–Mills theory [11], where $v$ plays the role of the Higgs field. But the analogy with the 't Hooft–Polyakov solution is not complete. First, the magnetic charge here is matrix-valued because the gauge group is larger than $SO(3)$. We therefore need the gauge transformation $\gamma(x)$ in (5) and (6) to place the solution into an $SO(3)$ subgroup of the gauge group. The second difference is the radial dependence of the Higgs field, which tends to a constant at infinity in the 't Hooft–Polyakov solution while $v^I$ here has the meaning of the coordinate grid in the de Sitter space. If we choose the static coordinate system, then it has the form

$$lv = \left(\sqrt{l^2 - r^2}\sinh\frac{t}{l},\, rn^i,\, \sqrt{l^2 - r^2}\cosh\frac{t}{l}\right), \tag{7}$$

where $i = 1,2,3$ and $n^i = x^i/|r|$ is the "hedgehog." In the particle rest frame, we can seek the solution in the form

$$A^{ij} = f(r)(n^i dn^j - n^j\, dn^i), \qquad A^{i0} = \sinh\frac{t}{l}\, g(r)\, dn^i, \quad A^{i4} = \cosh\frac{t}{l}\, g(r)\, dn^i. \tag{8}$$

Here, the solution is written in the gauge $A_r = 0$, and $f(r)$ and $g(r)$ are arbitrary functions of $r$ to be found. But these functions do not appear in the final result. By virtue of the known formula

$$\epsilon_{ijk}\, dn^i \wedge dn^j \wedge dn^k = 4\pi\delta^3(x), \tag{9}$$

we must finally obtain the Brouwer degree in (6),

$$K = \int_{\partial B} \gamma^{-1}(\epsilon_{IJklm}n^k\, dn^l \wedge dn^m)\gamma, \tag{10}$$

where $\gamma$ can be taken constant on $\partial B$. To cancel the "Higgs field" growth as $r$ in (7), the curvature must now decrease as $1/r^3$. It is known that this is indeed the case in the Schwarzschild solution.

We can now use the fact that formula (9) is invariant under arbitrary continuous norm-preserving transformations of the field $n^i$. Expression (10) can be essentially simplified if we transform $n^i$ into a constant field on the sphere ("comb the hedgehog") and move it outside the integral and apply the Stokes theorem to what remains in the integrand. If this were possible, then the result in (10) would be zero because the sphere has no boundary. But a transformation from a "hedgehog" to a constant field $n^i$ cannot be made continuous on the whole sphere, because it changes the winding number from 1 to 0. But it can be made continuous on any open subset of the sphere. We can therefore cut the sphere in half and "comb the hedgehog" on each of the halves [12] or, alternatively, puncture the sphere and "comb the hedgehog" on the punctured sphere [13]. The direction in which the constant $n^i$ points is the position of the origin (reference point). We can then apply the Stokes theorem to the curvature integrand in (6) ($v^I$ is now constant). But the integration region has now a boundary with a topologically nontrivial loop around it: the equator in the first case above or the puncture in the second. Both cases yield the same value of the Wilson loop:

$$g = h \exp\left[\epsilon_{abc0} 2Gm \frac{x^c}{x^2}\right] h^{-1}. \tag{11}$$

In the limit $l \to \infty$, we have $x^a = (t, x^i)$, $g$ is an element of the $SO(3,1)$ subgroup, and $K = lmT^{04}$.

The momentum space curvature varies with the distance to the origin. For a fixed $r = \sqrt{x^2}$, the curvature radius of the momentum space is the energy for which $r$ is the Schwarzschild radius. Loop (11) belongs to the conjugacy class of a purely rotational element of the Cartan subgroup. This means that one boost direction is absent and there are only two noncompact dimensions in the momentum space.

## 4. Conclusion

We can expect that the curvature of the momentum space will improve the ultraviolet behavior in the quantum theory of gravity and will even render it renormalizable. This effect is nonperturbative and would require an exact solution of the theory, which is hardly possible for four-dimensional gravity. An alternative possibility is to modify the perturbation theory to take the momentum space curvature into account. Such an approach for three-dimensional gravity coupled to a scalar field was used in [14]. It would be interesting to extend these results to (3+1)-dimensional gravity.

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## REFERENCES

1. J. Ambjørn, J. Jurkiewicz, and R. Loll, *Phys. Rev. Lett.*, **85**, 924–927 (2000); arXiv:hep-th/0002050v3 (2000); **93**, 131301 (2004); arXiv:hep-th/0404156v4 (2004); *Phys. Rev. D*, **72**, 064014 (2005); arXiv:hep-th/0505154v2 (2005).
2. S. Weinberg, "Ultraviolet divergences in quantum theories of gravitation," in: *General Relativity: An Einstein Centenary Survey* (S. W. Hawking and W. Israel, eds.), Cambridge Univ. Press, Cambridge (1979), pp. 790–831; M. Reuter and F. Saueressig, *Phys. Rev. D*, **65**, 065016 (2002); arXiv:hep-th/0110054v1 (2001); D. F. Litim, *Phys. Rev. Lett.*, **92**, 201301 (2004); arXiv:hep-th/0312114v2 (2003); M. Niedermaier, *Class. Q. Grav.*, **24**, R171–R230 (2007); arXiv:gr-qc/0610018v2 (2006).
3. S. Carlip, *AIP Conf. Proc.*, **1196**, 72–80 (2009); arXiv:0909.3329v1 [gr-qc] (2009).
4. M. Bronstein, *Phys. Z. Sowjetunion*, **9**, 140–157 (1936).
5. G. 't Hooft, *Class. Q. Grav.*, **10**, 1653–1664 (1993); **13**, 1023–1039 (1996).
6. H. J. Matschull and M. Welling, *Class. Q. Grav.*, **15**, 2981–3030 (1998); arXiv:gr-qc/9708054v2 (1997).
7. I. Ya. Aref'eva, *Theor. Math. Phys.*, **43**, 353–356 (1980).
8. A. Yu. Alekseev and A. Z. Malkin, *Commun. Math. Phys.*, **169**, 99–119 (1995); arXiv:hep-th/9312004v1 (1993); C. Meusburger and B. J. Schroers, *Nucl. Phys. B*, **738**, 425–456 (2006); arXiv:hep-th/0505143v1 (2005).

9. G. Amelino-Camelia and D. V. Ahluwalia, *Internat. J. Mod. Phys. D*, **11**, 35–59 (2002); arXiv:gr-qc/0012051v2 (2000); J. Kowalski-Glikman, "Introduction to doubly special relativity," in: *Planck Scale Effects in Astrophysics and Cosmology* (Lect. Notes Phys., Vol. 669, J. Kowalski-Glikman and G. Amelino-Camelia, eds.), Springer, Berlin (2005), pp. 131–159; arXiv:hep-th/0405273v1 (2004).
10. S. W. MacDowell and F. Mansouri, *Phys. Rev. Lett.*, **38**, 739–742 (1977); Erratum, **38**, 1376 (1977).
11. G. 't Hooft, *Nucl. Phys. B*, **79**, 276–284 (1974); A. M. Polyakov, *JETP Lett.*, **20**, 194–195 (1974).
12. T. T. Wu and C. N. Yang, *Phys. Rev. D*, **12**, 3845–3857 (1975).
13. P. A. M. Dirac, *Proc. Roy. Soc. A*, **133**, 60–72 (1931).
14. L. Freidel and E. R. Livine, *Phys. Rev. Lett.*, **96**, 221301 (2006); arXiv:hep-th/0512113v2 (2005).